

DIMORPHIC MERSENNE NUMBERS AND THEIR APPLICATIONS

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ABSTRACT. The Mersenne primes are primes which can be written as some prime power of 2 minus 1. These primes were studied from antiquity in that their close connection with perfect numbers and even to present day in that their easiness for primality test. In this paper, we introduce the dimorphic Mersenne numbers as a degenerate version of the Mersenne numbers and investigate some of their properties in connection with the degenerate Bernoulli polynomials and the incomplete Bell polynomials.

1. INTRODUCTION

In recent years, there have been active explorations for various degenerate versions of many special numbers and polynomials with diverse tools such as generating functions, combinatorial methods, p -adic analysis, umbral calculus, differential equations, probability theory, operator theory, analytic number theory and quantum physics. These were initiated by Carlitz in [3,4] and yielded many interesting results of arithmetical and combinatorial nature (see [7-9,12-14,16-18] and the references therein).

The aim of this paper is to introduce the dimorphic Mersenne numbers as a degenerate version of the Mersenne numbers and to find some of their applications in connection with the degenerate Bernoulli polynomials. The novelty of this paper is that it is the first paper which introduces the dimorphic Mersenne numbers and shows some of their applications associated with the degenerate Bernoulli numbers and the incomplete Bell polynomials.

The outline of this paper is as follows. In Section 1, we recall Mersenne numbers, the degenerate exponentials and the degenerate Bernoulli polynomials. We remind the reader of the degenerate Stirling numbers of the second kind, the incomplete Bell polynomials and the complete Bell polynomials. Section 2 is the main result of this paper. We first introduce the dimorphic Mersenne numbers as a degenerate version of the Mersenne numbers. Then, by using the generating function of the dimorphic Mersenne numbers in (16), in Theorem 1, we derive a recurrence formula for the degenerate Bernoulli polynomials involving the dimorphic Mersenne numbers. We express a factor of the generating function of the dimorphic Mersenne numbers in terms of the incomplete Bell polynomials with arguments given by the degenerate Bernoulli polynomials (see (16), (20)). In Theorem 2, we obtain from this expression a representation of the dimorphic Mersenne numbers in terms of those incomplete Bell polynomials with arguments given by the degenerate Bernoulli polynomials. Finally, in Theorem 3 we get an expression of the degenerate Bernoulli polynomials in terms of the same incomplete Bell polynomials with arguments given by the degenerate Bernoulli polynomials from Theorem 2. In the rest of this section, we recall the facts that are needed throughout this paper.

The Mersenne numbers are defined by

$$M_n = 2^n - 1, \quad (n \geq 0), \quad (\text{see [2]}).$$

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M_p is called a Mersenne prime if p is prime and M_p is also prime. For example, M_p is a Mersenne prime for $p = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521$ (see [2]). As of May 2022, the largest prime is the Mersenne prime $2^{82,589,933} - 1$ which has 24,862,048 digits when it is written in base 10. It is easy to show that

$$(1) \quad \frac{z}{1 - 3z + 2z^2} = \sum_{n=0}^{\infty} M_n z^n, \quad (\text{see [2]}).$$

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponentials are defined by

$$(2) \quad e_{\lambda}^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad e_{\lambda}(t) = e_{\lambda}^1(t), \quad (\text{see [7-9, 11-14]}),$$

where

$$(3) \quad (x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x - \lambda)(x - 2\lambda) \cdots (x - (n - 1)\lambda), \quad (n \geq 1).$$

Note that

$$\lim_{\lambda \rightarrow 0} e_{\lambda}^x(t) = e^{xt}, \quad \lim_{\lambda \rightarrow 0} (x)_{n,\lambda} = x^n, \quad (n \geq 0).$$

In [3,4], Carlitz introduced the degenerate Bernoulli polynomials defined by

$$(4) \quad \frac{t}{e_{\lambda}(t) - 1} e_{\lambda}^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}.$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$, $(n \geq 0)$, are called the degenerate Bernoulli numbers. Note that $\lim_{\lambda \rightarrow 0} \beta_{n,\lambda} = B_n$, where B_n are the ordinary Bernoulli numbers, (see [1-19]). From (4), we note that

$$\beta_{n,\lambda}(x) = \sum_{k=0}^n \binom{n}{k} (x)_{n-k,\lambda} \beta_{k,\lambda}, \quad (n \geq 0), \quad (\text{see [3,4]}).$$

