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Positive Periodic Solutions for a Stage Structured Predator Prey Model

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Abstract: Chen et al.[10] developed a non-autonomous predator-prey system, which we reviewed in this article. The existence of a positive periodic solution is demonstrated using the continuation theorem of coincidence degree theory. Also, we proved the uniqueness and global stability of the solution by creating an appropriate Lyapunov function.

Keywords: Predator- Prey model, Continuation theorem, Periodic solution, Global stability.

Mathematics subject classification: MSC 2020: 92D25, 34C25.

1. Introduction The dynamics of predator- prey models with time delay has been recieved much attention in recent years [1-2]. The investigation of the occurrence of periodic solutions for time-delayed systems is attractive as suggested by Freedman et al. [3] and Kuang [4]. Periodicity can be found in nature in a variety of settings. Several animals, for example, ovulate on a regular basis, and many insects have a predictable life cycle. The equilibria of autonomous systems [5]-[7] play the same role as periodic solutions. Furthermore, as many predator-prey systems exhibit long-term variations, predator-prey models that can provide periodic solutions are essential. Continuation theorem [8] is a useful tool for solving ordinary differential equations, delay differential equations, and boundary value problems.

Chen, Xie, Li [10] studiedthe following variable coefficients predator-prey systems with time delay.

$$\begin{aligned}
\frac{dp_1}{d\tau} &= s_1(\tau)p_2(\tau) - a_{11}p_1(\tau) - s_1(\tau-\tau_1)e^{-a_{11}\tau_1}p_2(\tau-\tau_1) \\
\frac{dp_2}{d\tau} &= s_1(\tau-\tau_1)e^{-a_{11}\tau_1}p_2(t-\tau_1) - d_{12}p_2(\tau) - m_1(\tau)p_2^2(\tau) - n_1(\tau)p_2(\tau)q_2(\tau) \\
\frac{dq_1}{d\tau} &= s_2(\tau)q_2(\tau) - a_{22}q_1(\tau) - s_2(\tau-\tau_2)e^{-a_{22}\tau_2}q_2(\tau-\tau_2) \\
\frac{dq_2}{d\tau} &= s_2(\tau-\tau_2)e^{-a_{22}\tau_2}q_2(\tau-\tau_2) - a_{21}q_2(\tau) - m_2(\tau)q_2^2(\tau) + n_2(\tau)q_2(\tau)p_2(\tau)
\end{aligned} \tag{1.1}$$

where $p_1(\tau)$ and $p_2(\tau)$ indicates the population of the mature and immature prey species at time τ, respectively; $q_1(\tau)$ and $q_2(\tau)$ represents the population densities of the mature and immature predator species at time τ, respectively; For every $\tau \geq 0$, $s_i(\tau), m_i(\tau), n_i(\tau)(i = 1,2)$ are all continuous functions bounded above and below by positive constants. $a_{ij}, \tau_i, i, j = 1,2$ are all positive constants. The extinction property of (1.1) were investigated by Chen.et.al [9], [10]. Here

$$p_i(\vartheta) = \varphi_i(\vartheta),\ q_i(\vartheta) = \psi_i(\vartheta),\ \phi_i(0) > 0,\ \Psi_i(0) > 0\ (i = 1,2), \vartheta \in [-\varpi, 0], \tag{1.2}$$

where $\varpi = max\{\tau_1, \tau_2\}$, $(\phi_1(\vartheta), \phi_2(\vartheta), \Psi_1(\vartheta), \Psi_2(\vartheta))$ defined in $C^+ = \{\varphi \in C([-\varpi, 0], R^4_{+0}\}$ where $R^4_{+0} = \{p_i : p_i \geq 0, i = 1,2,3,4\}$ are the initial conditions for system (1.1). For continuity, we need

$$p_1(0) = \int_{-\tau_1}^{0} s_1(r)\varphi_2(r)e^{a_{11}r}dr,\ q_1(0) = \int_{-\tau_2}^{0} s_1(r)\psi_2(r)e^{a_{22}s}ds \tag{1.3}$$

Lemma 1.1 [10] For every $\tau \geq 0$, solutions of (1.1) are positive and bounded.

