Proceedings of the Jangjeon Mathematical Society
25 (2022), No. 3, pp. 257 – 267

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2022.25.3.257

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A NEW APPROACH TO INFINITE DECISION-MAKING PROCESS

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Abstract. Classically, a countably infinite set and its infinite parts are indistinguishable relative to size. This fact has been exploited by several authors (by way of illustration, we focus on [2]) to introduce paradoxes in decision-making over infinitely many alternatives. In this paper, we observe that all such paradoxes arise from fixing the manner in which infinite processes are to be conceptualised. A standard conceptualisation, reliant upon sequential reasoning, handles infinite, completed processes by means of finite approximations: the ensuing deletion of information concerning achieved completion leads to all the difficulties detected so far. The latter can be eliminated through the introduction of an alternative conceptualisation proposed by Yaroslav Sergeyev (see in particular [11]) in which numerical specification of completions are available. We show how an extension of bounded arithmetic recently axiomatised in [9] can formalise some salient aspects of Sergeyevs approach and provide a logical framework to dissolve infinite decision-making paradoxes.

2010 Mathematics Subject Classification. 90C40, 40A15, 90C34.

Keywords and phrases. infinite decisions; numerical infinities; infinitesimal probabilities.

1. Introduction

Within classical set theory, and indeed already within Zermelo set theory, the set of natural numbers ω is represented as the intersection of the inductive subsets included in some inductive set I whose existence is guaranteed by the Axiom of Infinity[1]. The inductive set I is such that it is possible to single out some distinct elements from it in set-theoretic notation, for instance $\phi, \{\phi\}, \{\phi, \{\phi\}\}$. We may, upon request, identify further elements, which are also elements of ω, since it is easy to verify that they must belong to any inductive subset of I. This scenario presents us with a situation in which we can exhibit, in a notation based on the constant symbol ϕ and the set-theoretic operation of successor[2], a potential infinity of items within ω. However, ω is defined as an actually infinite collection, i.e., a completed collection obtained from I as the result of taking the set-theoretic operation of intersection. In short, the fact that ω is regarded as a completed infinity fails to be captured by the set-theoretic notation that may be employed to name some of its distinctive elements.

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[1] A set X is inductive if, and only if, $\phi \in X$ and, whenever $x \in X$, then $x \cup \{x\} \in X$

[2] This is the operation that allows us to form $x \cup \{x\}$ given x, and whose concretely given iterations starting from ϕ can be expressed in set-theoretic notation using curly brackets, commas and the constant symbol ϕ.

What the latter notation in fact allows is only the construction of some initial segment of a complete collection. We may effectively look at set-theoretic *notation* as a relatively unwieldy numeral system that enables us to name provably distinct items in ω. The notation is unwieldy because, in an alphabet using only $\phi, \{, \}$, the numeral corresponding to 10 in base ten is a string of 1, 536 symbols, and the latter numeral in base ten corresponds to a string of 21,535 3 symbols, which may be deemed an absolutely unfeasible string[3]. Such radical lack of feasibility goes together with the idea that, since ω is regarded as an actually infinite collection, it should contain items that may be named by arbitrary iterations of successor starting from ϕ. If we think of a concrete naming process as breaking down on account of feasibility constraints, we are left with an actually infinite object from which we are able to single out, by way of naming, only some initial items (in the natural ordering of ω). It may now be noted that fixing, for the sake of argument, a feasibility threshold at 21,535, a change of numerical notation, e.g., a switch to base ten numerals, allows us to name more initial items from ω. The fact that a transition to an alternative numeral system increases feasible expressibility is of special interest. It suggests that a suitable, further expansion of expressive resources might concretely increase naming capabilities to the extent that they could single out computationally serviceable, terminal items from ω. The demand for such an increase has a motivation clearly rooted in classical mathematical practice: ω is the semantic support for the description of sequential processes. It is, in particular, the reference for sequential processes over all of the natural numbers. These processes, however, can only be concretely studied in connection with the selection of a numeral system. A typical situation involves the comparison of sequences. Given two sequences whose generic terms are called n and $16n$ respectively, it is possible to say that the first three terms of the second sequence are obtained from the much longer initial segment 1, 2, . . ., 48, 49, 50 of the first sequence by omitting most listed items (namely, 47). By contrast, it is possible to list the items 16, 32, . . ., 800, which can be used to match the first fifty items from the first sequence. However, if we pushed listing to a feasibility threshold, we might see that the matching breaks down, i.e., we cannot generate as many items from the second sequence as needed for a full matching, since the factor 16 magnifies them to the point that an attempt at setting up a finite, one-to-one correspondences must fail. Equivalently, we reach the feasibility threshold more rapidly with the second sequence than we do with the first. If we transposed this argument into a numerical environment in which we were able to name terminal items from ω, then we could identify a scenario in which one sequence reaches a terminal threshold more rapidly than another and we may be able to conclude that the two sequences are not related by a one-to-one correspondence over ω, i.e., that there are fewer multiples of 16 than there are even numbers. We would, in short, be able to discriminate (at least some definable) sequences, something that is uniformly disallowed in Zermelo set theory, where all sequences are, by definition, countable.

