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HURWITZ TYPE RESULTS FOR CERTAIN REPRESENTATIONS OF INTEGERS AS SUMS OF SQUARES

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Abstract: Let $r_{\alpha,\beta}(n)$ denote the number of representations of n as a sum of α times a square and β times another square. In the recent past, a number of authors have obtained Hurwitz type results for representations of integers as sums of squares. Motivated by their works, in this paper we prove many results in which the generating function of $r_{2,3}(\lambda^k(an+b))$, $r_{2,4}(an+b)$ and $r_{1,5}(\lambda^k(an+b))$ for various non-negative integer values of λ, k, a and b are infinite products. To obtain our main results we use the theta function identities of Ramanujan found in chapter 16 of his second notebook.

Keywords: Sums of squares, theta function, infinite products.

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1 Introduction and some results

Throughout this paper we utilize some standard notations. For any complex number a and q with $|q| < 1$, we use the following q-product notation:

$$(a)_\infty := (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n),$$

$$(a)_n := (a;q)_n := \frac{(a)_\infty}{(aq^n)_\infty}, \text{ n, any integer.}$$

Ramanujan's general theta function $f(a,b)$ (cf. [1, 3, 8]) is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \ |ab| < 1. \tag{1.1}$$

Some of the properties of $f(a,b)$ are given by [3, Entry 18, p. 34]

$$f(a,b) = f(b,a), \tag{1.2}$$

$$f(1,a) = 2f(a,a^3) \tag{1.3}$$

and if n is an integer,

$$f(a,b) = a^{n(n+1)/2}b^{n(n-1)/2}f(a(ab)^n, b(ab)^{-n}). \tag{1.4}$$

The special cases of $f(a,b)$ are given by (cf. [1, 3, 8])

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

and

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

for $|q| < 1$.

For a non-negative integer n and a positive integer k, let $r_k(n)$ denote the number of representations of n as a sum of k squares and $r_{\alpha,\beta}(n)$ denote the number of representations of n as a sum of α times a square and β times another square. Hurwitz [6] proved eleven cases in which the generating function of $r_3(an+b)$ is a simple infinite product. Cooper and Hirschhorn [4] proved those eleven results along with twelve results of same sort and eighty infinite families of similar results. Hirschhorn and McGowan [5] have obtained many cases in which the generating function of $r_2(an+b)$ and $r_4(an+b)$ is a single infinite product. Barrucand, Cooper and Hirschhorn [2] established simple infinite product for the generating function of $r_k(an+b)$, for $k \in \{5,6,7\}$. Mahadeva Naika, Sumanth Bharadwaj and Hemanthkumar [7] established some linear relations for $r_{1,2}(n)$ and proved several Hurwitz type results. Somashekara and Vidya [9] proved many results for $r_2(2^k(an+b))$, $r_4(2^k(an+b))$, $r_{1,3}(2^k(an+b))$ for various values of a and b.

The main objective of this paper is to prove many Hurwitz type results for the generating function of $r_{2,3}(\lambda^k(an+b))$, $r_{2,4}(an+b)$ and $r_{1,5}(\lambda^k(an+b))$ for various non-negative integer values of λ, k, a and b which are infinite products.

2 Preliminary results

In the following lemma, we give some elementary properties of the Ramanujan's theta function and its special cases (cf. [1, 3, 8]).

Lemma. We have

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8), \tag{2.1}$$

$$\varphi(q) = \varphi(q^9) + 2qf(q^3, q^{15}), \tag{2.2}$$

$$\varphi(q) = \varphi(q^{25}) + 2qf(q^{15}, q^{35}) + 2q^4 f(q^5, q^{45}), \tag{2.3}$$

$$\psi(q) = f(q^6, q^{10}) + qf(q^2, q^{14}), \tag{2.4}$$

$$\psi(q) = f(q^3, q^6) + q\psi(q^9), \tag{2.5}$$

$$\psi(q) = f(q^{10}, q^{15}) + qf(q^5, q^{20}) + q^3\psi(q^{25}), \tag{2.6}$$

$$f(q, q^5) = f(q^8, q^{16}) + qf(q^4, q^{20}), \tag{2.7}$$

$$f(q, q^9) = f(q^{12}, q^{28}) + qf(q^8, q^{32}), \tag{2.8}$$

$$f(q, q^5) = f(q^{21}, q^{33}) + qf(q^{15}, q^{39}) + q^5 f(q^3, q^{51}). \tag{2.9}$$

