Proceedings of the Jangjeon Mathematical Society
25 (2022), No. 3, pp. 307 - 317

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2022.25.3.307

Solution of non-linear Fredholm integral equation via generalized complex metric space

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Abstract

In this paper, we introduce a new concept of generalized complex metric spaces and prove fixed point theorems. The presented results generalize and expand some of the literature's well-known results. We also explore some of the applications of our key results.

Key words: generalized complex metric space; standard complex metric space; fixed point.
AMS Subject Classification: 47H10, 54H25.

1 Introduction

In 2013, complex valued b-metric spaces introduced by Rao, Swamy, and Prasad [1]. The dislocate complex metric spaces introduced by Ege and Karaca [2], in 2018 and proved some fixed point theorems. Metric is not defined, many problems in fixed point theory can be reformulated in modular spaces (see [3-10] and references therein). In 2015, Jleli and Samet [11], proved fixed point theorems on generalized metric space. The complex valued metric space was introduced by Azam, Fisher and Khan [12] in 2011 as follows

Definition 1.1. *Let Λ be a nonempty set. Suppose that the mapping $\Gamma : \Lambda \times \Lambda \to \mathbb{C}$, satisfies:*

(B1) $0 \preceq \Gamma(\vartheta, \sigma)$, *for all* $\vartheta, \sigma \in \Lambda$ *and* $\Gamma(\vartheta, \sigma) = 0$ *iff* $\vartheta = \sigma$;

(B2) $\Gamma(\vartheta, \sigma) = \Gamma(\sigma, \vartheta)$ *for all* $\vartheta, \sigma \in \Lambda$;

(B3) $\Gamma(\vartheta, \sigma) \preceq \Gamma(\vartheta, \beta) + \Gamma(\beta, \sigma)$, *for all* $\vartheta, \sigma, \beta \in \Lambda$.

Then Γ is called a complex valued metric on Λ, and (Λ, Γ) is called a complex valued metric space.

Motivated by above Definition 1.1, we define generalized complex metric spaces. In this paper, we prove some fixed point theorems on generalized complex metric space with an application.

2 Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $\omega_1, \omega_2, \omega_3 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:
$\omega_1 \preceq \omega_2$ if and only if $\mathcal{R}(\omega_1) \leq \mathcal{R}(\omega_2)$, $\mathcal{I}(\omega_1) \leq \mathcal{I}(\omega_2)$.
Consequently, one can infer that $\omega_1 \preceq \omega_2$ if one of the following conditions is satisfied:
(i) $\mathcal{R}(\omega_1) = \mathcal{R}(\omega_2)$, $\mathcal{I}(\omega_1) < \mathcal{I}(\omega_2)$,
(ii)$\mathcal{R}(\omega_1) < \mathcal{R}(\omega_2)$, $\mathcal{I}(\omega_1) = \mathcal{I}(\omega_2)$,
(iii)$\mathcal{R}(\omega_1) < \mathcal{R}(\omega_2)$, $\mathcal{I}(\omega_1) < \mathcal{I}(\omega_2)$,
(iv)$\mathcal{R}(\omega_1) = \mathcal{R}(\omega_2)$, $\mathcal{I}(\omega_1) = \mathcal{I}(\omega_2)$.
In particular, we write $\omega_1 \precnsim \omega_2$ if $\omega_1 \neq \omega_2$ and one of (i), (ii) and (iii) is satisfied and we write $\omega_1 \prec \omega_2$ if only (iii) is satisfied. Notice that
(a) If $0 \preceq \omega_1 \precnsim \omega_2$, then $|\omega_1| < |\omega_2|$,
(b) If $\omega_1 \preceq \omega_2$ and $\omega_2 \prec \omega_3$ then $\omega_1 \prec \omega_3$,
(c) If $\eta, \gamma \in \mathbb{R}$ and $\eta \leq \gamma$ then $\eta\omega_1 \preceq \gamma\omega_1$ for all $0 \preceq \omega_1 \in \mathbb{C}$.
Let Λ be a nonempty set and $\Gamma : \Lambda \times \Lambda \to \mathbb{C}$ be given mapping. For every $\vartheta \in \Lambda$, let us define the set

$$\mathcal{C}(\Gamma, \Lambda, \vartheta) = \left\{ \{\vartheta_\alpha\} \subset \Lambda : \lim_{\alpha \to \infty} \Gamma(\vartheta_\alpha, \vartheta) = 0 \right\}.$$