Lemma 1.2 [9] If the assertions $p_2^i > 0$ and $q_2^i > 0$ hold, where $p_2^i = (s_1^i e^{-a_{11}\tau_1} - a_{12} - n_1^r q_2^r)(m_1^r)^{-1}$ and $q_2^i = (s_2^i e^{-a_{22}\tau_2} - a_{21} + n_2^i p_2^i)(m_2^r)^{-1}$, then the system (1.1) is permanent.

In the next section, the existence of periodic positive T solutions of (1.1) is provided. Section 3 deals with positive T periodic solutions' global stability of (1.1). Section 4 with conclusions completes the paper.

2. Existence of positive periodic solutions

Here we apply the method which involves the applications of the continuation theorem of coincidence degree.

Let M, N be real Banach spaces, $P : DomP \subset M \to N$ a linear and $L : M \to N$ a continuous, where P called a Fredholm mapping of index 0 if $dimKerP = CodimImP < +\infty$ and ImP is closed in N. Then $\exists\ A : M \to M$ and $B : N \to$ N such that $ImA = KerP, ImP = KerB = Im(I - B)$ then the constraint P_A of P into $DomP\ \cap KerA : (I - A)M \to ImP$ is invertible. Let the inverse of P_A be X_P. If Ω is an open bounded subset of M, then L is P compact on $\bar{\Omega}$ if $BL(\bar{\Omega})$ is bounded and $X_A(I - B)L : \bar{\Omega} \to M$ is compact. As ImB is isomorphic to $KerP$ $\exists\ J : ImB \to KerP$.

Theorem 2.1: *At least one strictly positive T periodic solution exists in system* (1.1).

Proof:

If $p_2(\tau)$ and $q_2(\tau)$ are bounded and $p_2(\tau) \to p_2^*$, $q_2(\tau) \to q_2^*$ as $\tau \to \infty$, then $p_1(\tau) \to \frac{f(p_2^*, p_2^*)}{a_{11}}$ as $\tau \to \infty$ and $q_1(\tau) \to \frac{f(q_2^*, q_2^*)}{a_{22}}$ as $\tau \to \infty$, that is the asymptotic behaviour of $p_1(\tau)$ and $q_1(\tau)$ is dependent on that of $p_2(\tau)$ and $q_2(\tau)$. Therefore we just need to consider the subsystem of system (1.1).

$$\begin{aligned} \dot{p_2}(\tau) &= s_1(\tau-\tau_1)e^{-a_{11}\tau_1}p_2(\tau-\tau_1)) - a_{12}p_2(\tau) - m_1(\tau)p_2^2(\tau) - n_1(\tau)p_2(\tau)q_2(\tau) \\ \dot{q_2}(\tau) &= s_2(\tau-\tau_2)e^{-a_{22}\tau_2}q_2(\tau-\tau_2) - a_{21}q_2(\tau) - m_2(\tau)q_2^2(\tau) + n_2(\tau)q_2(\tau)p_2(\tau) \end{aligned} \tag{2.1}$$

Let $w_1(\tau) = ln[p_2(\tau)]$, $w_2(\tau) = ln[q_2(\tau)]$. Then (2.1) becomes

$$\begin{aligned} \dot{w}_1(\tau) &= \frac{s_1(\tau-\tau_1)e^{w_1(\tau-\tau_1)}}{e^{a_{11}\tau_1+w_1(\tau)}} - a_{12} - m_1(\tau)e^{w_1(\tau)} - n_1(\tau)e^{w_2(\tau)} \\ \dot{w}_2(\tau) &= \frac{s_2(\tau-\tau_2)e^{w_2(\tau-\tau_2)}}{e^{a_{22}\tau_2+w_2(\tau)}} - a_{21} - m_2(\tau)e^{w_2(\tau)} + n_2(\tau)e^{w_1(\tau)} \end{aligned} \tag{2.2}$$