Proof. We have

$$\begin{aligned}
f(a,b) &= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \\
&= \sum_{n=-\infty}^{\infty} a^n (ab)^{n(n-1)/2} \\
&= \sum_{\substack{n=-\infty \\ n-even}}^{\infty} a^n (ab)^{n(n-1)/2} + \sum_{\substack{n=-\infty \\ n-odd}}^{\infty} a^n (ab)^{n(n-1)/2} \\
&= \sum_{n=-\infty}^{\infty} a^{2n} (ab)^{n(2n-1)} + \sum_{n=-\infty}^{\infty} a^{2n+1} (ab)^{(2n+1)n} \\
&= \sum_{n=-\infty}^{\infty} a^{2n^2+n} b^{2n^2-n} + a \sum_{n=-\infty}^{\infty} a^{2n^2+3n} b^{2n^2+n} \\
&= \sum_{n=-\infty}^{\infty} (a^3 b)^{n(n+1)/2} (ab^3)^{n(n-1)/2} + a \sum_{n=-\infty}^{\infty} (a^5 b^3)^{n(n+1)/2} \left(\frac{b}{a}\right)^{n(n-1)/2} \\
&= f(a^3 b, ab^3) + af(b/a, a^5 b^3), \qquad (2.10)
\end{aligned}$$

on using (1.1) and (1.2). Substituting $(a, b) = (q, q)$, (q, q^3), (q, q^5) and (q, q^9) in (2.10), we obtain (2.1), (2.4), (2.7) and (2.8) respectively.

$$
\begin{aligned}
f(a,b) &= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \\
&= \sum_{n=-\infty}^{\infty} a^{n}(ab)^{n(n-1)/2} \\
&= \sum_{\substack{n=-\infty \\ n\equiv 0(mod3)}}^{\infty} a^{n}(ab)^{n(n-1)/2} + \sum_{\substack{n=-\infty \\ n\equiv 1(mod3)}}^{\infty} a^{n}(ab)^{n(n-1)/2} + \sum_{\substack{n=-\infty \\ n\equiv 2(mod3)}}^{\infty} a^{n}(ab)^{n(n-1)/2} \\
&= \sum_{n=-\infty}^{\infty} a^{3n}(ab)^{3n(3n-1)/2} + \sum_{n=-\infty}^{\infty} a^{3n+1}(ab)^{(3n+1)3n/2} + \sum_{n=-\infty}^{\infty} a^{3n+2}(ab)^{(3n+2)(3n+1)/2} \\
&= \sum_{n=-\infty}^{\infty} a^{(9n^2+3n)/2} b^{(9n^2-3n)/2} + a\sum_{n=-\infty}^{\infty} a^{(9n^2+9n)/2} b^{(9n^2+3n)/2} + a^3 b \sum_{n=-\infty}^{\infty} a^{(9n^2+15n)/2} b^{(9n^2+9n)/2} \\
&= f(a^6b^3, a^3b^6) + af(b^3, a^9b^6) + a^3bf(a^{12}b^9, a^{-3}), \qquad (2.11)
\end{aligned}
$$

on using (1.1) and (1.2). Using (1.4) for $n = 1$, $a = q^3$, $b = q^{15}$ and substituting the resulting identity in (2.11) for $a = b = q$, we obtain (2.2). Substituting $a = 1$, $b = q$ in (2.11) and using (1.3), we obtain (2.5). Using (1.4) for $n = 1$, $a = q^3$, $b = q^{51}$ and sustituting the resulting identity in (2.11) for $a = q$, $b = q^5$, we obtain (2.9).

