Proceedings of the Jangjeon Mathematical Society
25 (2022), No. 3, pp. 287 – 296

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2022.25.3.287

ON LP-SASAKIAN MANIFOLDS ADMITTING GENERALIZED SYMMETRIC METRIC CONNECTION

G. SOMASHEKHARA, S. GIRISH BABU, P. SIVA KOTA REDDY, AND K. SHIVASHANKARA

Abstract. The objective of this paper is to study LP-Sasakian manifolds admitting generalized symmetric metric connection.

2010 Mathematics Subject Classification. 53C15, 53C25, 53C40.

Keywords and phrases. LP-Sasakian manifold, Generalized symmetric metric connection, ϕ-Projectively semisymmetric manifold, ϕ-Weyl semisymmetric manifold.

1. Introduction

A linear connection on a (semi-)Riemannian manifold M is suggested to be a generalized symmetric connection if its torsion tensor T is presented as follows:

$$T(X,Y) = \alpha\,(u(Y)X - u(X)Y) + \beta\,(u(Y)\phi X - u(X)\phi Y)\,, \tag{1}$$

for all vector fields X and Y on M , where α and β are smooth functions on M, ϕ can be viewed as a tensor of type $(1,1)$ and u is regarded as a 1-form connected with the vector field which has a non-vanishing smooth non-null unit.

In equation 1, if $\alpha = 0$, $\beta \neq 0$; $\alpha \neq 0$, $\beta = 0$, then the generalized symmetric connection is called β-quarter-symmetric connection; α-semi-symmetric connection, respectively. Additionally, the generalized symmetric connection reduces to a semi-symmetric, and quarter-symmetric connection when $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$, respectively. Thus, it can be suggested that generalizing semi-symmetric and quarter-symmetric connections paves the way for a generalized symmetric metric connection.

The notion of locally symmetry of a Riemannian manifold was started by Cartan [2]. Takahashi [9] introduced ϕ-symmetry on a Sasakian manifold. In [3], the author generalized the concept of ϕ-symmetry and introduced ϕ-recurrent Sasakian manifold. Dubey [5] initiated the notion of generalized recurrent manifold. Shaikh and Hui [8] introduced the notion of extended generalized ϕ-recurrent manifolds.

Ricci soliton in a Riemannian manifold (M^n, g) is a natural generalization of an Einstein metric and is defined as a triple (g, V, λ), where g is a

[1]Corresponding author: drksshankara@gmail.com

Riemannian metric, V is a vector field and λ is a real scalar such that

$$(L_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{2}$$

where S is the Ricci tensor, L_ξ denotes the Lie derivative along the vector field ξ.

2. Preliminaries

A differentiable manifold of dimension n is called a Lorentzian para-Sasakian manifold if it admits $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a 1-form η and Lorentzian metric g which satisfying (See [6, 7]):

$$\eta(\xi) = -1, \phi^2 X = X + \eta(X)\xi, \tag{3}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), g(X,\xi) = \eta(X), \tag{4}$$

$$\nabla_X \xi = \phi X, (\nabla_X \phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{5}$$

where ∇ is the covariant differentiation with respect to the Lorentzian metric g.

It can be easily seen that in a LP-Sasakian manifold the following relations hold:

$$\phi\xi = 0, \eta(\phi X) = 0, rank(\phi) = n-1. \tag{6}$$

If we write $\Phi(X,Y) = g(\phi X, Y)$ for all vector fields X, Y on M, then the tensor field $\Phi(X,Y)$ becomes a symmetric $(0,2)$ tensor field [6]. Also, the 1-form η is closed on an LP-Sasakian manifold and then we have:

$$(\nabla_X \eta)Y = \Phi(X,Y), \Phi(X,\xi) = 0, \tag{7}$$

for any vector fields X and Y on $M \in \Gamma(TM)$. Further, the following relations hold for LP-Sasakian manifolds [6, 7]:

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{8}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{9}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{10}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y) \tag{11}$$

for all vector fields X, Y on $M \in \Gamma(TM)$, in which R and S can be viewed as the Riemannian curvature tensor and the Ricci tensor of M respectively.