[3] The order of magnitude 21,000 is argued to identify an absolute feasibility limit, i.e. one independent of technological constraints, in Sasonov (1996).

Countability thus bypasses feasibility considerations from below and terminating evaluations from above.In certain mathematical contexts, these limitations are not seriously felt. Nonetheless, it is possible to identify contexts in which they stand out. The distinctive ways in which they emerge are impossibility results or paradoxes, which can be understood as appeals to countability theory where stronger numerical considerations are called for. A few elementary examples are worth exploring in order to make the last remarks compelling.

2. Some Paradoxes

A problem widely discussed since at least [7] is the construction of a probability model for the draw of a natural number from ω. The sought model, known as a De Finetti lottery, amounts to a uniform, discrete distribution over a countably infinite sample space. Classical attempts at constructing a De Finetti lottery evince typical difficulties attending the integration of probability modelling with classical countability theory. This integration problem is strikingly highlighted by a number of paradoxical results affecting elementary decision-theoretic scenarios built on De Finetti lotteries. A selection of illustrations will clarify what the issues are.

2.1. **Evaluating bets.** In [8], some speculation is made as to how certain bets over the natural numbers should be evaluated. If each draw of a number is assigned probability zero, then, given a finitely additive probability measure, betting for a payoff on a draw of a numbers between 1 and 103 is not more advantageous than betting on the draw of a number between 1 and 102. With zero probability assignments to individual draws, the question of evaluating larger events remains open. When set-theoretic inclusion orders them, it is tempting to say that the probability of a strict sub-event X of some event Y should be smaller than the probability of X. When, however, both events are sequences, probabilistic considerations suggest a form of distinguishability between whole and part that is undermined by countability theory. For instance, drawing a multiple of 6 is a strict sub-event of drawing a multiple of 2 but in both cases, one is drawing from a countably infinite sequence. If countability takes precedence, and the qualification countably infinite is supposed to suffice for a probability assignment, then a sequence is equiprobable with its infinite, strict sub-sequences. If, by contrast, set-theoretic inclusion takes precedence as an indication that some events are more restrictive than others, equiprobability must be rejected. This tension is left open as a puzzling problem in [8], but it is returned to in [2], which formulates a further problem that attends the comparison of sequences.

2.2. **Relabelling bets.** Unlike [8], [2] does not require a numerically defined probability assignment for single draws from a De Finetti lottery. A framework is deployed in which the equiprobability of two draws can hold even when their individual probabilities are left undefined4. Whilst individual draws from may not be assigned any probability value, it is plausible to assign probability values to certain events, e.g., the draw of an even number or the draw of a number whose remainder when divided by 4 is 1. The respective events should be assigned probability values 1/2 and 1/4, respectively. This way of assigning probabilities, even when restricted only to the remainder classes of ω modulo 2 and 4, leads to a problem. It is possible to make betting on a particular event more or less lucrative simply by renaming its elements. To see how, call ODD the event corresponding to a draw of an odd number and suppose that P(ODD) = 1/2. The same is true of P(EVEN), where EVEN is the draw of some even number. Consider, moreover, the events ONE, TWO, THREE and FOUR, each of which corresponds to the obvious remainder class of modulo 4. Consider a sequence $\{ai\}_{i\in\omega} = a1, a2, a3, ...$ and a sequence of indexed placeholders:

$$\square_1, \square_2, \square_3, ..., \square_n, ...$$

We may follow [2] and describe a way of filling the placeholders with items from $\{a_i\}_{i\in\omega}$. The first step consists in picking the placeholders with index in FOUR and filling them with the a_j, where $j \in EVEN$, in increasing order. The next step requires considering the remaining, empty placeholders with index in EVEN, which are filled with a_k, where $k \in THREE$. The final step of the procedure requires filling the empty placeholders with odd index with the a_h, where $h \in ONE$. According to the account offered in [2], this process fills every placeholder and produces a sequence that can be re-labelled as:

$$b_1, b_2, b_3, ..., b_n, ...$$

The probability of picking an item from this sequence with index in ODD should be 1/2. However, by the way the sequence has been constructed, this probability is 1/4 = P(ONE). It looks as if certain choices of odd indices may be made less convenient than others. An agent might be presented with a bet on the sequence $\{b_i\}_{i\in\omega}$ and suppose that the probability of winning, if an odd-indexed item was selected, is 1/2. After the construction of $\{b_i\}_{i\in\omega}$ is clarified to the agent, she may suddenly be persuaded that the original bet is less advantageous. It is clear that, in this context, every event under consideration is countably infinite, the union of disjoint events is countably infinite and any one of ONE, TWO, THREE and FOUR can be, on its own, used to fill the indexed placeholders initially given. This lead, in a less elaborate way, to a variant of the result just illustrated. The only scenario that has not been considered yet is one in which individual draws from are assigned probability values. If countability theory is made to interact with this choice, specific problems can be identified, as the next subsection illustrates.

2.3. **Packages of bets.** In [1], the setup of a De Finetti lottery is exploited to define a sequence of bets. The bet number n between agents A and B requires that A pay B \$ 2 if n is drawn and that A receive \$ 1/2 n from B if n is not drawn. It is suggested that A should consider each bet to be favourable, since it offers a presumably high chance of a finite gain and threatens a merely infinitesimal chance of a finite loss. It is being presupposed that the probability value assigned to a single draw is some infinitely small quantity. If this qualitative evaluation is correct, then the acceptance of each individual bet on the part of A should lead to the acceptance of an infinite sequence of bets. When this happens, however, A realises that the overall system of bets forces a payment of \$ 2 to B while offering a strictly smaller payoff. The special difficulty that arises here concerns the failure of amalgamation between individual bets and sequences of bets. Countability plays a slightly subtler role in this case, because it prevents a numerical specification of the probability value that should be assigned an individual draw (there is no multiplicative inverse to □0). Moreover, there is no way of describing any more than the finite head of the process whereby A sequentially accepts all bets. The evaluations of advantages and disadvantages that A can practically make only concern early stages of the decision process. There is never any consideration, because there is no numerical specification, of remote stages in the tail of this process, which must come to an end for the hypothetical betting setup to make any sense. If A engaged in an endless process of acceptance staying within the finite head of a sequence, at any one stage in this process the package of bets that is accepted is correctly recognised as convenient, but, by contrast, no decision is made concerning the totality of bets.

3. Computational Reconstruction

The problems illustrated in the previous section arise from the computational limitations accompanying the employment of countability theory. If a set like ω had a numerical measure insertable into computations, then, provided this numerical measure belonged to a field, it should be easy to construct a probability model for a De Finetti lottery. This has been done in a number of ways: most notably, measures called numerosities have been employed in [12,?] to this end. The use of numerosities has indeed proved useful to offer ways of generalising uniform, discrete distributions to infinite collections. Nevertheless, the problems highlighted in sections 2.1 and 2.2 require that certain subsets of ω be measured by submultiples of the measure assigned to and such submultiples must be guaranteed to be integers. The use of numerosities encounters difficulties with ensuring this result because their construction does not evenly split into the odd and even numbers (one of these two sets has a measure that equals the successor of the others measure) and it does not specify how one might compute the measures of certain cells in a definable partition of the natural numbers, for instance the partition into the remainder classes *modulo* 4 considered above. One reason for these difficulties is that numerosities do not replace countability theory with a different computational framework but rather integrate it. Even though they allow original developments, numerosities have been originally presented in the restricted contexts of countable sets, where they act as assignments to such sets only after they have been endowed by a labelling (a finite-to-one function) and are be compared by looking at their finiyr labelled segments. Because numerosities grow on top of countability theory and leave it intact, they are affected by its distinctive feature, namely the impossibility of drawing certain systematic distinctions between infinite