Define $M = N = \{w(t) = (w_1(\tau), w_2(\tau))^T \in (R, R^2) : w_i(T+\tau) = w_i(\tau), i = 1, 2\}$ and

$$||w|| = ||(w_1(\tau), w_2(\tau))^T|| = \max_{\tau\in[0,T]} |w_1(\tau)| + \max_{\tau\in[0,T]} |w_2(\tau)|$$

for any $p \in M(orN)$. Set $P : DomP \cap M \to M, P(w_1(\tau), w_2(\tau))^T = (\frac{dw_1(\tau)}{d\tau}, \frac{dw_2(\tau)}{d\tau})^T$ where $DomP = \{(w_1(\tau), w_2(\tau))^T \in C^1(R, R^2)\}$ and $L : M \to M$,

$$L\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \frac{s_1(\tau-\tau_1)e^{w_1(\tau-\tau_1)}}{e^{a_{11}\tau_1+w_1(\tau)}} - a_{12} - m_1(\tau)e^{w_1(\tau)} - n_1(\tau)e^{w_2(\tau)} \\ \\ \frac{s_2(\tau-\tau_2)e^{w_2(\tau-\tau_2)}}{e^{a_{22}\tau_2+w_2(\tau)}} - a_{21} - m_2(\tau)e^{w_2(\tau)} + n_2(\tau)e^{m_1(\tau)} \end{pmatrix}$$

(2.2) can be written as $Pw = Lw, w \in M$. Now define

$$A\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = B\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\mathbb{T}}\int_0^{\mathbb{T}} w_1(\tau)dt \\ \\ \frac{1}{\mathbb{T}}\int_0^{\mathbb{T}} w_2(\tau)dt \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in M = N.$$

Clearly $KerP = \{z_1 | z_1 \in M, z_1 = h, h \in R^2\}$, $ImP = \{z_2 | z_2 \in N, \int_0^{\mathbb{T}} z_1(\tau)d\tau = 0\}$ is closed in N and $dimKerP = CodimImP = 2$. Therefore P is a Fredholm mapping of index zero. Also $ImA = KerP, KerB = ImP = Im(I - B)$. Hence inverse (to P) $X_A : ImP \to KerA \cap DomP$ is of the form

$$X_A(z_2) = \int_0^{\tau} z_2(r)dr - \tfrac{1}{\mathbb{T}}\int_0^{\mathbb{T}}\int_0^{\tau} z_2(r)drd\tau$$

Then $BL : M \to N$ and $X_A(I - B)L : M \to M$ are given respectively by

$$BLz_1 = \begin{pmatrix} \frac{1}{\mathbb{T}}\int_0^{\mathbb{T}} \left(\frac{s_1(\tau-\tau_1)e^{w_1(\tau-\tau_1)}}{e^{a_{11}\tau_1+w_1(\tau)}} - a_{12} - m_1(\tau)e^{w_1(\tau)} - n_1(\tau)e^{w_2(\tau)}\right) d\tau \\ \frac{1}{\mathbb{T}}\int_0^{\mathbb{T}} \left(\frac{s_2(\tau-\tau_2)e^{w_2(\tau-\tau_2)}}{e^{a_{22}\tau_2+u_2(\tau)}} - a_{21} - b_2(\tau)e^{w_2(\tau)} + n_2(\tau)e^{w_1(\tau)}\right) dt \end{pmatrix}$$

and $$X_A(I-B)Lz_1 = \begin{pmatrix} \int_0^\tau \Delta_1(r)dr \\ \int_0^\tau \Delta_2(r)dr \end{pmatrix} - \begin{pmatrix} \frac{1}{\mathbb{T}}\int_0^{\mathbb{T}}\int_0^\tau \Delta_1(r)drd\tau \\ \frac{1}{\mathbb{T}}\int_0^{\mathbb{T}}\int_0^\tau \Delta_2(r)drd\tau \end{pmatrix} - \begin{pmatrix} (\frac{\mathbb{T}}{\tau} - \frac{1}{2})\int_0^{\mathbb{T}} \Delta_1(r)dr \\ (\frac{\mathbb{T}}{\tau} - \frac{1}{2})\int_0^{\mathbb{T}} \Delta_2(r)dr \end{pmatrix}$$