In a LP-Sasakian manifold admitting generalized symmetric metric connection $\bar{\nabla}$, the following relations hold (See [1]):

$$\bar{R}(X,\xi)Z = (1-\beta+\beta^2)(\eta(Z)X - g(X,Z)\xi) + \alpha(1-\beta)g(\phi X, Z)\xi, \tag{12}$$

$$\begin{aligned}\bar{R}(X,Y)\xi &= (1-\beta+\beta^2)(\eta(Y)X - \eta(X)Y) + \\ &\alpha(1-\beta)(\eta(X)\phi Y - \eta(Y)\phi X),\end{aligned} \tag{13}$$

$$\begin{aligned}\bar{R}(\xi,Y)Z &= \left(-\alpha\Phi(Y,Z) + (1-\beta)g(Y,Z) - \beta^2\eta(Y)\eta(Z)\right)\xi - \\ &(1-\beta+\beta^2)\eta(Z)Y + \alpha(1-\beta)\eta(Z)\phi Y,\end{aligned} \tag{14}$$

$$\bar{R}(\xi,Y)\xi = (1-\beta+\beta^2)(\eta(Y)\xi + Y) + \alpha(\beta-1)\phi Y, \tag{15}$$

$$\begin{aligned}\bar{S}(Y,Z) &= S(Y,Z)+\\ &(-\alpha\beta + (n-2)\alpha(\beta-1) + \beta(\beta-2)trace\Phi)\,\Phi(Y,Z)+\\ &\left(-2\alpha^2 + \beta - \beta^2 + n\alpha^2 + \alpha(\beta-1)trace\Phi\right) g(Y,Z)+\\ &\left(-2\alpha^2 + n(\alpha^2+\beta-\beta^2)\right)\eta(Y)\eta(Z),\end{aligned} \tag{16}$$

$$\begin{aligned}\bar{Q}Y &= QY+\\ &(-\alpha\beta + (n-2)\alpha(\beta-1) + \beta(\beta-2)trace\Phi)\,\phi Y+\\ &\left(-2\alpha^2 + \beta - \beta^2 + n\alpha^2 + \alpha(\beta-1)trace\Phi\right) Y+\\ &\left(-2\alpha^2 + n(\alpha^2+\beta-\beta^2)\right)\eta(Y)\xi,\end{aligned} \tag{17}$$

$$\bar{S}(Y,\xi) = \left((n-1)(1-\beta+\beta^2) + \alpha(\beta-1)trace\Phi\right)\eta(Y), \tag{18}$$

$$\begin{aligned}\bar{S}(\phi Y,\phi Z) &= \bar{S}(Y,Z)+\\ &\left((n-1)(1-\beta+\beta^2) + \alpha(\beta-1)trace\Phi\right)\eta(Y)\eta(Z),\end{aligned} \tag{19}$$

for every $X,\,Y,\,Z \in \Gamma(TM)$.

Lemma 2.1. *Let (M^n,g) be an LP-Sasakian manifold admitting generalized symmetric metric connection. Then for any vector fields Y,W the following relations hold:*

$$\begin{aligned}(\nabla_W\bar{S})(Y,\xi) &= \left((n-1)(1-\beta+\beta^2) + \alpha(\beta-1)trace\Phi\right) g(\phi W,Y)\\ &- \bar{S}(\phi W,Y).\end{aligned} \tag{20}$$

Proof. We know that

$$(\nabla_W\bar{S})(Y,\xi) = \nabla_W\bar{S}(Y,\xi) - \bar{S}(\nabla_WY,\xi) - \bar{S}(Y,\nabla_W\xi). \tag{21}$$

Using (5), (7) and (18) in above equation we get,

$$\begin{aligned}&(\nabla_W\bar{S})(Y,\xi) =\\ &\left((n-1)(1-\beta+\beta^2) + \alpha(\beta-1)trace\phi\right) g(\phi W,Y) - \bar{S}(\phi W,Y).\end{aligned} \tag{22}$$

□

Lemma 2.2. *Let (M^n,g) be an LP-Sasakian manifold admitting generalized symmetric metric connection. Then for any vector fields Y,W the following relations hold:*

$$(\nabla_W\bar{R})(\xi,Y)\xi = \beta^2 g(\phi W,Y)\xi + \alpha\beta g(W,Y)\xi + \alpha\beta\eta(Y)\eta(W)\xi. \tag{23}$$

