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INNER INVARIANCE OF THE KERNEL OF A BANACH SPACE PSEUDOREPRESENTATION OF A GROUP

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Abstract.

We prove that the kernel of an arbitrary pseudorepresentation in a dual Banach space with a sufficiently small defect is invariant with respect to all inner automorphisms of the group.

§ 1. Introduction

For the definitions, notation, and generalities concerning pseudocharacters, quasicharacters, pseudorepresentations, and quasirepresentations, see [1]–[3].

As is well known, the kernel of a pseudorepresentation need not be a subgroup. The same holds even for one-dimensional pseudorepresentations. For example, the kernel of the Rademacher pseudocharacter f [4] on $SL(2,\mathbb{Z})$ contains the finite-order generators of the group, and hence the one-dimensional pseudorepresentation of $SL(2,\mathbb{Z})$ given by $g \mapsto \exp(i\alpha f(g))$, $g \in G$, has the kernel which is not a subgroup of $SL(2,\mathbb{Z})$.

In this note we prove that, if a pseudorepresentation of a group is bounded and the defect of the pseudorepresentation is sufficiently small, then the kernel of the pseudorepresentation is invariant with respect to the inner automorphisms of the group.

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§ 2. Preliminaries

We need a well-known lemma. For the convenience of the reader, we present it with a proof.

Lemma. *Let T be a bounded linear operator on a dual Banach space E such that*

$$\|T^n - 1_E\| \leq q < 1, \qquad n \in \mathbb{N},$$

with respect to the operator norm. Then $T = 1_E$.

Proof. Let us apply an invariant mean I on $\mathbb{N}$ extended to the bounded linear operators on E as in [5]. Let $S = I(T^n)$. Then $\|S - I_E\| \leq q < 1$, and hence S is invertible. By the invariance of I, we have

$$TS = TI(T^n) = I(T^{n+1}) = S,$$

which implies that $T = 1_E$.

§ 3. Main result

Theorem. *Let G be a group and let π be a bounded pseudorepresentation of G in a dual Banach space E with defect ε. Let*

$$\|\pi(g)\| \leq C, \qquad g \in G,$$

with respect to the operator norm, and let

$$(C+1)\varepsilon < 1.$$

Then the kernel

$$K_\pi = \{g \in G, \pi(g) = 1_E\}$$

is invariant with respect to the inner automorphisms of G.

Proof. Let

$$a \in G, \qquad a \in K_\pi.$$

Then, since

$$\pi(g)\pi(a)\pi(g^{-1}) = 1_E$$

by the definition of a pseudorepresentation ($\pi(g)\pi(g^{-1}) = 1_E$ for all $g \in G$, because the restriction of π to the cyclic subgroup generated by a is an ordinary representation), we have

$$\|\pi(gag^{-1}) - 1_E\| \leq \|\pi(gag^{-1}) - \pi(g)\pi(ag^{-1})\|$$

$$+\|\pi(g)\pi(ag^{-1}) - \pi(g)\pi(a)\pi(g^{-1})\|$$

$$\leq \varepsilon + \|\pi(g)\|\|\pi(ag^{-1}) - \pi(a)\pi(g^{-1})\| \leq (C+1)\varepsilon,$$

and the same inequality holds for a^n, $n \in \mathbb{N}$:

$$\|\pi((gag^{-1}))^n - 1_E\| = \|\pi(ga^n g^{-1}) - 1_E\| \leq (C+1)\varepsilon,$$

and thus it follows from the assumption of the theorem that, for $(C+1)\varepsilon = q$, we arrive at the conditions of the lemma. Therefore,

$$\pi(gag^{-1}) = 1_E \quad \text{for every} \quad g \in G,$$

as was to be proved.

§ 4. Comments

Pseudorepresentations whose kernels are normal subgroups have special properties (see, e.g. [6–10]).

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