Definition 2.1. *We say that Γ is a generalized complex metric on Λ if*

(D1) *for every* $(\vartheta, \sigma) \in \Lambda \times \Lambda$, *we have* $\Gamma(\vartheta, \sigma) = 0 \Rightarrow \vartheta = \sigma$;

(D2) *for every* $(\vartheta, \sigma) \in \Lambda \times \Lambda$, *we have* $\Gamma(\vartheta, \sigma) = \Gamma(\sigma, \vartheta)$;

(D3) there exists $\mathfrak{C} > 0$ such that if $(\vartheta, \sigma) \in \Lambda \times \Lambda, \{\vartheta_\alpha\} \in \mathcal{C}(\Gamma, \Lambda, \vartheta)$, then

$$\Gamma(\vartheta, \sigma) \preceq \mathfrak{C} \lim_{\alpha \to \infty} \sup \Gamma(\vartheta_\alpha, \sigma).$$

We say that the pair (Λ, Γ) is a generalized complex metric space.

Definition 2.2. *Let (Λ, Γ) be a generalized complex metric space. Let $\{\vartheta_\alpha\}$ be a sequence in Λ and $\vartheta \in \Lambda$. We say that $\{\vartheta_\alpha\}$ Γ-converges to ϑ if*

$$\{\vartheta_\alpha\} \in \mathcal{C}(\Gamma, \Lambda, \vartheta).$$

Proposition 2.1. *Let (Λ, Γ) be a generalized complex metric space. Let $\{\vartheta_\alpha\}$ be a sequence in Λ and $(\vartheta, \sigma) \in \Lambda \times \Lambda$. If $\{\vartheta_\alpha\}$ Γ-converges to ϑ and $\{\vartheta_\alpha\}$ Γ-converges to σ, then $\vartheta = \sigma$.*

Proof. Using the property of (D3), we have

$$\Gamma(\vartheta, \sigma) \preceq \lim_{\alpha \to \infty} \sup \Gamma(\vartheta_\alpha, \sigma),$$

which implies that

$$|\Gamma(\vartheta, \sigma)| \preceq \lim_{\alpha \to \infty} \sup |\Gamma(\vartheta_\alpha, \sigma)| = 0.$$

By property (D1), we get $\vartheta = \sigma$. □

Definition 2.3. *Let (Λ, Γ) be a generalized complex metric space. Let $\{\vartheta_\alpha\}$ be a sequence in Λ. We say that $\{\vartheta_\alpha\}$ is a Γ-Cauchy sequence if*

$$\lim_{\beta, \alpha \to \infty} \Gamma(\vartheta_\alpha, \vartheta_{\alpha+\beta}) = 0.$$

Definition 2.4. *Let (Λ, Γ) be a generalized complex metric space. We say that Γ-complete if every Cauchy sequence in Λ is convergent to some element in Λ.*

Proposition 2.2. *$\mathcal{C}(\Gamma, \Lambda, \vartheta)$ is a nonempty set if and only if $\Gamma(\vartheta, \vartheta) = 0$.*

Proof. If $\mathcal{C}(\Gamma, \Lambda, \vartheta) \neq \emptyset$, then there exists a sequence $\{\vartheta_\alpha\} \subset \Lambda$ such that

$$\lim_{\alpha \to \infty} \Gamma(\vartheta_\alpha, \vartheta) = 0.$$

By property (D3), we obtain

$$\Gamma(\vartheta, \vartheta) \leq \mathfrak{C} \lim_{\alpha \to \infty} \sup \Gamma(\vartheta_\alpha, \vartheta),$$

and thus $\Gamma(\vartheta, \vartheta) = 0$. Conversely, assume that $\Gamma(\vartheta, \vartheta) = 0$. The sequence $\{\vartheta_\alpha\} \subset \Lambda$ defined by $\vartheta_\alpha = \vartheta$ for all $\alpha \in \mathbb{N}$ converges to ϑ. □

3 Main Results

Definition 3.1. *Let $(\varLambda, \varGamma)$ be a generalized complex metric space and $\varphi : \varLambda \to \varLambda$ be a mapping. Let $\mathfrak{u} \in (0,1)$. We say that φ is a $\mathfrak{u}$-contraction if*

$$\varGamma(\varphi(\vartheta), \varphi(\sigma)) \preceq \mathfrak{u}\varGamma(\vartheta, \sigma), \text{ for every } (\vartheta, \sigma) \in \varLambda \times \varLambda.$$

Proposition 3.1. *Let $(\varLambda, \varGamma)$ be a generalized complex metric space and $\varphi : \varLambda \to \varLambda$ be a mapping satisfying φ is a $\mathfrak{u}$-contraction for some $\mathfrak{u} \in (0,1)$. Then any fixed point $\varLambda \in \varLambda$ of φ satisfies*

$$\varGamma(\varLambda, \varLambda) \prec \infty \Rightarrow \varGamma(\varLambda, \varLambda) = 0.$$