Proof. We know that

$$\begin{aligned}&(\nabla_W\bar{R})(\xi,Y)\xi =\\ &\nabla_W\bar{R}(\xi,Y)\xi - \bar{R}(\nabla_W\xi,Y)\xi - \bar{R}(\xi,\nabla_WY)\xi - \bar{R}(\xi,Y)\nabla_W\xi.\end{aligned} \tag{24}$$

Using (3), (4), (5), (13), (14) and (15) in above equation we get,

$$\begin{aligned}(\nabla_W\bar{R})(\xi,Y)\xi &= (1-\beta+\beta^2)(\nabla_W\eta)Y\xi + \alpha(\beta-1)(\nabla_W\phi)Y+\\ &\alpha(1-\beta)\eta(Y)W + \alpha(1-\beta)\eta(Y)\eta(W)\xi + \alpha g(Y,W)\xi+\\ &\alpha\eta(Y)\eta(W)\xi - (1-\beta)g(Y,\phi W)\xi.\end{aligned} \tag{25}$$

Using (5) and (7) in above equation we get,

$$(\nabla_W \bar{R})(\xi, Y)\xi = \beta^2 g(\phi W, Y)\xi + \alpha\beta g(W, Y)\xi + \alpha\beta\eta(Y)\eta(W)\xi. \tag{26}$$

□

3. On extended generalized ϕ-recurrent LP-Sasakian manifold admitting generalized symmetric metric connection

Definition 3.1. *A LP-Sasakian manifold admitting generalized symmetric metric connection $M^n(\phi, \xi, \eta, g)$ is extended generalized ϕ-recurrent if (See [8])*

$$\phi^2((\nabla_W \bar{R})(X, Y)Z) = A(W)\phi^2(\bar{R}(X, Y)Z) + B(W)\phi^2(G(X, Y)Z), \tag{27}$$

for all $X, Y, Z, W \in \Gamma(\mathrm{TM})$, where A and B are two 1-forms such that $A(X) = g(X, \rho_1)$, $B(X) = g(X, \rho_2)$ and $G(X, Y)Z = g(Y, Z)X - g(X, Z)Y$. Hence ρ_1 and ρ_2 are the vector fields associated with 1-form A and B.

In view of equation (3) and (27) we have

$$\begin{aligned} &(\nabla_W \bar{R})(X, Y)Z + \eta((\nabla_W \bar{R})(X, Y)Z)\xi = \\ &A(W)\left(\bar{R}(X, Y)Z + \eta(\bar{R}(X, Y)Z)\xi\right) + \\ &B(W)\left(G(X, Y)Z + \eta(G(X, Y)Z)\xi\right). \end{aligned} \tag{28}$$

Taking inner product with U in (28), we get:

$$\begin{aligned} &g((\nabla_W \bar{R})(X, Y)Z, U) + g((\nabla_W \bar{R})(X, Y)Z, \xi)g(\xi, U) = \\ &A(W)\left(g(\bar{R}(X, Y)Z, U) + g(\bar{R}(X, Y)Z), \xi)g(U, \xi)\right) + \\ &B(W)\left(g(G(X, Y)Z, U) + g((G(X, Y)Z), \xi)g(U, \xi)\right). \end{aligned} \tag{29}$$

Let $\{e_1, e_2, \cdots, e_n\}$ be a local orthonormal basis of vector fields in M. Putting $X = U = e_i$ in (29), we obtain

$$\begin{aligned} &(\nabla_W \bar{S})(Y, Z) - \eta((\nabla_W \bar{R})(\xi, Y)Z) = A(W)\left(\bar{S}(Y, Z) - \eta(\bar{R}(\xi, Y)Z)\right) + \\ &B(W)\left((n-2)g(Y, Z) - \eta(Y)\eta(Z)\right). \end{aligned} \tag{30}$$

Putting $Z = \xi$ and using (15), (18) in (30) we get

$$\begin{aligned} &(\nabla_W \bar{S})(Y, \xi) = \eta((\nabla_W \bar{R})(\xi, Y)\xi) + A(W)\left((n-1)(1-\beta+\beta^2)\right) + \\ &A(W)\left(\alpha(\beta-1)trace\Phi\right)\eta(Y) + B(W)(n-1)\eta(Y). \end{aligned} \tag{31}$$