It is known that the Stirling numbers of the second kind are defined by

$$(5) \quad x^n = \sum_{k=0}^n S_2(n, k) (x)_k, \quad (n \geq 0), \quad (\text{see [8]}),$$

where $(x)_0 = 1$, $(x)_n = x(x - 1) \cdots (x - n + 1)$, $(n \geq 1)$. From (5), we note that

$$(6) \quad \frac{1}{k!} (e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n, k) \frac{t^n}{n!}, \quad (k \geq 0), \quad (\text{see [8]}).$$

The incomplete Bell polynomials are defined by

$$(7) \quad \frac{1}{k!} \left(\sum_{i=1}^{\infty} x_i \frac{t^i}{i!} \right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, \dots, x_{n-k+1}) \frac{t^n}{n!}, \quad (k \geq 0), \quad (\text{see [1-8, 10-13, 15]}),$$

More explicitly, they are given by

$$(8) \quad \begin{aligned} & B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \\ &= \sum_{\substack{l_1 + \dots + l_{n-k+1} = k \\ l_1 + 2l_2 + \dots + (n-k+1)l_{n-k+1} = n}} \frac{n!}{l_1! l_2! \cdots l_{n-k+1}!} \left(\frac{x_1}{1!} \right)^{l_1} \left(\frac{x_2}{2!} \right)^{l_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!} \right)^{l_{n-k+1}}, \end{aligned}$$

where the sum runs over all nonnegative integers l_1, \dots, l_{n-k+1} satisfying $l_1 + \dots + l_{n-k+1} = k$ and $l_1 + 2l_2 + \dots + (n - k + 1)l_{n-k+1} = n$.

The complete Bell polynomials are given by

$$(9) \quad \exp\left(\sum_{i=1}^{\infty} x_i \frac{t^i}{i!}\right) = \sum_{n=0}^{\infty} B_n(x_1, \dots, x_n) \frac{t^n}{n!} \\ = 1 + \sum_{n=1}^{\infty} B_n(x_1, x_2, \dots, x_n) \frac{t^n}{n!}, \quad (\text{see [1-8, 10-13, 15]}).$$

Then we have

$$B_0(x_1, \dots, x_n) = 1, \quad B_n(x_1, \dots, x_n) = \sum_{l_1+2l_2+\dots+nl_n=n} \frac{n!}{l_1! \dots l_n!} \left(\frac{x_1}{1!}\right)^{l_1} \dots \left(\frac{x_n}{n!}\right)^{l_n}, \quad (n \geq 1),$$

where the sum runs over all nonnegative integers l_1, \dots, l_n satisfying $l_1 + 2l_2 + \dots + nl_n = n$.

From (7) and (8), we note that

$$(10) \quad \exp\left(\sum_{i=1}^{\infty} x_i \frac{t^i}{i!}\right) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\sum_{i=1}^{\infty} x_i \frac{t^i}{i!}\right)^k \\ = 1 + \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \frac{t^n}{n!} \\ = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^n B_{n,k}(x_1, \dots, x_{n-k+1}) \frac{t^n}{n!}.$$

By (9) and (10), we get

$$(11) \quad B_n(x_1, x_2, \dots, x_n) = \sum_{k=1}^n B_{n,k}(x_1, \dots, x_{n-k+1}), \quad (n \geq 1).$$

It is known that the Bell polynomials are given by

$$\phi_n(x) = \sum_{k=0}^n S_2(n, k) x^k, \quad (n \geq 0), \quad (\text{see [16, 17]}).$$

From (6), (7) and (11), we note that

$$B_{n,k}(1, 1, \dots, 1) = S_2(n, k), \quad B_n(x, x, \dots, x) = \phi_n(x), \quad (n \geq 0).$$

2. DIMORPHIC MERSENNE NUMBERS AND THEIR APPLICATIONS

Now, we consider the dimorphic Mersenne numbers given by

$$(12) \quad M_{n,\lambda} = (2)_{n,\lambda} - (1)_{n,\lambda}, \quad (\lambda \in \mathbb{R}, \quad n \geq 0).$$