Proof. Let $\varLambda \in \varLambda$ be a fixed point of φ such that $\varGamma(\varLambda, \varLambda) \prec \infty$. Since φ is a $\mathfrak{u}$-contraction, we have

$$\varGamma(\varLambda, \varLambda) = \varGamma(\varphi(\varLambda), \varphi(\varLambda)) \preceq \mathfrak{u}\varGamma(\varLambda, \varLambda),$$

which implies that $\varGamma(\varLambda, \varLambda) = 0$ since $\mathfrak{u} \in (0,1)$ and $\varGamma(\varLambda, \varLambda) \prec \infty$. □

For every $\vartheta \in \varLambda$, let

$$\delta(\varGamma, \varphi, \vartheta) = \sup\left\{\varGamma(\varphi^i(\vartheta), \varphi^j(\vartheta)) : i, j \in \mathbb{N}\right\}.$$

Theorem 3.2. *Let $(\varLambda, \varGamma)$ be a generalized complex metric space and $\varphi : \varLambda \to \varLambda$ be a mapping such that*

(1) $(\varLambda, \varGamma)$ is complete;

(2) φ is a $\mathfrak{u}$-contraction for some $\mathfrak{u} \in (0,1)$;

(3) we can find $\vartheta_0 \in \varLambda$ satisfying $\delta(\varGamma, \varphi, \vartheta_0) \prec \infty$.

Then $\{\varphi^\alpha(\vartheta_0)\}$ converges to $\varLambda \in \varLambda$, a fixed point of φ. Moreover, if $\varLambda' \in \varLambda$ is another fixed point of φ such that $\varGamma(\varLambda, \varLambda') \prec \infty$, then $\varLambda = \varLambda'$.

Proof. Let $\alpha \in \mathbb{N}(\alpha \geq 1)$. Since φ is a $\mathfrak{u}$-contraction, $\forall\, i, j \in \mathbb{N}$,

$$\varGamma(\varphi^{\alpha+i}(\vartheta_0), \varphi^{\alpha+j}(\vartheta_0)) \preceq \mathfrak{u}\varGamma(\varphi^{\alpha-1+i}(\vartheta_0), \varphi^{\alpha-1+j}(\vartheta_0)),$$

consequently,

$$\delta(\varGamma, \varphi, \varphi^\alpha(\vartheta_0)) \preceq \mathfrak{u}\delta(\varGamma, \varphi, \varphi^{\alpha-1}(\vartheta_0)).$$

Then, for every $\alpha \in \mathbb{N}$, we have

$$\delta(\varGamma, \varphi, \varphi^\alpha(\vartheta_0)) \preceq \mathfrak{u}^\alpha\delta(\varGamma, \varphi, \vartheta_0).$$

Using the above inequality, for every $\alpha, \beta \in \mathbb{N}$, we have

$$\varGamma(\varphi^\alpha(\vartheta_0), \varphi^{\alpha+\beta}(\vartheta_0)) \preceq \delta(\varGamma, \varphi, \varphi^\alpha(\vartheta_0)) \preceq \mathfrak{u}^\alpha\delta(\varGamma, \varphi, \vartheta_0).$$

Since $\delta(\Gamma,\varphi,\vartheta_0) \prec \infty$ and $\mathfrak{u} \in (0,1)$, we obtain

$$\lim_{\alpha,\beta\to\infty} \Gamma(\varphi^{\alpha}(\vartheta_0), \varphi^{\alpha+\beta}(\vartheta_0)) = 0,$$

consequently, $\{\varphi^{\alpha}(\vartheta_0)\}$ is a Γ-Cauchy sequence.
Since (Λ,Γ) is Γ-complete, we can find some $\Lambda \in \Lambda$ satisfying $\{\varphi^{\alpha}(\vartheta_0)\}$ is a Γ-convergent to Λ. On the other hand, since φ is a $\mathfrak{u}$-contraction, $\forall\ \alpha \in \mathbb{N}$,

$$\Gamma(\varphi^{\alpha+1}(\vartheta_0), \varphi(\Lambda)) \preceq \mathfrak{u}\Gamma(\varphi^{\alpha}(\vartheta_0), \Lambda).$$

Letting $\alpha \to \infty$ in the above inequality, we get

$$\lim_{\alpha\to\infty} \Gamma(\varphi^{\alpha+1}(\vartheta_0), \varphi(\Lambda)) = 0.$$