Clearly BL and $X_A(I-B)L$ are continuous and $BL(\bar{\Omega})$ is bounded, $X_A(I-B)L(\bar{\Omega})$ is compact $\Omega \subset M$. So L will be P compact.
For $Pu = \lambda Lu,\ \lambda \in (0,1)$, we need a open bounded subset $\Omega \subset M$, we get

$$\begin{aligned} \frac{dw_1(\tau)}{d\tau} &= \lambda\left(\frac{s_1(\tau-\tau_1)e^{w_1(\tau-\tau_1)}}{e^{a_{11}\tau_1+w_1(\tau)}} - a_{12} - m_1(\tau)e^{w_1(\tau)} - n_1(\tau)e^{w_2(\tau)}\right) \\ \frac{dw_2(\tau)}{d\tau} &= \lambda\left(\frac{s_2(\tau-\tau_2)e^{w_2(\tau-\tau_2)}}{e^{a_{22}\tau_2+w_2(\tau)}} - a_{21} - m_2(\tau)e^{w_2(\tau)} + n_2(\tau)e^{w_1(\tau)}\right) \end{aligned} \tag{2.3}$$

Assume that $(w_1(\tau), w_2(\tau))^{\mathbb{T}} \in M$ is a solution of (2.3) for $\lambda \in (0,1)$. Now integrate (2.3) , which gives

$$\int_0^{\mathbb{T}} \left(\frac{s_1(\tau-\tau_1)e^{w_1(\tau-\tau_1)}}{e^{a_{11}\tau_1+w_1(\tau)}} - a_{12} - m_1(\tau)e^{u_1(\tau)} - n_1(\tau)e^{w_2(\tau)}\right) d\tau = 0 \tag{2.4}$$

$$\int_0^{\mathbb{T}} \left(\frac{s_2(\tau-\tau_2)e^{w_2(\tau-\tau_2)}}{e^{a_{22}\tau_2+w_2(\tau)}} - a_{21} - m_2(\tau)e^{w_2(\tau)} + n_2(\tau)e^{w_1(\tau)}\right) dt = 0 \tag{2.5}$$

Thus

$$\begin{aligned} \int_0^{\mathbb{T}} \frac{s_1(\tau-\tau_1)e^{w_1(\tau-\tau_1)}}{e^{a_{11}\tau_1+w_1(\tau)}} - m_1(\tau)e^{w_1(\tau)} - n_1(\tau)e^{w_2(\tau)}d\tau &= \int_0^{\mathbb{T}} a_{12}d\tau \\ &= d_{12}\mathbb{T} \end{aligned} \tag{2.6}$$

$$\begin{aligned} \int_0^{\mathbb{T}} \frac{s_2(\tau-\tau_2)e^{w_2(\tau-\tau_2)}}{e^{a_{22}\tau_2+w_2(\tau)}} - m_2(\tau)e^{w_2(\tau)} + n_2(\tau)e^{w_1(\tau)}d\tau &= \int_0^{\mathbb{T}} a_{21}d\tau \\ &= a_{21}\mathbb{T} \end{aligned} \tag{2.7}$$

From (2.3), (2.4), (2.6) and (2.7)

$$\begin{aligned} \int_0^{\mathbb{T}} |w_1'(\tau)|dt &= \lambda \int_0^{\mathbb{T}} \left|\frac{s_1(\tau-\tau_1)e^{w_1(\tau-\tau_1)}}{e^{a_{11}\tau_1+w_1(\tau)}} - a_{12} - m_1(\tau)e^{w_1(\tau)} - n_1(\tau)e^{w_2(\tau)}\right| d\tau \\ &< \int_0^{\mathbb{T}} |a_{12}|d\tau + \int_0^{\mathbb{T}} \left|\frac{s_1(\tau-\tau_1)e^{w_1(\tau-\tau_1)}}{e^{a_{11}\tau_1+w_1(\tau)}} + m_1(\tau)e^{w_1(\tau)} + n_1(\tau)e^{w_2(\tau)}\right| d\tau \\ &= a_{12}\mathbb{T} \end{aligned} \tag{2.8}$$