$$
\begin{aligned}
f(a,b) &= \sum_{n=-\infty}^{\infty} a^{n}(ab)^{n(n-1)/2} \\
&= \sum_{\substack{n=-\infty \\ n\equiv 0(mod5)}}^{\infty} a^{n}(ab)^{n(n-1)/2} + \sum_{\substack{n=-\infty \\ n\equiv 1(mod5)}}^{\infty} a^{n}(ab)^{n(n-1)/2} + \sum_{\substack{n=-\infty \\ n\equiv 2(mod5)}}^{\infty} a^{n}(ab)^{n(n-1)/2} \\
&\quad + \sum_{\substack{n=-\infty \\ n\equiv 3(mod5)}}^{\infty} a^{n}(ab)^{n(n-1)/2} + \sum_{\substack{n=-\infty \\ n\equiv 4(mod5)}}^{\infty} a^{n}(ab)^{n(n-1)/2} \\
&= \sum_{n=-\infty}^{\infty} a^{5n}(ab)^{5n(5n-1)/2} + a\sum_{n=-\infty}^{\infty} a^{5n}(ab)^{5n(5n+1)/2} + a^2\sum_{n=-\infty}^{\infty} a^{5n}(ab)^{(5n+2)(5n+1)/2} \\
&\quad + a^3\sum_{n=-\infty}^{\infty} a^{5n}(ab)^{(5n+3)(5n+2)/2} + a^4\sum_{n=-\infty}^{\infty} a^{5n}(ab)^{(5n+4)(5n+3)/2} \\
&= f(a^{15}b^{10}, a^{10}b^{15}) + af(a^5b^{10}, a^{20}b^{15}) + a^3bf(b^5, a^{25}b^{20}) + a^6b^3f(a^{30}b^{25}, a^{-5}) \\
&\quad + a^{10}b^6f(a^{35}b^{30}, a^{-10}b^{-5}), \qquad (2.12)
\end{aligned}
$$

on using (1.1) and (1.2). Using (1.4) for $a = q^5$, $b = q^{45}$, $n = 1$ and $a = q^{15}$, $b = q^{35}$, $n = 1$ and substituting the resulting identities in (2.12) for $a = b = q$, we obtain (2.3). Using (1.4) for $n = 1$, $a = q^5$, $b = q^{20}$ and substituting the resulting identity in (2.12) for $a = 1$, $b = q$ and using (1.3), we obtain (2.6). □

3 Main results

Motivated by the works of Cooper and Hirschhorn [4], Hirschhorn and McGowan [5], Barrucand, Cooper and Hirschhorn [2], Mahadeva Naika, Sumanth Bharadwaj and Hemanthkumar [7], Somashekara and Vidya [9], we obtain a number of Hurwitz type results for generating function of $r_{2,3}(\lambda^k(an+b))$, $r_{2,4}(an+b)$ and $r_{1,5}(\lambda^k(an+b))$ for various non-negative integer values of λ, k, a and b which are infinite products. To obtain our results, we use the lemma proved in section 2 which contains various theta function identities found in chapter 16 of Ramanujan's second notebook (cf. [1, 3, 8]).