Then $\{\varphi^{\alpha}(\vartheta_0)\}$ is a Γ-convergent to $\varphi(\Lambda)$. By the uniqueness of the limit, we get $\Lambda = \varphi(\Lambda)$, that is Λ is a fixed point of φ.
Let $\Lambda^{'} \in \Lambda$ be a fixed point of φ satisfying $\Gamma(\Lambda,\Lambda^{'}) \prec \infty$. Since φ is a $\mathfrak{u}$-contraction,

$$\Gamma(\Lambda,\Lambda^{'}) = \Gamma(\varphi(\Lambda), \varphi(\Lambda^{'})) \preceq \mathfrak{u}\Gamma(\Lambda,\Lambda^{'}).$$

Hence, $\Lambda = \Lambda^{'}$. □

Observe that we can replace condition (iii) in Theorem 3.2 by

$(iii)^{'}$ we can find $\vartheta_0 \in \Lambda$ satisfying $\sup\{\Gamma(\vartheta_0, \varphi^{\mathfrak{r}}(\vartheta_0)) : \mathfrak{r} \in \mathbb{N}\} \prec \infty$.

Since φ is a $\mathfrak{u}$-contraction,

$$\delta(\Gamma,\varphi,\vartheta_0) \preceq \sup\{\Gamma(\vartheta_0, \varphi^{\mathfrak{r}}(\vartheta_0)) : \mathfrak{r} \in \mathbb{N}\}.$$

So condition $(iii)^{'}$ implies condition (iii).

Definition 3.2. *Let (Λ,Γ) be a generalized complex metric space and $\varphi : \Lambda \to \Lambda$ be a mapping. Let $\mathfrak{u} \in (0,1)$. We say that φ is a $\mathfrak{u}$-quasi contraction if*

$$\Gamma(\varphi(\vartheta), \varphi(\sigma)) \preceq \mathfrak{u} \max\{\Gamma(\vartheta,\sigma), \Gamma(\vartheta,t\vartheta), \Gamma(\sigma,t\sigma), \Gamma(\vartheta,t\sigma), \Gamma(\sigma,t\vartheta)\},$$

for every $(\vartheta,\sigma) \in \Lambda \times \Lambda$.

Proposition 3.3. *Let (Λ,Γ) be a generalized complex metric space and $\varphi : \Lambda \to \Lambda$ be a mapping. Suppose that φ is a $\mathfrak{u}$-quasi contraction for some $\mathfrak{u} \in (0,1)$. Then any fixed point $\Lambda \in \Lambda$ of φ satisfies*

$$\Gamma(\Lambda,\Lambda) \prec \infty \Rightarrow \Gamma(\Lambda,\Lambda) = 0.$$

Proof. Let $\Lambda \in \Lambda$ be a fixed point of φ such that $\Gamma(\Lambda,\Lambda) \prec \infty$. Since φ is a $\mathfrak{u}$-quasicontraction, we have

$$\Gamma(\Lambda,\Lambda) = \Gamma(\varphi(\Lambda), \varphi(\Lambda)) \preceq \mathfrak{u}\Gamma(\Lambda,\Lambda).$$

Since $\mathfrak{u} \in (0,1)$, we get $\Gamma(\Lambda,\Lambda) = 0$.

□

Theorem 3.4. *Let (Λ, Γ) be a generalized complex metric space and $\varphi : \Lambda \to \Lambda$ be a mapping such that*

(i) (Λ, Γ) *is complete;*

(ii) φ *is a* $\mathfrak{u}$*-quasi contraction for some* $\mathfrak{u} \in (0,1)$*;*

(iii) there exists $\vartheta_0 \in \Lambda$ *such that* $\delta(\Gamma, \varphi, \vartheta_0) \prec \infty$.

Then $\{\varphi^\alpha(\vartheta_0)\}$ *converges to some* $\Lambda \in \Lambda$*. If* $\Gamma(\vartheta_0, \varphi(\Lambda)) \prec \infty$ *and* $\Gamma(\Lambda, \varphi(\Lambda)) \prec \infty$ *, then* Λ *is a fixed point of* φ*. Moreover, if* $\Lambda^{'} \in \Lambda$ *is another fixed point of* φ *satisfying* $\Gamma(\Lambda, \Lambda^{'}) \prec \infty$ *and* $\Gamma(\Lambda^{'}, \Lambda^{'}) \prec \infty$*, then* $\Lambda = \Lambda^{'}$.