[4] This is because [2] works with betting quotients and relative betting quotients: the relative betting quotient depends on two events, and it may be defined even in the absence of a betting quotient for each event: for details see Bartha (2004, pp. 307310).

sequential processes. An alternative way of thinking, which has been introduced by Yaroslav Sergeyev and is remarkable because of its extensive impact on applications (see [11] for a recent survey), is afforded by a focus on the replacement of countability theory with an extended computational framework, in which the terminal features of many sequential processes are assigned computable numerical specifications. One leading idea underlying the approach is that the collection of all natural numbers should be regarded as completely counted in a numeral system that fixes its unit of measure ①(read: gross-one). Clearly, $① > n$ holds if n can be expressed in, say, base ten. Given that a completed count starting at 1 ends at ①, a count that omits only 1 and starts at 2 has ① - 1 stages, i.e. the length of a count that starts at 1 but ends at ① - 1. Note that not all intermediate steps can be effectively counted but the expansion of the store of computable terms afforded by the introduction of ① makes it possible to recognise the terminal stages of sequential processes and thus take into account more stages of such processes. Sergeyev also allows for e.g. alternating counts that skip blocks of numbers. If, for example, one starts from 1 skipping blocks of one number each, one is counting the odd numbers, whose computable measure is ①/2. Because 2 divides ①, it follows that ① is not odd and the count of the even numbers is an alternating count starting from 2 but stopping at ①, unlike the count of odd numbers (which stops at ① - 1), and thus has the same computable measure ①/2. Sergeyev works with a generalisation of this approach that can give rise to computable measures for many sequences. Whilst simple and easy to handle, Sergeyevs perspective affords a uniform technique to build probability models that is unaffected by the difficulties raised in section 2. From the present point of view, it is of special interest that this treatment can be rendered within the purely arithmetical theory axiomatized in [9]. What we are going to do in the remainder of this paper is offer a very concise presentation of the latter theory and show that the results concerning ① intuitively described in this section are theorems within it. With a few simple theorems in hand, we can offer a satisfactory resolution of the difficulties raised by the scenarios from section 2.

4. Arithmetic with distinguished orders of infinity

While numerosities are developed against the backdrop of Zermelo-Fraenkel set theory with additional assumptions, the standpoint presented in [9] relies on an arithmetical theory Γ, which is the union of a tower of theories Γ_i , with $i \in N$. In particular, the consistency of the theory Γ from [9] can be reduced to that of the theory of Bounded Arithmetic $I \vartriangle_0 \cup \{\Omega_1\}$, known as S_2 in [4][5]. A direct proof of the consistency of Γ can be obtained by showing that each Γ_i has a natural model. Since $\Gamma = \cup_i \Gamma_i$ and the Γ_i form a chain relative to inclusion, any finite subset of Γ is satisfiable and, by the compactness theorem of first-order logic, Γ itself does. Now, for some fixed i, Γ_i has in its language the constant symbols $\infty_{-i}, \infty_{-i+1}, ..., \infty_{-1}$, ①, which are intended to single out distinguished, infinitely large elements. The axioms of Γ_i include atomic formulae of the form $m < \infty_k$, with m a natural number and $k \in \{-i, -i+1, ..., -1\}$, as well as i equalities of the form $(\infty_{-k-1}!)^{\infty-k-1} = \infty_{-k}$, where $k \in \{-i, -i+1, ..., -1\}$.

