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A SPECIAL CASE OF AN UNBOUNDED PSEUDOREPRESENTATION OF AN AMENABLE GROUP

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Abstract. In the theory of pseudorepresentations, there is an old unsolved problem: whether or not every unbounded pseudorepresentation of an amenable group with sufficiently small defect admits a close ordinary representation of the group (and the correction is small together with the defect). Till now, only partial results of this kind are known in this area. In this note, we improve one of the known partial results in this direction.

§ 1. Introduction

For the definitions, notation, and generalities concerning pseudorepresentations and quasirepresentations, see [1]–[3].

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§ 2. Preliminaries

Let us cite one of the main results of [5] which is used below. Let the symbol $\mathcal{L}(E)$ stand for the Banach algebra of bounded linear operators on a dual Banach space E and let $\tau(g) = I_h(\pi(gh)\pi(h^{-1}))$, $h \in G$, for some invariant mean I on G extended to bounded linear operators (as is defined, e.g., in [5]).

Theorem 1. *Let π be a quasirepresentation (not necessarily bounded) of an amenable group G in a dual Banach space E with defect ε. Let there be a subalgebra $A \subset \mathcal{L}(E)$ with zero multiplication which contains all operators $\pi(h)\pi(k) - \pi(hk)$, $h, k \in G$. Then the mapping $\Sigma(g) = \Sigma_\pi(g) = \tau(g) - \pi(g)\tau(e) + \pi(g)$, $g \in G$ is an ordinary representation of G.*

For the proof, see [5].

It should be added that, obviously, $\|\Sigma(g) - \pi(g)\|$ is uniformly small if and only if $\|\pi(g)(1 - \tau(e))\|$ is uniformly small. In fact, the last condition is strong. In many cases, this condition means that $\tau(e) = 1_E$.

§ 3. Main theorem

Theorem 2. *Let E be a dual Banach space, let F be a closed subspace of E, and let π be a pseudorepresentation (not necessarily bounded) of an amenable group G on E with defect ε with an invariant subspace F such that the restriction of π to F is an ordinary representation and the mapping defined by π on the quotient space E/F is also an ordinary representation. Let I be an invariant mean on G and let $\tau(g) = I_h(\pi(gh)\pi(h^{-1}))$, $h \in G$, for some standard extension of an invariant mean I on G to bounded linear operators (cf. [5]). Then the mapping $\Sigma(g) = \Sigma_\pi(g) = \tau(g) - \pi(g)\tau(e) + \pi(g)$, $g \in G$ is an ordinary representation of G. It is close to π if and only if $\|\pi(g)(1 - \tau(e))\|$ is uniformly small.*

Proof. It follows immediately from the conditions of the theorem that the defect operators $\pi(gh) - \pi(g)\pi(h)$, $g, h \in G$, act from E to F and their kernel contains F. Hence, every product of the form $(\pi(gh) - \pi(g)\pi(h))(\pi(uv) - \pi(u)\pi(v))$, where $g, h, u, v \in G$, is the zero operator on E. Thus, Theorem 1 can be applied, which implies the assertion of Theorem 2 immediately.

There is a special case of Theorem 2 in which both the restriction of π to F and the representation defined by π on the quotient space E/F are identity representations. Even this special case is not covered by [4], since it is assumed in [4] that E is a direct sum of two subspaces.

§ 4. Concluding remarks

It seems that Theorem 2 has consequences concerning finite step-like structure instead of the two-step structure discussed in Theorem 2. This more general case will be treated elsewhere

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