Theorem 3.1. *For $\lambda = 4, 6, 9$ and $k \geq 0$ we have*

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(n))q^n = \varphi(q^2)\varphi(q^3), \tag{3.1}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(2n))q^n = \varphi(q)\varphi(q^6), \tag{3.2}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(2n+1))q^n = 2q\varphi(q)\psi(q^{12}), \tag{3.3}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(3n))q^n = \varphi(q)\varphi(q^6), \tag{3.4}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(3n+2))q^n = 2\varphi(q)f(q^2, q^{10}), \tag{3.5}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(4n+1))q^n = 4q\psi(q^4)\psi(q^6), \tag{3.6}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(4n+2))q^n = 2\varphi(q^3)\psi(q^4), \tag{3.7}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(4n+3))q^n = 2\varphi(q^2)\psi(q^6), \tag{3.8}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(6n+2))q^n = 2f(q, q^5)\varphi(q^2), \tag{3.9}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(6n+3))q^n = 2\psi(q^4)\varphi(q^3), \tag{3.10}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(6n+5))q^n = 4\psi(q^4)f(q, q^5), \tag{3.11}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(8n+2))q^n = 2\varphi(q^6)\psi(q^2), \tag{3.12}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(8n+3))q^n = 2\varphi(q)\psi(q^3), \tag{3.13}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(8n+5))q^n = 4\psi(q^2)\psi(q^3), \tag{3.14}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(8n+6))q^n = 4q\psi(q^2)\psi(q^{12}), \tag{3.15}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(9n+3))q^n = 2\varphi(q^2)f(q,q^5), \tag{3.16}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(12n+2))q^n = 2\varphi(q)f(q^4,q^8), \tag{3.17}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(12n+3))q^n = 2\varphi(q^6)\psi(q^2), \tag{3.18}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(12n+5))q^n = 4f(q^4,q^8)\psi(q^2), \tag{3.19}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(12n+11))q^n = 4\psi(q^2)f(q^2,q^{10}), \tag{3.20}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(16n+2))q^n = 2\varphi(q^3)\psi(q), \tag{3.21}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(16n+5))q^n = 4\psi(q)f(q^9,q^{15}), \tag{3.22}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(16n+14))q^n = 4\psi(q)\psi(q^6), \tag{3.23}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(18n+3))q^n = 2\varphi(q)f(q^4,q^8), \tag{3.24}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(24n+2))q^n = 2\varphi(q^2)f(q^2,q^4), \tag{3.25}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(24n+3))q^n = 2\varphi(q^3)\psi(q), \tag{3.26}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(24n+5))q^n = 4\psi(q)f(q^2,q^4), \tag{3.27}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(24n+11))q^n = 4\psi(q)f(q,q^5), \tag{3.28}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(24n+14))q^n = 4\psi(q^4)f(q^2,q^4), \tag{3.29}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(24n+21))q^n = 4\psi(q)\psi(q^6), \tag{3.30}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(32n+14))q^n = 4f(q^3,q^5)\psi(q^3), \tag{3.31}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(32n+30))q^n = 4f(q,q^7)\psi(q^3), \tag{3.32}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(36n+3))q^n = 2\varphi(q^2)f(q^2,q^4), \tag{3.33}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(36n+21))q^n = 4\psi(q^4)f(q^2,q^4), \tag{3.34}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(48n+2))q^n = 2\varphi(q)f(q,q^2), \tag{3.35}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(48n+5))q^n = 4f(q,q^2)f(q^3,q^5), \tag{3.36}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(48n+14))q^n = 4\psi(q^2)f(q,q^2), \tag{3.37}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(48n+21))q^n = 4\psi(q^3)f(q^3,q^5), \tag{3.38}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(48n+29))q^n = 4f(q,q^7)f(q,q^2), \tag{3.39}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(48n+45))q^n = 4\psi(q^3)f(q,q^7), \tag{3.40}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(72n+3))q^n = 2\varphi(q)f(q,q^2), \tag{3.41}$$

$$\sum_{n=0}^{\infty} r_{2,3}(\lambda^k(72n+21))q^n = 4\psi(q^2)f(q,q^2). \tag{3.42}$$