The equality $(\infty_{-1}!)^{\infty-1} = ①$ completes the list of axioms that govern the behaviour of distinguished, infinitely large elements. The remaining axioms (for a complete list, see [9], pp.56-57) characterise the relation $\leq$ as a linear ordering with maximum ① and provide the recursive definitions of familiar operations like addition, multiplication, bounded sums and products, truncating them at ①. A clear instance of how truncations are handled is offered by the successor axioms, which, if s is taken as the successor function symbol, may be formulated as follows:

$$*\neg(s(x) = 0);$$
$$*s(x) = s(y) \to (x = ① \vee y = ① \vee x = y);$$
$$*s(①) = ①.$$

It may be noted that the successor is injective unless the maximum of a model is attained, which is a fixed point for the function s. The theory Γ shares with each Γ_i the axioms that provide truncated arithmetical operations but in addition includes a full sequence of distinguished, infinitely large elements below ①. An interesting metatheoretical feature of both Γ and each Γ_i is that the truncation of a model at a distinguished, infinitely large element is, under a natural interpretation of the non-logical vocabulary, still a model of the same theory. This makes it possible to project a model of Γ (Γ_i) into a lower order of infinity and opens the way to codifying mathematical objects other than nonnegative integers into a model. For instance, call N a model of $\Gamma 3$. Then N has a lower order projection in Nat whose domain is $[0, \infty_{-2})$. It is then of interest to codify the integers Int, including all integers between $-\infty_{-2}$ and ∞_{-2} and, indeed, the rational numbers within this interval. For many practical purposes, it suffices to consider those with a fixed denominator ∞^2_{-2} and a numerator whose absolute value lies in $[0, \infty^2_{-2}]$. The simultaneous coding of integers and rational numbers can be carried out within $[0, 2.\infty^2_{-2}]$ which is not only a subset of N, but also of $[0, \infty_{-1}]$, on account of the axioms. Thus, within N one can work with the model **Nat** and, in addition, encode integers and rationals relative to **Nat**. This is not all, since definable subsets of bounded sets like **Nat** $= [0, \infty_{-2})$ also possess a natural codification. Let X be one such set: the axioms of Γ_3 imply that the characteristic function C_X of X is a term in the language. The code of X can then be obtained as the unique summation over $i \in [0, \infty_{-2})$ of the terms $2^i C_X(i)$. This number is bounded above by $2^0 + 2^1 + ... + 2^{\infty_{-2}}$, i.e. by:

$$2.2^{\infty-2} - 1 \leq 2.2^{\infty-2} \leq (\infty_{-2}!/2)^{\infty-2}.2^{\infty-2} = (\infty_{-2}!)^{\infty-2} = \infty_{-1}.$$

Thus, the initial segment $[0, \infty_{-1}]$ can code the integers and rationales relative to **Nat**, as well as its definable subsets. Because the axioms adopted in [9] include pairing functions, it is possible to code ordered pairs and,

[5] Bounded Arithmetic was originally introduced in [10], as pointed out in [5]. It is the axiomatic theory obtained from Peano Arithmetic by restricting the induction schema to $\triangle_0$ formulae. Its supplementation by the axiom $\forall x \forall y \exists z (z = x^{\llcorner y \lrcorner})$, known in the literature as 1 was extensively studied by Sam Buss.

thus, by what we have just observed, definable sets of ordered pairs (membership into a set is also codifiable). In particular, definable functions can be coded. With this coding apparatus in place, it is possible, by repeated use of induction, to build a uniform, discrete distribution over **Nat** and assign probabilities to its definable events without facing any of the problems described in section 2. For this to be possible, a way of counting the elements of a definable subset X of **Nat** must be used, since a probability measure for any such subset can then be naturally defined as the (code of the) rational number with the fixed denominator ∞_{-2} and with numerator the count over X. Because C_X is a term, it is possible to define the summation of its values over **Nat** as a completed count of X. With this version of cardinality-as-count in place, we are in a position fruitfully to revisit the decision problems examined earlier.

5. Decision Problems Revisited

The De Finetti lottery over Nat assigns the rational number $1/\infty_{-2}$ to any individual draw. For any definable subset X of **Nat**, we follow [9], p.66 in defining the probability measure:

$$P(X) = \frac{Card(X)}{\infty_{-2}}$$

Where $Card(X)$ denotes a count of X, defined as at the end of the previous section. To illustrate how P works, consider the subsets of Nat defined by the formulae:

$$x \leq \infty_{-2} \wedge \exists y(x = 4.y + k),$$

with $k = 0, 1, 2, 3$. Following [2], we may call the sets just defined **FOUR, ONE, TWO** and **THREE** respectively. $Fixk = 1$, which singles out **ONE.** The characteristic function C_{ONE} is a term in the language of Γ_3, so it is possible to write the following formula, which is proved by an induction on x:

$$(X < \infty_{-2} \wedge C_{ONE}(x)) = 1 \rightarrow 4.\sum_{i=0}^{x} C_{ONE}(i) = x + 3$$