Similarly

$$\int_0^{\mathbb{T}} |w_2'(\tau)| d\tau \le a_{21}\mathbb{T} \tag{2.9}$$

From (2.5),

$$\int_0^{\mathbb{T}} a_{21} d\tau + \int_0^{\mathbb{T}} m_2(\tau) e^{w_2(\tau)} - \int_0^{\mathbb{T}} n_2(\tau) e^{w_1(\tau)} d\tau = \int_0^{\mathbb{T}} \frac{s_2(\tau-\tau_2) e^{w_2(\tau-\tau_2)}}{e^{a_{22}\tau_2 + w_2(\tau)}} d\tau$$
$$\le s_2^M \int_0^{\mathbb{T}} e^{w_2(\tau)} d\tau$$

$$(m_2^M - s_2^M) \int_0^{\mathbb{T}} e^{w_2(\tau)} - n_2^M \int_0^{\mathbb{T}} e^{w_1(\tau)} d\tau \le -a_{21} T \tag{2.10}$$

From (2.4),

$$\int_0^{\mathbb{T}} a_{12} d\tau + \int_0^{\mathbb{T}} m_1(\tau) e^{w_1(\tau)} + \int_0^{\mathbb{T}} n_1(\tau) e^{w_2(\tau)} d\tau = \int_0^{\mathbb{T}} \frac{s_1(\tau-\tau_1) e^{w_1(\tau-\tau_1)}}{e^{a_{11}\tau_1 + w_1(\tau)}} dt$$
$$\le s_1^M \int_0^{\mathbb{T}} e^{w_1(\tau)} d\tau$$

$$(m_1^M - s_1^M) \int_0^{\mathbb{T}} e^{w_1(\tau)} + n_1^M \int_0^{\mathbb{T}} e^{w_2(\tau)} d\tau \le (-a_{12}) T \tag{2.11}$$

From (2.10) and (2.11),

$$\int_0^{\mathbb{T}} e^{w_2(\tau)} d\tau \le \frac{(m_1^M - s_1^M)(-a_{21}\mathbb{T}) + n_2^M((-a_{12}\mathbb{T})}{(m_1^L - s_1^L)(m_2^L - s_2^L) + n_1^L n_2^L} \tag{2.12}$$

Also

$$\int_0^{\mathbb{T}} e^{w_1(\tau)} d\tau \le \frac{-a_{12}T - n_1^M\left(\frac{(m_1^M - s_1^M)(-a_{21}\mathbb{T}) + n_2^M((-a_{12}\mathbb{T})}{(m_1^L - s_1^L)(m_2^L - s_2^L) + n_1^L n_2^L}\right)}{(m_1^L - s_1^L)} \tag{2.13}$$

As $w(\tau) = (w_1(\tau), w_2(\tau))^T \in X$, there exists $\varepsilon \in [0, \mathbb{T}]$ such that

$$e^{w_2(\varepsilon_2)} \textstyle\int_0^{\mathbb{T}} d\tau \le \frac{(m_1^M - s_1^M)(-a_{21}\mathbb{T}) + n_2^M((-a_{12}\mathbb{T})}{(m_1^L - s_1^L)(m_2^L - s_2^L) + n_1^L n_2^L}$$

which implies $w_2(\varepsilon_2) \le ln \frac{(m_1^M - s_1^M)(-a_{21}\mathbb{T}) + n_2^M((-a_{12}T)}{(m_1^L - s_1^L)(m_2^L - s_2^L) + n_1^L n_2^L}$.

$$w_2(\tau) \le w_2(\varepsilon_2) + \int_0^{\mathbb{T}} |w_2'(\tau)| d\tau \le ln \frac{(m_1^M - s_1^M)(-a_{21}\mathbb{T}) + n_2^M((-a_{12}\mathbb{T})}{(m_1^L - s_1^L)(m_2^L - s_2^L) + n_1^L n_2^L} + a_{21}\mathbb{T} = H_{21} \tag{2.14}$$