Proof. The generating function for $r_{2,3}(n)$ is

$$\sum_{n=0}^{\infty} r_{2,3}(n)q^n = \varphi(q^2)\varphi(q^3). \tag{3.43}$$

Using (2.1) in (3.43) and extracting the coefficients of q^{4n} and then changing q^4 to q and iterating, we obtain (3.1) for $\lambda = 4$.
Using (2.1) in (3.43) and extracting the coefficients of q^{2n} and then changing q^2 to q, also using (2.2) in (3.43) and extracting the coefficients of q^{3n} and then changing q^3 to q, we obtain

$$\sum_{n=0}^{\infty} r_{2,3}(\gamma n)q^n = \varphi(q)\varphi(q^6) \tag{3.44}$$

for $\gamma = 2, 3$ respectively. Using (2.2) in (3.44) and extracting the coefficients of q^{3n} and then changing q^3 to q and iterating, we obtain (3.1) for $\lambda = 6$ at $\gamma = 2$ and $\lambda = 9$ at $\gamma = 3$, completing the proof of (3.1).

From (2.1) and (3.1), we obtain (3.2) and (3.3). (2.2) and (3.1) give (3.4) and (3.5). (2.2) and (3.2) give (3.9). (2.1) in (3.1) give (3.6), (3.7) and (3.8). From (2.2) and (3.3) we get (3.10) and (3.11). (2.1) and (3.7) give (3.12) and (3.15). Using (2.1) in (3.3) we get (3.13) and (3.14). (2.2) and (3.4) give (3.16). (2.7) and (3.9) give (3.17). (2.2) and (3.8) give (3.18).

Using (2.7) in (3.11) we get (3.19) and (3.20). (3.21) directly follow from (3.12). Using (2.4) in (3.14) we get (3.22). (3.23) directly follow from (3.15). Using (2.5) in (3.10) we get (3.24). Using (2.5) in (3.12) we get (3.25). From (2.2) and (3.13) we get (3.26) and (3.28). (2.5) and (3.14) give (3.27) and (3.30). (2.1) and (3.17) give (3.29). Using (2.4) in (3.23) we get (3.31) and (3.32). Using (2.1) in (3.24) give (3.33) and (3.34). (3.35) directly follow from (3.25). (2.5) and (3.22) give (3.36) and (3.38). (2.4) and (3.27) give (3.39). (2.4) and (3.30) give (3.40). (3.37) directly follow from (3.29). (3.41) and (3.42) directly follow from (3.33) and (3.34) respectively. □

Theorem 3.2. *For $\lambda = 4,\ 5$ and $k \geq 0$ we have*

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(n))q^n = \varphi(q)\varphi(q^5), \tag{3.45}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(3n+1))q^n = 2f(q,q^5)\varphi(q^{15}), \tag{3.46}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(3n+2))q^n = 2q\varphi(q^3)f(q^5,q^{25}), \tag{3.47}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(4n+2))q^n = 4q\psi(q^2)\psi(q^{10}), \tag{3.48}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(5n+1))q^n = 2\varphi(q)f(q^3,q^7), \tag{3.49}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(5n+4))q^n = 2\varphi(q)f(q,q^9), \tag{3.50}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(8n+6))q^n = 4\psi(q)\psi(q^5), \tag{3.51}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(9n+1))q^n = 2f(q^7,q^{11})\varphi(q^5), \tag{3.52}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(9n+4))q^n = 2f(q^5,q^{13})\varphi(q^5), \tag{3.53}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(9n+7))q^n = 2qf(q,q^{17})\varphi(q^5), \tag{3.54}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(15n+11))q^n = 4q^2f(q,q^5)f(q^3,q^{27}), \tag{3.55}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(15n+14))q^n = 4f(q^9,q^{21})f(q,q^5), \tag{3.56}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(20n+6))q^n = 4f(q^4,q^6)\psi(q^2), \tag{3.57}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(20n+14))q^n = 4\psi(q^2)f(q^2,q^8), \tag{3.58}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(40n+6))q^n = 4f(q^2,q^3)\psi(q), \tag{3.59}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(40n+14))q^n = 4f(q,q^4)\psi(q), \tag{3.60}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(45n+11))q^n = 4qf(q,q^9)f(q^5,q^{13}), \tag{3.61}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(45n+14))q^n = 4f(q^3,q^7)f(q^7,q^{11}), \tag{3.62}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(45n+29))q^n = 4f(q^3,q^7)f(q^5,q^{13}), \tag{3.63}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(45n+41))q^n = 4f(q,q^9)f(q^7,q^{11}), \tag{3.64}$$

$$\sum_{n=0}^{\infty} r_{1,5}(\lambda^k(45n+44))q^n = 4qf(q^3,q^7)f(q,q^{17}). \tag{3.65}$$