From Lemmas 2.1 and 2.2, equation (31) becomes

$$\begin{aligned} &\left((n-1)(1-\beta+\beta^2) + \alpha(\beta-1)trace\Phi\right)g(\phi W, Y) - \bar{S}(\phi W, Y) = \\ &-\left(\beta^2 g(\phi W, Y) + \alpha\beta(g(Y, W) + \eta(W)\eta(Y))\right) + B(W)(n-1)\eta(Y) \\ &+A(W)\left((n-1)(1-\beta+\beta^2) + \alpha(\beta-1)trace\Phi\right)\eta(Y). \end{aligned} \tag{32}$$

Substituting $Y = \xi$ in (32), we obtain

$$\begin{aligned} &\left((n-1)(1-\beta+\beta^2) + \alpha(\beta-1)trace\Phi\right)g(\phi W, \xi) - \bar{S}(\phi W, \xi) = \\ &-\left(\beta^2 g(\phi W, \xi) + \alpha\beta(g(\xi, W) + \eta(W)\eta(\xi))\right) + B(W)(n-1)\eta(\xi) \\ &+ A(W)\left((n-1)(1-\beta+\beta^2) + \alpha(\beta-1)trace\Phi\right)\eta(\xi). \end{aligned} \tag{33}$$

Using (3), (4), (6) in (33), we have

$$A(W) = -kB(W), \tag{34}$$

where

$$k = \frac{n-1}{((n-1)(1-\beta+\beta^2)+\alpha(\beta-1)trace\Phi)}.$$

Thus we can state the following Corollary:

Corollary 3.1. *In an n-dimensional extended generalized ϕ-recurrent LP-Sasakian manifold (M^n, g) admitting generalized symmetric metric connection, the 1-forms $A(W)$ and $B(W)$ satisfies the relation $A(W) = -kB(W)$.*

By the virtue of Bianchi's identity and (28), we have

$$\begin{aligned}
&A(W)\left(\bar{R}(X,Y)Z + \eta(\bar{R}(X,Y)Z)\xi\right) + \\
&B(W)\left(G(X,Y)Z + \eta(G(X,Y)Z)\xi\right) + \\
&A(X)\left(\bar{R}(Y,W)Z + \eta(\bar{R}(Y,W)Z)\xi\right) + \\
&B(X)\left(G(Y,W)Z + \eta(G(Y,W)Z)\xi\right) + \\
&A(Y)\left(\bar{R}(W,X)Z + \eta(\bar{R}(W,X)Z)\xi\right) + \\
&B(Y)\left(G(W,X)Z + \eta(G(W,X)Z)\xi\right) = 0.
\end{aligned} \tag{35}$$

Taking inner product with U

$$\begin{aligned}
&A(W)(g(\bar{R}(X,Y)Z,U) + \eta(\bar{R}(X,Y)Z)g(\xi,U)) \\
&+ B(W)(g(G(X,Y)Z,U) + \eta(G(X,Y)Z)g(U,\xi)) \\
&+ A(X)(g(\bar{R}(Y,W)Z,U) + \eta(\bar{R}(Y,W)Z)g(\xi,U)) + \\
&B(X)(g(G(Y,W)Z,U) + \eta(G(Y,W)Z)g(\xi,U)) \\
&+ A(Y)(g(\bar{R}(W,X)Z,U) + \eta(\bar{R}(W,X)Z)g(\xi,U)) \\
&+ B(Y)\left(g(G(W,X)Z,U) + \eta(G(W,X)Z)g(\xi,U)\right) = 0.
\end{aligned} \tag{36}$$

Putting $Y = Z = e_i$ in equation (36) and taking summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned}
&A(W)\left(\bar{S}(X,U) + \bar{S}(X,\xi)g(U,\xi)\right) \\
&+ B(W)\left((n-1)g(X,U) + (n-1)g(X,\xi)g(U,\xi)\right) + \\
&A(X)\left(-\bar{S}(W,U) - \bar{S}(W,\xi)g(U,\xi)\right) \\
&+ B(X)\left((1-n)g(W,U) + (1-n)g(W,\xi)g(U,\xi)\right) \\
&+ A(e_i)\left(g(\bar{R}(W,X)e_i,U) + g(\bar{R}(W,X)e_i,\xi)g(U,\xi)\right) \\
&+ B(e_i)(g(X,e_i)g(W,U) - g(W,e_i)g(X,U) \\
&+ g(X,e_i)g(W,\xi)g(U,\xi) - g(W,e_i)g(X,\xi)g(U,\xi)) = 0.
\end{aligned} \tag{37}$$