Note that $\lim_{\lambda \rightarrow 0} M_{n,\lambda} = M_n$. We observe that

$$(13) \quad e_{\lambda}^2(t) - e_{\lambda}(t) = \sum_{n=0}^{\infty} (2)_{n,\lambda} \frac{t^n}{n!} - \sum_{n=0}^{\infty} (1)_{n,\lambda} \frac{t^n}{n!} \\ = \sum_{n=0}^{\infty} \left((2)_{n,\lambda} - (1)_{n,\lambda} \right) \frac{t^n}{n!}.$$

Thus, by (12) and (13), we have

$$(14) \quad e_{\lambda}^2(t) - e_{\lambda}(t) = \sum_{n=0}^{\infty} M_{n,\lambda} \frac{t^n}{n!}.$$

Thus, by (14) and noting that $M_{0,\lambda} = 0$, we see that

$$(15) \quad \frac{1}{t} (e_{\lambda}^2(t) - e_{\lambda}(t)) = \frac{1}{t} \sum_{n=1}^{\infty} M_{n,\lambda} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{M_{n+1,\lambda}}{n+1} \frac{t^n}{n!}.$$

From (15), we have

$$(16) \quad \begin{aligned} \sum_{n=0}^{\infty} \frac{M_{n+1,\lambda}}{n+1} \frac{t^n}{n!} &= \frac{1}{t} (e_{\lambda}^2(t) - e_{\lambda}(t)) = \frac{e_{\lambda}(t)}{t} (e_{\lambda}(t) - 1) \\ &= \frac{e_{\lambda}^{x+1}(t)}{t e_{\lambda}^x(t)} (e_{\lambda}(t) - 1). \end{aligned}$$

Now, we observe that

$$(17) \quad \begin{aligned} e_{\lambda}^{x+1}(t) &= \frac{t}{e_{\lambda}(t) - 1} e_{\lambda}^x(t) \frac{e_{\lambda}^{x+1}(t)}{t e_{\lambda}^x(t)} (e_{\lambda}(t) - 1) \\ &= \sum_{l=0}^{\infty} \beta_{l,\lambda}(x) \frac{t^l}{l!} \sum_{m=0}^{\infty} \frac{M_{m+1,\lambda}}{m+1} \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^n \binom{n}{l} \beta_{l,\lambda}(x) \frac{M_{n-l+1,\lambda}}{n-l+1} \frac{t^n}{n!}. \end{aligned}$$

On the other hand, by (2), we also have

$$(18) \quad e_{\lambda}^{x+1}(t) = \sum_{n=0}^{\infty} (x+1)_{n,\lambda} \frac{t^n}{n!}.$$

From (17) and (18), we note that

$$(19) \quad \begin{aligned} (x+1)_{n,\lambda} &= \sum_{l=0}^n \binom{n}{l} \beta_{l,\lambda}(x) \frac{M_{n-l+1,\lambda}}{n-l+1} \\ &= \beta_{n,\lambda}(x) + \sum_{l=0}^{n-1} \binom{n}{l} \beta_{l,\lambda}(x) \frac{M_{n-l+1,\lambda}}{n-l+1}. \end{aligned}$$

Therefore, by (19), we obtain the following theorem.

Theorem 1. For $n \geq 0$, we have

$$\beta_{n,\lambda}(x) = (x+1)_{n,\lambda} - \sum_{l=0}^{n-1} \binom{n}{l} \beta_{l,\lambda}(x) \frac{M_{n-l+1,\lambda}}{n-l+1}.$$

From (7), we note that

$$\begin{aligned}
 (20) \quad \frac{e_\lambda(t) - 1}{te_\lambda^x(t)} &= \frac{1}{\frac{t}{e_\lambda(t)-1}e_\lambda^x(t)} = \frac{1}{1 + \frac{t}{e_\lambda(t)-1}e_\lambda^x(t) - 1} \\
 &= \frac{1}{1 + \sum_{n=1}^\infty \beta_{n,\lambda}(x) \frac{t^n}{n!}} = \sum_{k=0}^\infty (-1)^k \left(\sum_{i=1}^\infty \beta_{i,\lambda}(x) \frac{t^i}{i!} \right)^k \\
 &= 1 + \sum_{k=1}^\infty (-1)^k k! \frac{1}{k!} \left(\sum_{i=1}^\infty \beta_{i,\lambda}(x) \frac{t^i}{i!} \right)^k \\
 &= \sum_{k=1}^\infty (-1)^k k! \sum_{n=k}^\infty B_{n,k}(\beta_{1,\lambda}(x), \beta_{2,\lambda}(x), \dots, \beta_{n-k+1,\lambda}(x)) \frac{t^n}{n!} + 1 \\
 &= \sum_{n=1}^\infty \sum_{k=1}^n (-1)^k k! B_{n,k}(\beta_{1,\lambda}(x), \beta_{2,\lambda}(x), \dots, \beta_{n-k+1,\lambda}(x)) \frac{t^n}{n!} + 1.
 \end{aligned}$$