Similarly

$$w_1(\tau) \le w_1(\varepsilon_1) + \int_0^{\mathbb{T}} |w_1'(\tau)| d\tau \le ln \frac{-a_{12}T - n_1^M\left(\frac{(m_1^M - s_1^M)(-a_{21}\mathbb{T}) + n_2^M((-a_{12}\mathbb{T})}{(m_1^L - s_1^L)(m_2^L - s_2^L) + n_1^L n_2^L}\right)}{(m_1^L - s_1^L)} + a_{12}\mathbb{T}$$
$$= H_{11} \tag{2.15}$$

Also from (2.4), we have

$$s_1(\tau-\tau_1)e^{w_1(\tau-\tau_1)}d\tau \geq \int_0^{\mathbb{T}} a_{12}d\tau$$

which yields

$$w_1(\eta_1) \geq ln\frac{a_{12}}{s_1^L} \tag{2.16}$$

$$w_1(\tau) \geq w_1(\eta_1) - \int_0^{\mathbb{T}} |w_1'(\tau)|d\tau \geq ln\frac{a_{12}}{s_1^M} - a_{12}\mathbb{T} = H_{12} \tag{2.17}$$

From (2.15) and (2.17),

$$\max_{\tau\in[0,\mathbb{T}]} |m_1(\tau)| \leq \max\{|H_{11}, H_{12}\} := H_1 \tag{2.18}$$

Similarly

$$w_2(\eta_2) \geq ln\frac{a_{21}}{s_2^L} \tag{2.19}$$

$$w_2(\tau) \geq w_2(\eta_2) - \int_0^{\mathbb{T}} |w_2'(t)|dt \geq ln\frac{a_{21}}{s_2^M} - a_{21}\mathbb{T} = H_{22} \tag{2.20}$$

From (2.14) and (2.20),

$$\max_{\tau\in[0,\mathbb{T}]} |u_2(\tau)| \leq \max\{|H_{21}, H_{22}\} := H_2 \tag{2.21}$$

Let $H = H_1 + H_2 + H_0$, then $(w^*, v^*)^T$ of the system,

$$\begin{aligned} s_1(t-\tau_1)e^{-a_{11}\tau_1} - a_{12} - m_1e^w - n_1e^v = 0 \\ s_2(t-\tau_2)e^{-a_{21}\tau_2} - a_{21} - m_2e^v + c_2e^w = 0 \end{aligned} \tag{2.22}$$

satisfies $||(w^*, v^*)|| = |w^*| + |v^*| < H$. Choose $\Omega = \{(w_1(\tau), w_2(\tau))^{\mathbb{T}} \in X : ||(w_1, w_2)^{\mathbb{T}}|| < H$.

When $(w_1(\tau), w_2(\tau))^{\mathbb{T}} \in \partial\Omega \cap R^2, (w_1, w_2)^{\mathbb{T}}$ is a constant vector in R^2 with $|w_1|+|w_2| = H$. Therefore

$$QN\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} s_1(\tau-\tau_1)e^{-a_{11}\tau_1} - a_{12} - m_1e^{w_1} - n_1e^{w_2} \\ s_2(\tau-\tau_2)e^{-a_{21}\tau_2} - a_{21} - m_2e^{w_2} + n_2e^{w_1} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

Now choose $\mathbb{J} = \mathbb{I} : Im\mathbb{Q} \to Ker\mathbb{L}, (w_1, w_2)^{\mathbb{T}}$. After simplification, we get

$$\begin{aligned} deg\{\mathbb{JQN}, \Omega \cap Ker\mathbb{L}, 0\} &= sgn\ det\begin{pmatrix} -m_1e^{w_1} & -n_1e^{w_2} \\ n_2e^{w_1} & -m_2e^{w_2} \end{pmatrix} \\ &= sgn(m_1m_2e^{w_1+w_2} + n_1n_2e^{w_1+w_2}) \\ &\neq 0 \end{aligned}$$

where $(w_1^*, w_2^*)^T$ is the unique solution of (2.22). Also $\mathbb{X}^P(\mathbb{I}-\mathbb{Q})\mathbb{L} : \bar{\Omega} \to M$ is compact. So$\mathbb{L}$ is $\mathbb{P}-$ compact. Hence (2.2) has at least one $\mathbb{T}$ periodic solution. .