By the same clue, we obtain:
$*(x < \infty_{-2} \wedge C_{TWO}(x)) = 1 \rightarrow 4.\sum_{i=0}^{x} C_{TWO}(i) = x + 2$
$*(x < \infty_{-2} \wedge C_{THREE}(x)) = 1 \rightarrow 4.\sum_{i=0}^{x} C_{THREE}(i) = x + 1$
$*(x < \infty_{-2} \wedge C_{FOUR}(x)) = 1 \rightarrow 4.\sum_{i=0}^{x} C_{TOUR}(i) = x + 4.$
Now note that the axioms of Γ_3 imply $= (\infty_{-3}!)^{\infty-3}$. By the definition of factorial and the fact that ∞_{-3} is an integer, ∞_{-2} is a multiple of 4. It is therefore meaningful to adopt the notation $\infty_{-2}/4$ to denote an integer. It also follows ∞_{-2} that $\infty_{-2} - 4$ is a multiple of 4 and, in particular, that it is the largest multiple of 4 in **Nat**. By this clue we can exploit the last theorem from the above list to conclude, setting $x = \infty_{-2} - 4$, that:

$$4.\sum_{i=0}^{x=\infty_{-2}-4} C_{FOUR}(i) = \infty_{-2} \text{ or, equivalently } 4.\sum_{i=0}^{x=\infty_{-2}-4} C_{FOUR}(i) = \infty_{-2}/4$$

The last equality does not change if we take the summation over the whole of **Nat**, since none of the remaining three numbers is in FOUR. We may therefore conclude that:

$$Card(FOUR) = \frac{\infty_{-2}}{4}$$

and, by the definition of $P, P(FOUR) = 1/4$.
A further induction shows that the following equality holds:

$$x < \infty_{-2} \wedge C_{ONE}(x) = 1 \rightarrow \sum_{i=0}^{x} C_{FOUR}(i) = \sum_{i=0}^{x} C_(ONE(i)$$

Since the antecedent of the last conditional is satisfied when $x = \infty_{-2} - 1$, it is possible to take the summations in the consequent over **Nat**, thus obtaining:

$$Card(ONE) = Card(FOUR) = \infty_{-2}/4.$$

Suitable variations of the argument show that the remainder classes *modulo* 4 in Nat have the some number of elements, which is one fourth of the number of elements in Nat. The events **ONE, TWO, THREE** and **FOUR** are thus equiprobable in a De Finetti lottery that, unlike the one in [2], assigns the same infinitesimal probability to every individual draw. Moreover, **EVEN** and **ODD** are proved, much in the same way as above, to have the same measure $\infty_{-2}/2$ and, as a consequence, the same probability measure 1/2. The procedure giving rise to a relabelling paradox in section 2 can now be described within a new context, in which numerical specification of terminating features of sequential processes make it possible to register the completion of operations repeated an infinite number of times. The fundamental point is that, in a sequence of place-holders indexed over **Nat**, there are $\infty_{-2}/4$ place-holders with an index in **FOUR**. Thus, to insert the even-indexed elements from a complete sequence, i.e. a sequence indexed over the whole of Nat, into a sequence of place-holders indexed over **FOUR** is to fail to insert half of the given elements. More precisely, only:

$$a_2 a_2, a_4, , a_6, .., a_{\frac{\infty_{-2}}{2}-4}, a_{\frac{\infty_{-2}}{2}-2}, a_{\frac{\infty_{-2}}{2}}$$

are inserted. Moreover, since the a_i with index in **ONE** are inserted into the odd-indexed place-holders, they can occupy only a half of them. Thus, at the end of the procedure described by Bartha, only $3(\infty_{-2}/4)$ place-holders are filled. Since the empty ones are all odd-indexed, the probability of drawing a_i with an index in **ONE** from the relabelled sequence remains the same if the empty place-holders are kept and made to correspond to vacuous draws. Otherwise, the sequence $\{b_i\}_{i \in Nat}$ contains $3(\infty_{-2}/4)$ elements and the probability of extracting an odd-indexed one after relabelling is 3/8, but this value depends on the fact that now the odd-indexed items are a strict superset of **ONE** .