Replacing $X = U = \xi$ and using (4), (18), equation (37) becomes

$$\begin{aligned}
&A(W)\left(\bar{r} - (n-1)(1-\beta+\beta^2) + \alpha(\beta-1)trace\phi\right) \\
&+ B(W)(2-3n) + \eta(W)\eta(\rho_2) = 0.
\end{aligned} \tag{38}$$

Thus we state the following Corollary:

Corollary 3.2. *In an n-dimensional extended generalized ϕ-recurrent LP-Sasakian manifold (M^n, g) admitting generalized symmetric metric connection, the 1-forms $A(W)$ and $B(W)$ satisfies the relation:*

$$\begin{aligned} &A(W)\left(\bar{r}-(n-1)(1-\beta+\beta^2)+\alpha(\beta-1)trace\phi\right)+\\ &B(W)(2-3n)+\eta(W)\eta(\rho_2)=0. \end{aligned} \tag{39}$$

4. ϕ-Projectively Semisymmetric LP-Sasakian Manifold admitting Generalized Symmetric Metric Connection

Let (M^n, g) be a ϕ-projectively semisymmetric LP-Sasakian manifold admitting generalized symmetric connection. Therefore $\bar{P}(X,Y).\phi = 0$ becomes

$$(\bar{P}(X,Y).\phi)Z = \bar{P}(X,Y)\phi Z - \phi\bar{P}(X,Y)Z = 0, \tag{40}$$

for any vector fields $X, Y, Z \in \Gamma(TM)$. Putting $Y = \xi$ in equation (40) and it becomes:

$$(\bar{P}(X,\xi).\phi)Z = \bar{P}(X,\xi)\phi Z - \phi\bar{P}(X,\xi)Z = 0, \tag{41}$$

The ϕ-projectively semisymmetric LP-Sasakian manifold admitting generalized symmetric metric connection [4] is

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{n-1}\left(\bar{S}(Y,Z)X - \bar{S}(X,Z)Y\right). \tag{42}$$

Putting $Y = \xi$ and $Z = \phi Z$, equation (42) becomes

$$\bar{P}(X,\xi)\phi Z = \bar{R}(X,\xi)\phi Z - \frac{1}{n-1}\left(\bar{S}(\xi,\phi Z)X - \bar{S}(X,\phi Z)\xi\right). \tag{43}$$

In view of the equations (4), (10), (12), (16), (18) and (43), we get:

$$\begin{aligned} \bar{P}(X,\xi)\phi Z &= \left(1-(1-\beta+\beta^2)\right)g(X,\phi Z)\xi\\ &+\left(\frac{1}{n-1}\left(-2\alpha^2+\beta-\beta^2+n\alpha^2\right)\right)\\ &g(X,\phi Z)\xi + (\alpha(\beta-1)trace\phi)\,g(X,\phi Z)\xi\\ &+(\alpha(1-\beta))\left(g(X,Z)+\eta(X)\eta(Z)\right)\xi+\\ &\left(\frac{1}{n-1}\left(-\alpha\beta+(n-2)\alpha(\beta-1)\right)\right)\left(g(X,Z)+\eta(X)\eta(Z)\right)\xi+\\ &\frac{1}{n-1}\left(\beta(\beta-2)trace\phi\right)\left(g(X,Z)+\eta(X)\eta(Z)\right)\xi. \end{aligned} \tag{44}$$

Similarly we proceed to find,

$$\phi\bar{P}(X,\xi)Z = \phi\left(\bar{R}(X,\xi)Z - \frac{1}{n-1}\left(\bar{S}(\xi,Z)X - \bar{S}(X,Z)\xi\right)\right). \tag{45}$$

In view of the equations (6), (12), (18) and (45), we have:

$$\phi\bar{P}(X,\xi)Z = -\frac{1}{n-1}\left(\alpha(\beta-1)\eta(z)\phi X trace\phi\right). \tag{46}$$