Let$(p_2^*(\tau), q_2^*(\tau))^{\mathbb{T}}$ be a positive $\mathbb{T}$ periodic solution of system (2.1). Similarly $p_1^*(\tau)$ and $q_1^*(\tau)$ are also T periodic. Thus $(p_1^*(\tau), p_2^*(\tau), q_1^*(\tau), q_2^*(\tau))^{\mathbb{T}}$ is a positive $\mathbb{T}$ periodic solution of system (1.1) .

3. Uniqueness and global stability Here, we provide the uniqueness and the global stability of positive periodic solution of (1.1).

Definition 3.1: System (1.1) is globally asymptotically stable if

$$\lim_{\tau\to\infty}|p_1(\tau)-w_1(\tau)|=0,\ \lim_{\tau\to\infty}|p_2(\tau)-w_2(\tau)|=0,\ \lim_{\tau\to\infty}|q_1(\tau)-v_1(\tau)|=0,\ \lim_{\tau\to\infty}|q_2(\tau)-v_2(\tau)|=0$$

hold for any two solutions $(p_1(\tau), p_2(\tau), q_1(\tau), q_2(\tau))$ and $(w_1(\tau), w_2(\tau), v_1(\tau), v_2(\tau))$ of (1.1).

Theorem 3.1: *System* (1.1) *with initial conditions* (1.2) *and* (1.3) *is globally asymptotically stable.*

Lemma 3.1: *If* $a_1 > 0$ *and* $a_2 > 0$ *exists in such a way that* $\delta_i(\tau)(i = 1, 2)$ *are positive on* $[0, \infty)$ *and one has* $\sum_{k=1}^{\infty}\int_{g_k}^{h_k} B_i(t)dt = \infty$*, for any interval sequence* $\{[g_i, h_i]\}_1^{\infty}, [g_i, h_i] \cap [g_j, h_j] = \phi$ *and* $h_i - g_i = h_j - g_j > 0$ *for all* $i, j = 1, 2..$ *and* $i \neq j,$*, then system* (2.1) *is globally asymptotically stable. Here,*

$$\begin{aligned}\delta_1(\tau) &= a_1(a_{12} + 2mm_1(\tau) + n_1(\tau)M - s_1(\tau)e^{-a_{11}\tau_1}) + a_2n_2(\tau)M \\ \delta_2(\tau) &= a_2(a_{21} + 2mm_2(\tau) - n_2(\tau)M - r_2(\tau)e^{-a_{22}\tau_2}) + a_1n_1(\tau)M\end{aligned} \tag{3.1}$$

Proof: Assume that $(p_2(\tau), q_2(\tau))$ and $(w_2(\tau), v_2(\tau))$ are any two solutions of system (2.1).

Define $W_1(\tau) = |ln\ p_2(\tau) - ln\ w_2(\tau)|$ and $W_2(\tau) = |ln\ q_2(\tau) - ln\ v_2(\tau)|$. Then

$$\begin{aligned}\mathbb{D}^+W_1(\tau) &= \left(\frac{\dot{p_2}(\tau)}{p_2(\tau)} - \frac{\dot{w_2}(\tau)}{w_2(\tau)}\right) sgn(p_2(\tau) - w_2(\tau)) \\ &\leq (s_1(\tau)e^{-a_{11}\tau_1} - a_{12} - 2mm_1(\tau) - n_1(\tau)M)|p_2(\tau) - w_2(\tau)| + n_1(\tau)M|q_2(\tau) - v_2(\tau)|\end{aligned} \tag{3.2}$$

$$\begin{aligned}\mathbb{D}^+W_2(\tau) &= \left(\frac{\dot{q_2}(\tau)}{q_2(\tau)} - \frac{\dot{v_2}(\tau)}{v_2(t\tau)}\right) sgn(q_2(\tau) - v_2(\tau)) \\ &\leq -Mc_2(\tau)|x_2(\tau) - w_2(\tau)| + (s_2(\tau)e^{-a_{22}\tau_2} - a_{21} - 2mm_2(\tau) + n_2(\tau)M)|q_2(\tau) - v_2(\tau)|\end{aligned} \tag{3.3}$$