Proof. Let $\alpha \in \mathbb{N}(\alpha \geq 1)$. Since φ is a $\mathfrak{u}$-quasi contraction, $\forall\ i, j \in \mathbb{N}$, we have

$$\begin{aligned}\Gamma(\varphi^{\alpha+i}(\vartheta_0), \varphi^{\alpha+j}(\vartheta_0)) \preceq \mathfrak{u} \max\{&\Gamma(\varphi^{\alpha-1+i}(\vartheta_0), \varphi^{\alpha-1+j}(\vartheta_0)), \Gamma(\varphi^{\alpha-1+i}(\vartheta_0), \varphi^{\alpha+i}(\vartheta_0)),\\ &\Gamma(\varphi^{\alpha-1+i}(\vartheta_0), \varphi^{\alpha+j}(\vartheta_0)), \Gamma(\varphi^{\alpha-1+j}(\vartheta_0), \varphi^{\alpha+j}(\vartheta_0)),\\ &\Gamma(\varphi^{\alpha-1+j}(\vartheta_0), \varphi^{\alpha+i}(\vartheta_0))\},\end{aligned}$$

which implies that

$$\delta(\Gamma, \varphi, \varphi^\alpha(\vartheta_0)) \preceq \mathfrak{u}\delta(\Gamma, \varphi, \varphi^{\alpha-1}(\vartheta_0)).$$

Hence, for any $\alpha \geq 1$, we have

$$\delta(\Gamma, \varphi, \varphi^\alpha(\vartheta_0)) \preceq \mathfrak{u}^\alpha \delta(\Gamma, \varphi, \vartheta_0).$$

Using the above inequality, for every $\alpha, \beta \in \mathbb{N}$, we have

$$\Gamma(\varphi^\alpha(\vartheta_0), \varphi^{\alpha+\beta}(\vartheta_0)) \preceq \delta(\Gamma, \varphi, \varphi^\alpha(\vartheta_0)) \preceq \mathfrak{u}^\alpha \delta(\Gamma, \varphi, \vartheta_0).$$

Since $\delta(\Gamma, \varphi, \vartheta_0) \prec \infty$ and $\mathfrak{u} \in (0,1)$, we obtain

$$\lim_{\alpha,\beta\to\infty} \Gamma(\varphi^\alpha(\vartheta_0), \varphi^{\alpha+\beta}(\vartheta_0)) = 0.$$

Therefore $\{\varphi^\alpha(\vartheta_0)\}$ is a Γ-Cauchy sequence. Since (Λ, Γ) is Γ-complete, we can find some $\Lambda \in \Lambda$ satisfying $\{\varphi^\alpha(\vartheta_0)\}$ is Γ-convergent to Λ. Suppose that $\Gamma(\vartheta_0, \varphi(\Lambda)) \prec \infty$. Using the inequality

$$\Gamma(\varphi^\alpha(\vartheta_0), \varphi^{\alpha+\beta}(\vartheta_0)) \preceq \mathfrak{u}^\alpha \delta(\Gamma, \varphi, \vartheta_0), \tag{1}$$

for every $\alpha, \beta \in \mathbb{N}$, we can find some constant $\mathfrak{C} > 0$ satisfying

$$\Gamma(\Lambda, \varphi^\alpha(\vartheta_0)) \preceq \mathfrak{C} \lim_{\beta\to\infty} \sup \Gamma(\varphi^\alpha(\vartheta_0), \varphi^{\alpha+\beta}(\vartheta_0)) \preceq \mathfrak{C}\mathfrak{u}^\alpha \delta(\Gamma, \varphi, \vartheta_0), \tag{2}$$

for every $\alpha \in \mathbb{N}$.
On the other hand,

$$\Gamma(\varphi(\vartheta_0), \varphi(\Lambda)) \preceq \mathfrak{u} \max\{\Gamma(\vartheta_0, \Lambda), \Gamma(\vartheta_0, \varphi(\vartheta_0)), \Gamma(\Lambda, \varphi(\Lambda)), \Gamma(\varphi(\vartheta_0), \Lambda), \Gamma(\vartheta_0, \varphi(\Lambda))\}.$$

Using (1) and (2), we get

$$\Gamma(\varphi(\vartheta_0), \varphi(\Lambda)) \preceq \max\{\mathfrak{u}\mathfrak{C}\delta(\Gamma, \varphi, \vartheta_0), \mathfrak{u}\delta(\Gamma, \varphi, \vartheta_0), \mathfrak{u}\Gamma(\Lambda, \varphi(\Lambda)), \mathfrak{u}\Gamma(\vartheta_0, \varphi(\Lambda))\}.$$