Proof. The generating function for $r_{1,5}(n)$ is

$$\sum_{n=0}^{\infty} r_{1,5}(n)q^n = \varphi(q)\varphi(q^5). \tag{3.66}$$

Using (2.1) in (3.66) and extracting the coefficients of q^{4n} and then changing q^4 to q and iterating we get (3.45) for $\lambda = 4$. Also using (2.3) in (3.66) and extracting the coefficients of q^{5n} and then changing q^5 to q and iterating we get (3.45) for $\lambda = 5$, completing the proof of (3.45).

(2.2) and (3.45) give (3.46) and (3.47). (2.1) and (3.45) lead to (3.48). (2.3) and (3.45) lead to (3.49) and (3.50). (3.51) directly follow from (3.48). (2.9) and (3.46) give (3.52), (3.53) and (3.54). Using (2.3) in (3.47) we get (3.55) and (3.56).

(2.6) and (3.48) give (3.57) and (3.58). (2.6) in (3.51) lead to (3.59) and (3.60). (2.9) and (3.55) give (3.61) and (3.64). (2.9) and (3.56) lead to (3.62), (3.63) and (3.65). □

Theorem 3.3. *We have*

$$\sum_{n=0}^{\infty} r_{2,4}(2n)q^n = \varphi(q)\varphi(q^2), \tag{3.67}$$

$$\sum_{n=0}^{\infty} r_{2,4}(3n+1)q^n = 2q\varphi(q^6)f(q^4,q^{20}), \tag{3.68}$$

$$\sum_{n=0}^{\infty} r_{2,4}(3n+2)q^n = 2f(q^2,q^{10})\varphi(q^{12}), \tag{3.69}$$

$$\sum_{n=0}^{\infty} r_{2,4}(4n+2)q^n = 2\varphi(q)\psi(q^4), \tag{3.70}$$

$$\sum_{n=0}^{\infty} r_{2,4}(6n+2)q^n = 2f(q,q^5)\varphi(q^6), \tag{3.71}$$

$$\sum_{n=0}^{\infty} r_{2,4}(6n+4)q^n = 2\varphi(q^3)f(q^2,q^{10}), \tag{3.72}$$

$$\sum_{n=0}^{\infty} r_{2,4}(8n+2)q^n = 2\varphi(q^2)\psi(q^2), \tag{3.73}$$

$$\sum_{n=0}^{\infty} r_{2,4}(8n+6)q^n = 4\psi(q^2)\psi(q^4), \tag{3.74}$$

$$\sum_{n=0}^{\infty} r_{2,4}(12n+2)q^n = 2f(q^4,q^8)\varphi(q^3), \tag{3.75}$$

$$\sum_{n=0}^{\infty} r_{2,4}(12n+4)q^n = 2f(q,q^5)\varphi(q^6), \tag{3.76}$$

$$\sum_{n=0}^{\infty} r_{2,4}(12n+8)q^n = 2f(q^2,q^{10})\varphi(q^3), \tag{3.77}$$

$$\sum_{n=0}^{\infty} r_{2,4}(12n+10)q^n = 4qf(q,q^5)\psi(q^{12}), \tag{3.78}$$

$$\sum_{n=0}^{\infty} r_{2,4}(16n+2)q^n = 2\varphi(q)\psi(q), \tag{3.79}$$

$$\sum_{n=0}^{\infty} r_{2,4}(16n+6)q^n = 4\psi(q)\psi(q^2), \tag{3.80}$$

$$\sum_{n=0}^{\infty} r_{2,4}(18n+2)q^n = 2\varphi(q^2)f(q^7,q^{11}), \tag{3.81}$$