Using (44) and (46) in (41), we get:

$$A_1 g(X,\phi Z)\xi + A_2\left(g(X,Z)\xi + \eta(X)\eta(Z)\xi\right) + A_3\eta(Z)\phi X = 0, \tag{47}$$

where

$$A_1 = (1-(1-\beta+\beta^2)+ \frac{1}{n-1}\left(-2\alpha^2+\beta-\beta^2+n\alpha^2+\alpha(\beta-1)trace\phi\right), \tag{48}$$

$$A_2 = \alpha(1-\beta)+\frac{1}{n-1}\left(-\alpha\beta+(n-2)\alpha(\beta-1)+\beta(\beta-2)trace\phi\right), \tag{49}$$

$$A_3 = \frac{1}{n-1}\alpha(\beta-1)trace\phi. \tag{50}$$

Replacing $Z=\phi Z$ and using (3), (6), the equation (47) becomes

$$g(X,\phi Z)\xi = -\frac{A_1}{A_2}\left(g(X,Z)\xi+\eta(Z)\eta(X)\xi\right). \tag{51}$$

Substituting (51) in (47) we get,

$$\left(A_2-\frac{A_1^2}{A_2}\right)\left(g(X,Z)\xi+\eta(X)\eta(Z)\xi\right)+A_3\eta(Z)\phi X=0. \tag{52}$$

Again putting $Z=\phi Z$ and using (6), equation (52) becomes

$$\left(A_2-\frac{A_1^2}{A_2}\right)g(X,\phi Z)\xi=0, \tag{53}$$

which implies that $A_2=\frac{A_1^2}{A_2}$. Hence we can state the following result:

Theorem 4.1. *If an n-dimensional LP-Sasakian manifold admitting generalized symmetric metric connection is ϕ-projectively semisymmetric then $A_1^2-A_2^2=0$, where A_1 and A_2 are given in equations* (48) *and* (49).

5. ϕ-Weyl Semisymmetric LP-Sasakian Manifold admitting Generalized Symmetric Metric Connection

Let (M^n,g) be a ϕ-Weyl semisymmetric LP-Sasakian manifold admitting generalized symmetric connection. Therefore $(\bar{C}(X,Y).\phi)Z=0$ becomes

$$(\bar{C}(X,Y).\phi)Z=\bar{C}(X,Y)\phi Z-\phi\bar{C}(X,Y)Z=0, \tag{54}$$

for any vector fields $X,Y,Z\in\Gamma(TM)$. Putting $Y=\xi$, equation (54) becomes

$$(\bar{C}(X,\xi).\phi)Z=\bar{C}(X,\xi)\phi Z-\phi\bar{C}(X,\xi)Z=0, \tag{55}$$

The Weyl conformal curvature tensor admitting generalized symmetric metric connection [4] is

$$\begin{aligned}\bar{C}(X,Y)Z=&\\ \bar{R}(X,Y)Z&-\frac{1}{n-2}\left(\bar{S}(Y,Z)X-\bar{S}(X,Z)Y\right)\\ &+\frac{1}{n-2}\left(g(Y,Z)\bar{Q}X-g(X,Z)\bar{Q}Y\right)\\ &+\frac{r}{(n-1)(n-2)}\left(g(Y,Z)X-g(X,Z)Y\right).\end{aligned} \tag{56}$$

Putting $Y=\xi$ and $Z=\phi Z$, equation (56) becomes:

$$\bar{C}(X,\xi)\phi Z = \bar{R}(X,\xi)\phi Z - \frac{1}{n-2}\left(\bar{S}(\xi,\phi Z)X - \bar{S}(X,\phi Z)\xi\right) +$$

$$\frac{1}{n-2}\left(g(\xi,\phi Z)\bar{Q}X - g(X,\phi Z)\bar{Q}\xi\right) - \frac{r}{(n-1)(n-2)}\left(g(X,\phi Z)\xi\right). \tag{57}$$

In view of (3), (4), (6), (10), (12), (17) and (57), we have

$$\bar{C}(X,\xi)\phi Z = -\left(1-\beta+\beta^2\right)g(X,\phi Z)\xi + \alpha(1-\beta)g(\phi X,\phi Z)\xi$$
$$-\frac{1}{n-2}(((n-1)g(X,\phi Z) + (-\alpha\beta + (n-2)\alpha(\beta-1) + \beta(\beta-2)trace\phi)$$
$$g(\phi X,\phi Z) + (-2\alpha^2 + \beta - \beta^2 + n\alpha^2 + \alpha(\beta-1)trace\phi)g(X,\phi Z))\xi -$$
$$g(X,\phi Z)((n-1)\xi + (-2\alpha^2 + \beta - \beta^2 + n\alpha^2 + \alpha(\beta-1)trace\phi)\xi -$$
$$(-2\alpha^2 + n(\alpha^2 + \beta - \beta^2))\xi)) - \frac{r}{(n-1)(n-2)}g(X,\phi Z)\xi.$$

