Advanced Studies in Contemporary Mathematics
32 (2022), No. 3, pp. 403 - 422

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2022.32.3.403

Further Hypergeometric Functions in Four Variables and Their Associated Properties

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June 7, 2022

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Abstract

In this article, we define nine new hypergeometric functions in four variables. For all the quadruple functions we have defined, we obtain their integral representations, operational formulas and generating functions.

Mathematics Subject Classification 2010. Primary 33C20; Secondary 33C65.

keywords: Beta and Gamma functions, quadruple hypergeometric functions, integrals of Euler type, Laplace integral, inverse pairs of symbolic operators, generating functions.

1 Introduction

Hypergeometric functions have been attract the attention of many researchers due to their importance and applications in diverse areas of mathematical, physical, engineering and statistical sciences [1, 2, 3, 4, 5, 6, 8, 9, 19, 20]. Multiple hypergeometric functions occur in various fields of pure and applied mathematics such as approximation theory, partition theory, representation theory, group theory, mirror symmetry, difference equations and mathematical physics etc. They possess important properties such as recurrence and explicit relations, summation formulae, symmetric and convolution identities, algebraic properties etc. Hasanov et al. [18] studied some of the properties of the Horn type second-order double hypergeometric function H_2^* involving integral representations, differential equations, and generating functions. Choi et al. [15] introduced certain integral representations for Srivastava's triple hypergeometric functions H_A, H_B and H_C. Bin-Saad and Younis [10, 12, 13] established new hypergeometric functions of four variables together with their basic properties. Younis and Nisar [28] introduced new integral representations of Euler-type for Exton's hypergeometric functions of four variables D_1, D_2, D_3, D_4 and D_5.

We begin with the Gauss hypergeometric function ${}_2F_1$ which is defined as [26]

$$ {}_2F_1\left(a, b; c; x\right) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!}, (|x| < 1). $$

Here, $(a)_m$ is the Pochhammer symbol defined by

$$ (a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1)...(a+m-1), $$

for $m \geq 1$, $(a)_0 = 1$, where the notation Γ is used for the gamma function.

Appell hypergeometric functions of two variables F_3 and F_4 are respectively defined by (see [26])

$$ F_3\left(a, b, c, d; e; x, y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_m (b)_n (c)_m (d)_n}{(e)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} $$

and

$$ F_4\left(a, b; c, d; x, y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n}}{(c)_m (d)_n} \frac{x^m}{m!} \frac{y^n}{n!}. $$

Horn's functions H_3 and H_4 of two variables [11] are given as

$$ H_3(a, b; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n} (b)_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} $$

and

$$ H_4(a, b; c, d; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n} (b)_n}{(c)_m (d)_n} \frac{x^m}{m!} \frac{y^n}{n!}. $$

The Exton's triple functions X_7, X_{11}, X_{12} and X_{18} (see [12]) are defined as follows:

$$ X_7\left(a, b, c; d, e; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p} (b)_n (c)_p}{(d)_m (e)_{n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, $$

$$ X_{11}\left(a, b; c, d; x, y, \right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n} (b)_{n+2p}}{(c)_{m+p} (d)_n} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, $$

$$ X_{12}\left(a, b; c, d, e; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n} (b)_{n+2p}}{(c)_m (d)_n (e)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} $$

and

$$ X_{18}\left(a_1, a_2, a_3, a_4; c; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n} (a_2)_n (a_3)_p (a_4)_p}{(c)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}. \quad (1) $$

The Lauricella functions of three variables $F_B^{(3)}$ and F_R [14] are defined as:

$$F_B^{(3)}(a_1,a_2,a_3,b_1,b_2,b_3;c;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m(a_2)_n(a_3)_p(b_1)_m(b_2)_n(b_3)_p}{(c)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}$$

and

$$F_R(a_1,a_2,a_1,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+p}(a_2)_n(b_1)_{m+p}(b_2)_n}{(c_1)_m(c_2)_{n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}.$$

The Sharma and Parihar hypergeometric function of four variables $F_{73}^{(4)}$ is defined by [18]

$$F_{73}^{(4)}(a_1,a_1,a_2,a_3,a_4,a_4,a_5,a_6;c_1,c_2,c_1,c_1;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_p(a_3)_q(a_4)_{m+n}(a_5)_p(a_6)_q}{(c_1)_{m+p+q}(c_2)_n} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}.$$

More recently, Bin-Saad and Younis [11] defined thirty functions of four variables and they denoted them by $X_1^{(4)}, X_2^{(4)}, ..., X_{30}^{(4)}$. Here, we give one of them

$$X_5^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_2,a_2;c_1,c_1,c_1,c_2;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{q+n+2p}}{(c_1)_{m+n+p}(c_2)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}.$$