Again, using the above inequality, we have

$$\Gamma(\varphi^2(\vartheta_0), \varphi(\Lambda)) \preceq \max\{\mathfrak{u}^2\mathfrak{C}\delta(\Gamma, \varphi, \vartheta_0), \mathfrak{u}^2\delta(\Gamma, \varphi, \vartheta_0), \mathfrak{u}\Gamma(\Lambda, \varphi(\Lambda)), \mathfrak{u}^2\Gamma(\vartheta_0, \varphi(\Lambda))\}.$$

Continuing this process, by induction we get

$$\Gamma(\varphi^\alpha(\vartheta_0), \varphi(\Lambda)) \preceq \max\{\mathfrak{u}^\alpha\mathfrak{C}\delta(\Gamma, \varphi, \vartheta_0), \mathfrak{u}^\alpha\delta(\Gamma, \varphi, \vartheta_0), \mathfrak{u}\Gamma(\Lambda, \varphi(\Lambda)), \mathfrak{u}^\alpha\Gamma(\vartheta_0, \varphi(\Lambda))\},$$

for every $\alpha \geq 1$. Therefore, we have

$$\lim_{\alpha\to\infty} \sup \Gamma(\varphi^\alpha(\vartheta_0), \varphi(\Lambda)) \preceq \mathfrak{u}\Gamma(\Lambda, \varphi(\Lambda)),$$

since $\Gamma(\vartheta_0, \varphi(\Lambda)) \prec \infty$ and $\delta(\Gamma, \varphi, \vartheta_0) \prec \infty$. Using the property $(D3)$, we get

$$\Gamma(\varphi(\Lambda), \Lambda) \preceq \lim_{\alpha\to\infty} \sup \Gamma(\varphi^\alpha(\vartheta_0), \varphi(\Lambda)) \preceq \mathfrak{u}\Gamma(\Lambda, \varphi(\Lambda)),$$

which implies that $\Gamma(\varphi(\Lambda), \Lambda) = 0$, since $\Gamma(\Lambda, \varphi(\Lambda)) \prec \infty$ and $\mathfrak{u} \in (0, 1)$. Then Λ is a fixed point of φ. By Proposition 3.3, we have $\Gamma(\Lambda, \Lambda) = 0$.
Let $\Lambda' \in \Lambda$ is another fixed point of φ satisfying $\Gamma(\Lambda, \Lambda') \prec \infty$ and $\Gamma(\Lambda', \Lambda') \prec \infty$. By Proposition 3.3, $\Gamma(\Lambda', \Lambda') = 0$. Since φ is a $\mathfrak{u}$-quasi contraction,

$$\Gamma(\Lambda, \Lambda') = \Gamma(\varphi(\Lambda), \varphi(\Lambda')) \preceq \mathfrak{u}\Gamma(\Lambda, \Lambda'),$$

which implies that $\Lambda = \Lambda'$. □

Example 3.5. *Let $\Lambda = [0, 1]$, and let $\Gamma : \Lambda \times \Lambda \to \mathbb{C}$ be the mapping defined by*

$$\begin{cases} \Gamma(\vartheta, \sigma) = (\vartheta + \sigma)e^{i\theta} & \text{if } \vartheta \neq 0 \text{ and } \sigma \neq 0 \\ \Gamma(0, \vartheta) = \Gamma(\vartheta, 0) = \frac{\vartheta e^{i\theta}}{2} & \text{for all } \vartheta \in \Lambda, \end{cases}$$

where $0 \leq \theta \leq \frac{\pi}{2}$. Conditions $(D1)$ and $(D2)$ are trivially satisfied. By Proposition 2.2 we need to verify condition (D3) only for elements ϑ of Λ such that $\Gamma(\vartheta, \vartheta) = 0$. For all $\alpha \in \mathbb{N}$ and $\sigma \in \Lambda$, we obtain:

$$\Gamma(\vartheta_\alpha, \sigma) = \begin{cases} (\vartheta_\alpha + \sigma)e^{i\theta} & \text{if } \vartheta_\alpha \neq 0, \\ \frac{\vartheta e^{i\theta}}{2} & \text{if } \vartheta_\alpha = 0. \end{cases}$$

Then

$$\frac{\vartheta e^{i\theta}}{2} \preceq \Gamma(\vartheta_\alpha, \sigma),$$

which implies that

$$\Gamma(0, \sigma) = \frac{\vartheta e^{i\theta}}{2} \preceq \lim_{\alpha\to\infty} \sup \Gamma(\vartheta_\alpha, \sigma).$$