(58)

Similarly we proceed to find

$$\phi(\bar{C}(X,\xi)Z) = \phi\left(\bar{R}(X,\xi)Z - \frac{1}{n-2}\left(\bar{S}(\xi,Z)X - \bar{S}(X,Z)\xi + g(\xi,Z)\bar{Q}X - g(X,Z)\bar{Q}\xi\right)\right) +$$
$$\frac{r}{(n-1)(n-2)}\phi\left(g(\xi,Z)X - g(X,Z)\xi\right).$$

(59)

In view of (3), (6) (12), (16), (17), (18) and (59), we have:

$$\phi\bar{C}(X,\xi)Z = (1-\beta+\beta^2)\eta(Z)\phi X - \frac{1}{n-2}(((n-1)(1-\beta+\beta^2) + \alpha(\beta-1)trace\phi)$$
$$\eta(Z)\phi X + \eta(Z)((n-1)\phi X + (-\alpha\beta + (n-2)\alpha(\beta-1) + \beta(\beta-2)trace\phi)(X + \eta(X)\xi) +$$
$$(-2\alpha^2 + \beta - \beta^2 + n\alpha^2 + \alpha(\beta-1)trace\phi)\phi X)) + \frac{r}{(n-1)(n-2)}\eta(Z)\phi X.$$

(60)

Substituting (58), (60) in (55) and taking inner product with W, we get

$$-\left(1-\beta+\beta^2\right)g(X,\phi Z)g(W,\xi) + \alpha(1-\beta)g(\phi X,\phi Z)g(W,\xi)$$
$$-\frac{1}{n-2}(((n-1)g(X,\phi Z) + (-\alpha\beta + (n-2)\alpha(\beta-1) + \beta(\beta-2)trace\phi)g(\phi X,\phi Z) +$$
$$(-2\alpha^2 + \beta - \beta^2 + n\alpha^2 + \alpha(\beta-1)trace\phi)g(X,\phi Z))g(W,\xi) -$$
$$g(X,\phi Z)((n-1)g(W,\xi) + (-2\alpha^2 + \beta - \beta^2 + n\alpha^2 + \alpha(\beta-1)trace\phi)g(W,\xi) -$$
$$(-2\alpha^2 + n(\alpha^2 + \beta - \beta^2))g(W,\xi))) - \frac{r}{(n-1)(n-2)}g(X,\phi Z)g(W,\xi)$$
$$-(1-\beta+\beta^2)\eta(Z)g(\phi X,W) + \frac{1}{n-2}(((n-1)(1-\beta+\beta^2) + \alpha(\beta-1)trace\phi)$$
$$\eta(Z)g(\phi X,W) + (n-1)\eta(Z)g(\phi X,W) + (-\alpha\beta + (n-2)\alpha(\beta-1) + \beta(\beta-2)trace\phi)$$
$$(\eta(Z)g(X,W) + \eta(X)\eta(Z)g(W,\xi)) + (-2\alpha^2 + \beta - \beta^2 + n\alpha^2 + \alpha(\beta-1)trace\phi)$$
$$g(\phi X,W)\eta(Z)) - \frac{r}{(n-1)(n-2)}\eta(Z)\phi X = 0.$$

(61)

Putting $W = \xi$ and using (3), (4) in (61), we get

$$B_1 g(X, \phi Z) + B_2 \left(g(X, Z) + \eta(X)\eta(Z)\right) = 0, \tag{62}$$

where

$$\begin{aligned} B_1 &= (1 - \beta + \beta^2) + \\ &\frac{1}{n-2}\left(-2\alpha^2 + (n)(\alpha^2 + \beta - \beta^2)\right) + \frac{r}{(n-1)(n-2)}, \end{aligned} \tag{63}$$

$$B_2 = \frac{1}{n-2}\left(-\alpha\beta + (n-2)\alpha(\beta - 1) + \beta(\beta - 2)trace\phi\right) - \alpha(1 - \beta). \tag{64}$$