The computations carried out so far extend to other definable subsets of **Nat**, for instance the subset of even numbers and the strictly smaller subset of multiples of six. Exploiting the numerical specifications now assignable to completed counts of these subsets, it is possible to conclude that it is more likely to draw an even number than a multiple of six from the De Finetti lottery over Nat. This addresses one of the few similar problems raised in [8] and discussed in Section 2.1. Countability theory systematically precluded answers to them because it stood in the way of articulating the numerical specifications available in our current arithmetical model. It is for the same reason that we can resolve the problem from Section 2.3 by fairly elementary means.

Because we have been able to describe a uniform, discrete distribution over Nat, we are in a position to determine the expected gain and the expected loss from the De Finetti lottery from Section 2.3. The expected loss is clearly \$2. The expected gain can be computed because 1/2 is a codifiable rational number (since it is a/∞_{-2}^2 for $a = \infty_{-2}/2$) and all definable series of rational numbers have a (coded) sum in the theory we have been working with (see [9], p.67-68 for details). In the present case, the sum in question is:

$$\$(2 - (\frac{1}{2})^{\infty_{-2}-1})$$

Since the latter payoff is strictly smaller than the expected gain, the system of bets indexed over **Nat** is globally disadvantageous. In the present context, however, we can also look at whether every individual bet is advantageous without restrictive ourselves to bets within the finite head of the sequence Nat. Thus, for instance, the bet number $\infty_{-2} - 1$ produces the expected loss $2/\infty_{-2}$, which is infinitely small. At the same time, the corresponding, expected gain amounts to:

$$(\frac{1}{2})^{\frac{\infty_{-2}}{2}-1}\,(1 - \frac{1}{\infty_{-2}}) < (\frac{1}{2})^{\frac{\infty_{-2}}{2}-1}$$

We wish to compare: $(\frac{2}{\infty_{-2}}) = (\frac{1}{2})^{-1}.(\frac{1}{\infty_{-2}})$ and $(\frac{1}{2})^{\frac{\infty_{-2}}{2}-1} = (\frac{1}{2})^{-1}(\frac{1}{2})^{\frac{\infty_{-2}}{2}-1}$

Because $1/a > (1/b)^n$ iff $a < b^n$, it suffices to verify that $(2)^{(\frac{\infty_{-2}}{2})} > \infty_{-2}$ in order to establish that the expected loss of bet number $\frac{\infty_{-2}}{2} - 1$ is larger than the expected gain to be hoped for from the same bet. We may proceed by contradiction and assume:

$(2)^{\infty_{-2}/2} - 1 \leq \infty_{-2}$, which implies $2^{\infty_{-2}} \leq \infty_{-2}^2$.

Induction within the bound ∞_{-1} from the base $x = 5$ shows that $x^2 < 2^x$, which yields a contradiction at ∞_{-2}. We are now in a position to conclude that, for a distinguished, individual bet, the expected loss is larger than the expected gain. This continues to hold for every subsequent bet within Nat so the whole package of bets contains at least $\infty_{-2}/2 + 1$ that are disadvantageous. This reflects the global situation. Without an arithmetical model like Nat, only a narrow inspection of individual bets is allowed to determine the local behaviour of the infinite package. When the narrow perspective is forced upon the better, a suspicion of the conflict between local and global odds may be aroused, but this suspicion is dispelled as soon as the question of determining whether or not a conflict arises can be posed in concrete, numerical terms.

6. Summary

The goal of this paper was to clarify that several, and arguably all, difficulties associated with infinite decision-making processes depend on two factors: reliance on accountability theory and the associated restriction of numerical specifications (which affects the symbolic arithmetic of numerosities as well). The arithmetical theory in [9] provides a way to move beyond these restrictions and develop a numerically more discerning theory of cardinalities. Its resources are certainly sufficient to resolve the problems that may be encountered within the decision-theoretic literature. It is of interest to see whether the same framework is adequate for the development of a sufficiently rich fragment of probability theory, which includes a workable account of random variables and basic statistics. In particular, the question as to whether the amount of probability theory developed along novel lines in [6]can actually be rigorously developed within the axiomatic theory used in the present paper.

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