It follows that (Λ, Γ) is a generalized complex metric space that is not a standard complex metric space since the triangular does not hold. If $\vartheta, \sigma \in \Lambda - \{0\}$, then we have $\Gamma(\vartheta, \sigma) = (\vartheta + \sigma)e^{i\theta}$ and $\Gamma(\vartheta, 0) + \Gamma(0, \sigma) = \frac{(\vartheta+\sigma)e^{i\theta}}{2}$ and thus

$$\Gamma(\vartheta, \sigma) \succ \Gamma(\vartheta, 0) + \Gamma(0, \sigma).$$

Note that (Λ, Γ) is Γ-complete. Define the mapping t on Λ by

$$t(\vartheta) = \frac{\vartheta}{\vartheta + 2} \text{ for all } x \in \Lambda.$$

For any $\vartheta \in \Lambda$, we have:

$$\begin{aligned} \Gamma(t(\vartheta), t(0)) = \Gamma(\frac{\vartheta}{\vartheta + 2}, 0) &= \frac{\vartheta e^{i\theta}}{2(\vartheta + 2)} \\ &\preceq \frac{\vartheta e^{i\theta}}{4} \\ &= \frac{1}{2}\Gamma(\vartheta, 0). \end{aligned}$$

Then

$$\Gamma(t(\vartheta), t(0)) \preceq \mathfrak{u}\Gamma(\vartheta, 0),\ \mathfrak{u} \in (0, 1).$$

For any $\vartheta, \sigma \in \Lambda - \{0\}$, we obtain

$$\begin{aligned} \Gamma(t(\vartheta), t(\sigma)) &= \Gamma\left(\frac{\vartheta}{\vartheta + 2}, \frac{\sigma}{\sigma + 2}\right) \\ &= \left(\frac{\vartheta}{2(\vartheta + 2)} + \frac{\sigma}{2(\sigma + 2)}\right)e^{i\theta} \\ &\preceq \frac{1}{2}(\vartheta + \sigma)e^{i\theta} \\ &= \mathfrak{u}\Gamma(\vartheta, \sigma). \end{aligned}$$

Then

$$\Gamma(t(\vartheta), t(\sigma)) \preceq \mathfrak{u}\Gamma(\vartheta, \sigma),\ \mathfrak{u} \in (0, 1).$$

Therefore hypotheses of Theorem 3.2 are satisfied. Hence t has a unique fixed point $\vartheta = 0 \in \Lambda$.

4 Application

Let $\Lambda = C[\eta_1, \eta_2]$ be a set of all real continuous functions on $[\eta_1, \eta_2]$ equipped with metric $\Gamma(\vartheta, \sigma) = |\vartheta(\tau) - \sigma(\tau)|(1 + i)$ for all $\vartheta, \sigma \in C[\eta_1, \eta_2]$. Then, (Λ, Γ)

is a complete generalized complex metric space. Now, we consider a non-linear Fredholm integral equation

$$\vartheta(\tau) = \mathfrak{v}(\tau) + \frac{1}{\eta_2 - \eta_1} \int_{\eta_1}^{\eta_2} \mathfrak{K}_1(\tau, \mathfrak{s}, \vartheta(s))ds$$

where $\tau, \mathfrak{s} \in [\eta_1, \eta_2]$. Assume that $\mathfrak{K}_1 : [\eta_1, \eta_2] \times [\eta_1, \eta_2] \times \Lambda \to \mathbb{R}$ and $\mathfrak{v} : [\eta_1, \eta_2] \to \mathbb{R}$ continuous, where $\mathfrak{v}(\tau)$ is a given function in Λ.

Theorem 4.1. *Suppose that (Λ, Γ) is a complete generalized complex metric space equipped with metric $\Gamma(\vartheta, \sigma) = |\vartheta(\tau) - \sigma(\tau)|(1+i)$ for all $\vartheta, \sigma \in C[\eta_1, \eta_2]$. and $t : \Lambda \to \Lambda$ be continuous operator on Λ defined by*

$$t\vartheta(t) = \mathfrak{v}(\tau) + \frac{1}{\eta_2 - \eta_1} \int_{\eta_1}^{\eta_2} \mathfrak{K}_1(\tau, \mathfrak{s}, \vartheta(s))ds. \tag{3}$$