On the other hand, by (16), we get

$$\begin{aligned}
 (21) \quad 1 + \sum_{n=1}^\infty \frac{M_{n+1,\lambda}}{n+1} \frac{t^n}{n!} &= \sum_{n=0}^\infty \frac{M_{n+1,\lambda}}{n+1} \frac{t^n}{n!} = \frac{e_\lambda^{x+1}(t)}{te_\lambda^x(t)} (e_\lambda(t) - 1) \\
 &= e_\lambda^{x+1}(t) \left(1 + \sum_{j=1}^\infty \sum_{k=1}^j (-1)^k k! B_{j,k}(\beta_{1,\lambda}(x), \dots, \beta_{j-k+1,\lambda}(x)) \frac{t^j}{j!} \right) \\
 &= \sum_{l=0}^\infty (x+1)_{l,\lambda} \frac{t^l}{l!} \left(1 + \sum_{j=1}^\infty \sum_{k=1}^j (-1)^k k! B_{j,k}(\beta_{1,\lambda}(x), \dots, \beta_{j-k+1,\lambda}(x)) \frac{t^j}{j!} \right) \\
 &= \sum_{n=0}^\infty (x+1)_{n,\lambda} \frac{t^n}{n!} + \sum_{n=1}^\infty \sum_{j=1}^n \sum_{k=1}^j \binom{n}{j} (x+1)_{n-j,\lambda} (-1)^k k! B_{j,k}(\beta_{1,\lambda}(x), \dots, \beta_{j-k+1,\lambda}(x)) \frac{t^n}{n!} \\
 &= 1 + \sum_{n=1}^\infty \left((x+1)_{n,\lambda} + \sum_{j=1}^n \sum_{k=1}^j \binom{n}{j} (x+1)_{n-j,\lambda} (-1)^k k! B_{j,k}(\beta_{1,\lambda}(x), \dots, \beta_{j-k+1,\lambda}(x)) \right) \frac{t^n}{n!}.
 \end{aligned}$$

Therefore, by comparing the coefficients on both sides of (21), we obtain the following theorem.

Theorem 2. For $n \in \mathbb{N}$ and any x , we have

$$\frac{M_{n+1,\lambda}}{n+1} = (x+1)_{n,\lambda} + \sum_{i=1}^n \sum_{k=1}^i \binom{n}{i} (x+1)_{n-i,\lambda} (-1)^k k! B_{i,k}(\beta_{1,\lambda}(x), \dots, \beta_{i-k+1,\lambda}(x)).$$

We note that

$$\begin{aligned}
 (22) \quad B_{n,1}(\beta_{1,\lambda}(x), \beta_{2,\lambda}(x), \dots, \beta_{n,\lambda}(x)) &= \sum_{\substack{l_1 + \dots + l_n = 1 \\ l_1 + 2l_2 + \dots + nl_n = n}} \frac{n!}{l_1! l_2! \dots l_n!} \left(\frac{\beta_{1,\lambda}(x)}{1!} \right)^{l_1} \dots \left(\frac{\beta_{n,\lambda}(x)}{n!} \right)^{l_n} \\
 &= n! \frac{\beta_{n,\lambda}(x)}{n!} = \beta_{n,\lambda}(x).
 \end{aligned}$$