Putting $Z = \phi Z$ and using (3), (6) in (62), we get

$$g(X, Z) + \eta(X)\eta(Z) = -\frac{B_2}{B_1} g(X, \phi Z) \tag{65}$$

Substituting (65) in (62), we get

$$\left(B_1 - \frac{B_2^2}{B_1}\right) g(X, \phi Z) = 0, \tag{66}$$

which implies that $B_1 = \frac{B_2^2}{B_1}$. Hence we can state the following Theorem:

Theorem 5.1. *If an n-dimensional LP-Sasakian manifold admitting generalized symmetric metric connection is ϕ-Weyl semisymmetric, then $B_1^2 - B_2^2 = 0$, where B_1 and B_2 are given as in equations* (60) *and* (61).

6. Ricci Soliton on LP-Sasakian Manifold admitting Generalized Symmetric Metric Connection

Theorem 6.1. *A Ricci soliton on LP-Sasakian manifold admitting generalized symmetric metric connection is not η-Einstein manifold.*

Proof. We know that

$$(L_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi). \tag{67}$$

Using (5) in (67), we get

$$(L_\xi g)(X, Y) = 2g(\phi X, Y). \tag{68}$$

In the view of (2) and (68) we have

$$g(\phi X, Y) + \bar{S}(X, Y) + \lambda g(X, Y) = 0. \tag{69}$$

In view of (16) and (69), we have

$$\begin{aligned} S(X, Y) &= (\alpha\beta - (n-2)\alpha(\beta - 1) - \beta(\beta - 2)trace\Phi - 1)\, g(\phi X, Y) \\ &\left(2\alpha^2 - \beta(1 - \beta) - n\alpha^2 - \alpha(\beta - 1)trace\Phi + 2\lambda\right) g(X, Y) + \\ &\left(2\alpha^2 - n(\alpha^2 + \beta - \beta^2)\right) \eta(X)\eta(Y). \end{aligned} \tag{70}$$

□

Acknowledgement

The authors would like to thank the referees for their valuable comments which helped to improve the manuscript.

References

[1] O. Bahadir, *LP*-Sasakian manifolds with generalized symmetric metric connection, *Novi Sad J. Math.*, accepted for publication.

[2] E. Cartan, Sur une classes remarquable d'espaces de Riemann, *Bull. Soc. Math. France*, 54 (1926), 214-264.

[3] U. C. De, A. A. Shaikh and S. Biswas, On ϕ-recurrent Sasakian manifolds, *Novi Sad J. Math.*, 33 (2003), 43-48.

[4] U. C. De and Pradip Majhi, ϕ-Semisymmetric Generalized Sasakian-space forms, *Arab J. Math. Sci.*, 21 (2015) 170-178.

[5] R. S. D. Dubey, Generalized recurrent spaces, *Indian J. Pure Appl. Math.*, 10(12) (1979), 1508-1513.

[6] K. Matsumoto, On Lorentzian paracontact manifolds, *Bull. Yamagata Univ. Natur. Sci.*, 12(2) (1989), 151-156.

[7] I. Mihai and R. Rosca, On Lorentzian *P*-Sasakian manifolds, In Classical analysis (Kazimierz Dolny, 1991), *World Sci. Publ.*, River Edge, NJ, 1992, 155–169.

[8] A.A. Shaikh and S. K. Hui, On extended generalized ϕ-recurrent β-Kenmotsu manifolds, *Publ. Inst. Math. (Beograd) (N.S.)*, 89(103) (2011), 77-88.

[9] T. Takahashi, Sasakian ϕ-symmetric spaces, *Tohoku Math. J.*, 29 (1977), 91-113.

Department of Mathematics and Statistics, M. S. Ramaiah University of Applied Sciences, Bangalore-560 054, INDIA

Email address: somashekhara.mt.mp@msruas.ac.in

Department of Mathematics, Acharya institute of Technology, Bangalore-560 107, INDIA

Email address: sgirishbabu84@gmail.com

Department of Mathematics, Sri Jayachamarajendra College of Engineering, JSS Science and Technology University, Mysuru-570 006, INDIA

Email address: pskreddy@jssstuniv.in; pskreddy@sjce.ac.in

Department of Mathematics, Yuvaraja's College, University of Mysore, Mysuru-570 005, INDIA

Email address: drksshankara@gmail.com