$$\sum_{n=0}^{\infty} r_{2,4}(18n+8)q^n = 2\varphi(q^2)f(q^5,q^{13}), \tag{3.82}$$

$$\sum_{n=0}^{\infty} r_{2,4}(18n+14)q^n = 2q\varphi(q^2)f(q,q^{17}), \tag{3.83}$$

$$\sum_{n=0}^{\infty} r_{2,4}(24n+2)q^n = 2f(q^2,q^4)\varphi(q^6), \tag{3.84}$$

$$\sum_{n=0}^{\infty} r_{2,4}(24n+4)q^n = 2f(q^4,q^8)\varphi(q^3), \tag{3.85}$$

$$\sum_{n=0}^{\infty} r_{2,4}(24n+8)q^n = 2f(q,q^5)\varphi(q^6), \tag{3.86}$$

$$\sum_{n=0}^{\infty} r_{2,4}(24n+14)q^n = 4qf(q^2,q^4)\psi(q^{12}), \tag{3.87}$$

$$\sum_{n=0}^{\infty} r_{2,4}(24n+16)q^n = 2f(q^2,q^{10})\varphi(q^3), \tag{3.88}$$

$$\sum_{n=0}^{\infty} r_{2,4}(24n+20)q^n = 4qf(q,q^5)\psi(q^{12}), \tag{3.89}$$

$$\sum_{n=0}^{\infty} r_{2,4}(32n+6)q^n = 4f(q^3,q^5)\psi(q), \tag{3.90}$$

$$\sum_{n=0}^{\infty} r_{2,4}(32n+22)q^n = 4f(q,q^7)\psi(q), \tag{3.91}$$

$$\sum_{n=0}^{\infty} r_{2,4}(36n+10)q^n = 4q^2\psi(q^4)f(q,q^{17}), \tag{3.92}$$

$$\sum_{n=0}^{\infty} r_{2,4}(36n+22)q^n = 4\psi(q^4)f(q^7,q^{11}), \tag{3.93}$$

$$\sum_{n=0}^{\infty} r_{2,4}(36n+34)q^n = 4\psi(q^4)f(q^5,q^{13}), \tag{3.94}$$

$$\sum_{n=0}^{\infty} r_{2,4}(48n+2)q^n = 2f(q,q^2)\varphi(q^3), \tag{3.95}$$

$$\sum_{n=0}^{\infty} r_{2,4}(48n+4)q^n = 2f(q^2,q^4)\varphi(q^6), \tag{3.96}$$

$$\sum_{n=0}^{\infty} r_{2,4}(48n+8)q^n = 2f(q^4,q^8)\varphi(q^3), \tag{3.97}$$

$$\sum_{n=0}^{\infty} r_{2,4}(48n+16)q^n = 2f(q,q^5)\varphi(q^6), \tag{3.98}$$

$$\sum_{n=0}^{\infty} r_{2,4}(48n+22)q^n = 4\psi(q^3)f(q^2,q^4), \tag{3.99}$$

$$\sum_{n=0}^{\infty} r_{2,4}(48n+32)q^n = 2f(q^2,q^{10})\varphi(q^3), \tag{3.100}$$

$$\sum_{n=0}^{\infty} r_{2,4}(48n+34)q^n = 4f(q,q^5)\psi(q^3), \tag{3.101}$$

$$\sum_{n=0}^{\infty} r_{2,4}(48n+38)q^n = 4f(q,q^2)\psi(q^6), \tag{3.102}$$

$$\sum_{n=0}^{\infty} r_{2,4}(96n+4)q^n = 2f(q,q^2)\varphi(q^3), \tag{3.103}$$

$$\sum_{n=0}^{\infty} r_{2,4}(96n+22)q^n = 4f(q^9, q^{15})f(q, q^2), \tag{3.104}$$