From Theorem 2 and (22), we further observe that

$$\begin{aligned}
 (23) \quad \frac{M_{n+1,\lambda}}{n+1} &= (x+1)_{n,\lambda} + \sum_{j=1}^n \sum_{k=1}^j \binom{n}{j} (x+1)_{n-j,\lambda} (-1)^k k! B_{j,k}(\beta_{1,\lambda}(x), \beta_{2,\lambda}(x), \dots, \beta_{j-k+1,\lambda}(x)) \\
 &= (x+1)_{n,\lambda} + \sum_{k=1}^n (-1)^k k! B_{n,k}(\beta_{1,\lambda}(x), \dots, \beta_{n-k+1,\lambda}(x)) \\
 &\quad + \sum_{j=1}^{n-1} \sum_{k=1}^j \binom{n}{j} (x+1)_{n-j,\lambda} (-1)^k k! B_{j,k}(\beta_{1,\lambda}(x), \dots, \beta_{j-k+1,\lambda}(x)) \\
 &= (x+1)_{n,\lambda} - B_{n,1}(\beta_{1,\lambda}(x), \dots, \beta_{n,\lambda}(x)) + \sum_{k=2}^n (-1)^k k! B_{n,k}(\beta_{1,\lambda}(x), \beta_{2,\lambda}(x), \dots, \beta_{n-k+1,\lambda}(x)) \\
 &\quad + \sum_{j=1}^{n-1} \sum_{k=1}^j \binom{n}{j} (x+1)_{n-j,\lambda} (-1)^k k! B_{j,k}(\beta_{1,\lambda}(x), \beta_{2,\lambda}(x), \dots, \beta_{j-k+1,\lambda}(x)) \\
 &= -\beta_{n,\lambda}(x) + (x+1)_{n,\lambda} + \sum_{k=2}^n (-1)^k k! B_{n,k}(\beta_{1,\lambda}(x), \beta_{2,\lambda}(x), \dots, \beta_{n-k+1,\lambda}(x)) \\
 &\quad + \sum_{j=1}^{n-1} \sum_{k=1}^j \binom{n}{j} (x+1)_{n-j,\lambda} (-1)^k k! B_{j,k}(\beta_{1,\lambda}(x), \beta_{2,\lambda}(x), \dots, \beta_{j-k+1,\lambda}(x)).
 \end{aligned}$$

Thus, by (23), we get

$$\begin{aligned}
 (24) \quad \beta_{n,\lambda}(x) &= (x+1)_{n,\lambda} - \frac{M_{n+1,\lambda}}{n+1} + \sum_{k=2}^n (-1)^k k! B_{n,k}(\beta_{1,\lambda}(x), \beta_{2,\lambda}(x), \dots, \beta_{n-k+1,\lambda}(x)) \\
 &\quad + \sum_{j=1}^{n-1} \sum_{k=1}^j \binom{n}{j} (x+1)_{n-j,\lambda} (-1)^k k! B_{j,k}(\beta_{1,\lambda}(x), \beta_{2,\lambda}(x), \dots, \beta_{j-k+1,\lambda}(x)).
 \end{aligned}$$

Therefore, by (24), we obtain the following theorem.

Theorem 3. For $n \in \mathbb{N}$, we have

$$\begin{aligned}
 \beta_{n,\lambda}(x) &= (x+1)_{n,\lambda} - \frac{M_{n+1,\lambda}}{n+1} + \sum_{k=2}^n (-1)^k k! B_{n,k}(\beta_{1,\lambda}(x), \beta_{2,\lambda}(x), \dots, \beta_{n-k+1,\lambda}(x)) \\
 &\quad + \sum_{j=1}^{n-1} \sum_{k=1}^j \binom{n}{j} (x+1)_{n-j,\lambda} (-1)^k k! B_{j,k}(\beta_{1,\lambda}(x), \beta_{2,\lambda}(x), \dots, \beta_{j-k+1,\lambda}(x)).
 \end{aligned}$$

3. CONCLUSION

Study of various degenerate versions of some special numbers and polynomials, which began with the papers [3,4] by Carlitz, regained recent interests of some mathematicians and led to unexpected introduction of degenerate gamma functions and degenerate umbral calculus (see [9,14]).

In this paper, the dimorphic Mersenne numbers were introduced as a degenerate version of the Mersenne numbers and some of their properties was investigated in connection with the degenerate Bernoulli polynomials and the incomplete Bell polynomials.

It is one of our future research projects to continue to study various degenerate versions of some special numbers and polynomials and to find their applications to physics, science and engineering as well as to mathematics.

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