Let $W(\tau) = a_1W_1(\tau) + a_2W_2(\tau)$, and from (3.2) and (3.3), we have

$$\mathbb{D}^+W(\tau) \leq -\delta_1(\tau)|p_2(\tau) - w_2(\tau)| - \delta_2(\tau)|q_2(\tau) - v_2(\tau)|\ \forall\ \tau \geq \mathbb{T}_4 \tag{3.4}$$

where $\mathbb{T}_4$ is defined in [9] and $\delta_i(\tau), (i = 1, 2)$ are in (3.1).
Integrating (3.4) from $\mathbb{T}_4$ to τ, we get

$$\begin{aligned}&\int_{\mathbb{T}_4}^{\tau}\{\delta_1(\tau)|p_2(\tau) - w_2(\tau)| + \delta_2(\tau)|q_2(\tau) - v_2(\tau)|\}d\tau \leq W(\mathbb{T}_4) - W(\tau) \\ \Rightarrow &\int_{\mathbb{T}_4}^{\tau}\{\delta_1(\tau)|p_2(\tau) - w_2(\tau)| + \delta_2(\tau)|q_2(\tau) - v_2(\tau)|\}d\tau < \infty\end{aligned} \tag{3.5}$$

Then from (3.5),

$$\lim_{\tau\to\infty} |p_2(\tau) - w_2(\tau)| = 0 \ and \ \lim_{\tau\to\infty} |q_2(\tau) - v_2(\tau)| = 0. \tag{3.6}$$

Therefore (2.1) is globally asymptotically stable.

Proof of Theorem 3.1: Since $p_1(\tau) = \int_{\tau-\tau_1}^{\tau} s_1(r)e^{-a_{11}(\tau-r)}p_2(r)dr$, we have

$$\begin{aligned} |p_1(\tau) - w_1(\tau)| &\leq \int_{\tau-\tau_1}^{\tau} s_1(r)e^{-a_{11}(\tau-r)}|q_2(r) - w_2(r)|dr \\ &\leq \int_{\tau-\tau_1}^{\tau} s_1^M |p_2(s) - w_2(r)|dr, \end{aligned} \tag{3.7}$$

which together with (3.6) yields,

$$\lim_{\tau\to\infty} |p_1(\tau) - w_1(\tau)| = 0$$

Similarly, we can obtain

$$\lim_{\tau\to\infty} |q_1(\tau) - v_1(\tau)| = 0$$

Hence system (1.1) with initial conditions (1.2) and (1.3) is globally asymptotically stable.

Corollary 3.1: *If there exists* $a_1 > 0$ *and* $a_2 > 0$ *such that* $\liminf_{\tau\to\infty} \delta_i(\tau) > 0, (i = 1, 2)$ *where* $\delta_i(\tau)$ *are given by* (3.1)*, then system* (2.1) *is globally asymptotically stable.*

Hence the functions and the time delay induce effect on the global asymptotic stability, which may rule out any complicated behaviour of the system. From Theorem 2.1 and the above results, we have the following corollary.

Corollary 3.2: *If system* (1.1) *is* $\mathbb{T}$ *periodic and conditions in Lemma 3.1 and Theorem 3.1 holds, then there exists a unique positive* $\mathbb{T}$ *periodic solution that is globally asymptotically stable.*

4. Conclusion Chen et al.'s [10] non-autonomous stage structured predator-prey system is investigated here. The existence of the positive T periodic solutions of system (1.1) under certain conditions was demonstrated. By constructing a suitable Lyapunov functional, sufficient conditions for the uniqueness and global stability of positive T periodic solutions of system (1.1) were obtained.

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