$$\sum_{n=0}^{\infty} r_{2,4}(96n+70)q^n = 4qf(q, q^2)f(q^3, q^{21}), \tag{3.105}$$

$$\sum_{n=0}^{\infty} r_{2,4}(144n+34)q^n = 4\psi(q)f(q^7, q^{11}), \tag{3.106}$$

$$\sum_{n=0}^{\infty} r_{2,4}(144n+82)q^n = 4\psi(q)f(q^5, q^{13}), \tag{3.107}$$

$$\sum_{n=0}^{\infty} r_{2,4}(144n+130)q^n = 4q\psi(q)f(q, q^{17}). \tag{3.108}$$

Proof. The generating function for $r_{2,4}(n)$ is

$$\sum_{n=0}^{\infty} r_{2,4}(n)q^n = \varphi(q^2)\varphi(q^4). \tag{3.109}$$

Equation (3.67) directly follow from (3.109). (2.2) in (3.109) give (3.68) and (3.69). (2.1) in (3.67) give (3.70). (3.72) and (3.71) directly follow from (3.68) and (3.69) respectively. From (2.1) and (3.70) we obtain (3.73) and (3.74). (2.7) and (3.71) give (3.75) and (3.77). (2.1) and (3.72) give (3.76) and (3.78). (3.79) and (3.80) directly follow from (3.73) and (3.74) respectively. (2.9) and (3.71) give (3.81), (3.82) and (3.83). (2.7) in (3.76) lead to (3.85) and (3.88). (2.1) and (3.75) give (3.84) and (3.87). (2.1) and (3.77) give (3.86) and (3.89).

Using (2.4) in (3.80) we get (3.90) and (3.91). (2.9) and (3.78) give (3.92), (3.93) and (3.94). Using (2.2) and (2.5) in (3.79) we get (3.95) and (3.101). (2.5) and (3.80) lead to (3.99) and (3.102). (2.1) in (3.85) and (3.88) give (3.96) and (3.98) respectively. (2.7) and (3.86) give (3.97) and (3.100). (2.4) in (3.99) lead to (3.104) and (3.105). (3.103) directly follow from (3.96). (2.9) and (3.101) give (3.106), (3.107) and (3.108). □

Theorem 3.4. *We have*

$$r_{2\alpha,2\beta}(2n) = r_{\alpha,\beta}(n). \tag{3.110}$$

Proof. We have

$$\sum_{n=0}^{\infty} r_{2\alpha,2\beta}(n)q^n = \varphi(q^{2\alpha})\varphi(q^{2\beta}).$$

By extracting the coefficients of q^{2n} and then changing q^2 to q, we obtain

$$\sum_{n=0}^{\infty} r_{2\alpha,2\beta}(2n)q^n = \varphi(q^\alpha)\varphi(q^\beta)$$
$$= \sum_{n=0}^{\infty} r_{\alpha,\beta}(n)q^n.$$

Now, comparing the coefficientss of q^n on both sides of the above equation, we get (3.110). □

Remark: From (3.2), it follow that $r_{2,3}(2n) = r_{1,6}(n)$.

Conclusion: In this paper, we gave simple proofs to some theta function identities found in chapter 16 of Ramanujan's second notebook. We then used those results to obtain Hurwitz type results for the generating function of $r_{2,3}(\lambda^k(an+b))$, $r_{2,4}(an+b)$ and $r_{1,5}(\lambda^k(an+b))$ for various non-negative integer values of λ, k, a and b which are infinite products. Our work complements the work of Cooper and Hirschhorn [4], Hirschhorn and McGowan [5], Barrucand, Cooper and Hirschhorn [2], Mahadeva Naika, Sumanth Bharadwaj and Hemanthkumar [7], Somashekara and Vidya [9]. On the same lines one can obtain similar Hurwitz type results for various representations of integers as sums of squares.

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