If there exists $\wr \in (0, 1)$ such that for all $\vartheta, \sigma \in \Lambda$ with $\vartheta \neq \sigma$ and $\mathfrak{s}, \tau \in [\eta_1, \eta_2]$ satisfying the following inequality

$$|\mathfrak{K}_1(\tau, \mathfrak{s}, \vartheta(s)) - \mathfrak{K}_1(\tau, \mathfrak{s}, \sigma(\mathfrak{s}))| \leq \wr\, |\vartheta(s) - \sigma(\mathfrak{s})|, \tag{4}$$

then the integral operators defined by (3) *has a unique solution.*

Proof. Consider,

$$\begin{aligned}
\Gamma(t(\vartheta), t(\sigma)) &= |t\vartheta(\tau) - t\sigma(\tau)|(1+i) \\
&= \frac{1}{|\eta_2 - \eta_1|} \bigg| \int_{\eta_1}^{\eta_2} \mathfrak{K}_1(\tau, \mathfrak{s}, \vartheta(s))ds \\
&\quad - \int_{\eta_1}^{\eta_2} \mathfrak{K}_2(\tau, \mathfrak{s}, \sigma(\mathfrak{s}))ds \bigg| (1+i) \\
&\leq \frac{1}{|\eta_2 - \eta_1|} \bigg| \int_{\eta_1}^{\eta_2} |\mathfrak{K}_1(\tau, \mathfrak{s}, \vartheta(s)) \\
&\quad - \mathfrak{K}_2(\tau, \mathfrak{s}, \sigma(\mathfrak{s}))| ds \bigg| (1+i) \\
&\leq \frac{\wr}{|\eta_2 - \eta_1|} \left(\int_{\eta_1}^{\eta_2} |\vartheta(s) - \sigma(\mathfrak{s})| ds \right)(1+i) \\
&\leq \frac{\wr |\vartheta(s) - \sigma(\mathfrak{s})|}{|\eta_2 - \eta_1|} \left(\int_{\eta_1}^{\eta_2} ds \right)(1+i) \\
&= \wr |\vartheta(s) - \sigma(\mathfrak{s})|(1+i) \\
&= \wr \Gamma(\vartheta, \sigma)
\end{aligned}$$

Hence, all the conditions of Theorem 3.2 are satisfied and so, the integral operators t defined by (3) has a unique solution. □

5 Conclusion and future work

In this paper, we proved fixed point theorems on generalized complex metric space. An illustrative example and application on generalized complex metric space is given. Recently, ElKouch and Marhrani [13], proved fixed point theorems in generalized metric spaces. It is an interesting open problem to study the fixed point theorems in generalized complex metric spaces instead of fixed point theorems in generalized metric spaces.

References

[1] K. P. R. Rao, P. R. Swamy, and J. R. Prasad, *A common fixed point theorem in complex valued b-metric spaces*, Bulletin of Mathematics and Statistics Research, vol. 1, no. 1, 2013.

[2] O. Ege and I. Karaca, *complex valued dislocate metric space*, Korean J. Math. 26 (2018), No. 4, pp. 809–822 https://doi.org/10.11568/kjm.2018.26.4.809

[3] H. Nakano, *Modular semi-ordered spaces*, Tokyo, Japan (1959).

[4] J. Musielak, W. Orlicz, *On modular spaces*, Stud. Math. 18, 49-65 (1959).

[5] T. Dominguez Benavides, M.A. Khamsi, S. Samadi, *Uniformly Lipschitzian mappings in modular function spaces*, Nonlinear Anal. 46(2), 267-278 (2001).

[6] A. Hajji, E. Hanebaly, *Fixed point theorem and its application to perturbed integral equations in modular function spaces*, Electron. J. Differ. Equ. 2005, 105 (2005).

[7] M.A. Khamsi, *Nonlinear semigroups in modular function spaces*, Math. Jpn. 37, 291-299 (1992).

[8] M.A. Khamsi, W.M. Kozlowski, S. Reich, *Fixed point theory in modular function spaces*, Nonlinear Anal. 14, 935-953(1990).

[9] W.M. Kozlowski, *Modular Function Spaces*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 122. Dekker, New York (1988).

[10] A. Razani, S.H. Pour, E. Nabizadeh, M.B. Mohamadi, *A new version of the Ćirić quasi-contraction principle in the modular space*, Novi Sad J. Math. 43(2), 1-9 (2003).

[11] M. Jleli, B. Samet, *A generalized metric space and related fixed point theorems*, Fixed Point Theory Appl 2015, 61 (2015).

[12] A. Azam, B. Fisher, M. Khan, *Common fixed point theorems in complex valued metric spaces*, Numer. Funct. Anal. Optim. 32 (3) (2011) 243–253.

[13] Y. ElKouch, E. M. Marhrani, *On some fixed point theorems in generalized metric spaces*, Fixed Point Theory Appl 2017, 23 (2017).