Now, we give new quadruple hypergeometric functions as follows:

$$X_{91}^{(4)}(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;c_1,c_1,c_2,c_1;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_p(a_4)_q(a_5)_q}{(c_1)_{m+n+q}(c_2)_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \quad (2)$$

$$X_{92}^{(4)}(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;c_1,c_2,c_1,c_1;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_p(a_4)_q(a_5)_q}{(c_1)_{m+p+q}(c_2)_n} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \quad (3)$$

$$X_{93}^{(4)}(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;c_2,c_1,c_1,c_1;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_p(a_4)_q(a_5)_q}{(c_1)_{n+p+q}(c_2)_m} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \quad (4)$$

$$X_{94}^{(4)}(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;c,c,c,c;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_p(a_4)_q(a_5)_q}{(c)_{m+n+p+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \quad (5)$$

$$X_{95}^{(4)}\left(a_1,a_1,a_3,a_5,a_1,a_2,a_4,a_6;c_2,c_1,c_1,c_1;x,y,z,u\right)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_n(a_3)_p(a_4)_p(a_5)_q(a_6)_q}{(c_1)_{n+p+q}(c_2)_m}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},\quad (6)$$

$$X_{96}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c_2,c_1,c_2;x,y,z,u\right)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{2p+n}(a_3)_q(a_4)_q}{(c_1)_{m+p}(c_2)_{n+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},\quad (7)$$

$$X_{97}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c_1,c_2,c_1;x,y,z,u\right)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{2p+n}(a_3)_q(a_4)_q}{(c_1)_{m+n+q}(c_2)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},\quad (8)$$

$$X_{98}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c_2,c_1,c_1;x,y,z,u\right)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{2p+n}(a_3)_q(a_4)_q}{(c_1)_{m+p+q}(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},\quad (9)$$

$$X_{99}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c,c,c,c;x,y,z,u\right)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{2p+n}(a_3)_q(a_4)_q}{(c)_{m+n+p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}.\quad (10)$$

In the present paper, we introduce nine new hypergeometric functions in four variables and investigate certain properties of them. In Section 2, we derive integral representations of Laplace-type for the quadruple functions $X_{91}^{(4)}, X_{92}^{(4)}, ..., X_{99}^{(4)}$. In Section 3, we establish integral representations of Euler-type for these new quadruple functions. In Section 4, we give some operational formulas involving the functions $X_{91}^{(4)}, X_{92}^{(4)}, ..., X_{99}^{(4)}$. Finally, in Section 5, we present certain generating functions for our quadruple functions.

2 Integral representations of Laplace-type

In this section, we investigate certain integral representations of Laplace-type that are related to the new hypergeometric functions in four variables defined above.

Theorem 2.1. *For the hypergeometric functions in four variables $X_{91}^{(4)}$, $X_{92}^{(4)}, ..., X_{99}^{(4)}$, we obtain*

$$
\begin{aligned}
&X_{91}^{(4)}\left(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;c_1,c_1,c_2,c_1;x,y,z,u\right)\\
&=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_4)\Gamma(a_5)}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v+w)}s^{a_1-1}t^{a_2-1}\\
&\times v^{a_4-1}w^{a_5-1}{}_0F_1\left(-;c_1;s^2x+sty+vwu\right){}_1F_1\left(a_3;c_2;tz\right)dsdtdvdw,\\
&\qquad\left(Re(a_1)>0,Re(a_2)>0,Re(a_4)>0,Re(a_5)>0\right),\qquad(11)
\end{aligned}
$$

$$
\begin{aligned}
&X_{92}^{(4)}\left(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;c_1,c_2,c_1,c_1;x,y,z,u\right)\\
&\quad=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_5)}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)}s^{a_1-1}t^{a_2-1}v^{a_5-1}\\
&\qquad\times\Phi_3^{(3)}\left(a_3,a_4;c_1;tz,vu,s^2x\right){}_0F_1\left(-;c_2;sty\right)\ dsdtdv,\\
&\qquad\left(Re(a_1)>0,Re(a_2)>0,Re(a_5)>0\right),\qquad(12)
\end{aligned}
$$

$$
\begin{aligned}
&X_{93}^{(4)}\left(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;c_2,c_1,c_1,c_1;x,y,z,u\right)\\
&\quad=\frac{1}{\Gamma(a_1)\Gamma(a_3)\Gamma(a_5)}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)}s^{a_1-1}t^{a_3-1}v^{a_5-1}\\
&\qquad\times\Phi_2\left(a_2,a_4;c_1;sy+tz,vu\right){}_0F_1\left(-;c_2;s^2x\right)\ dsdtdv,\\
&\qquad\left(Re(a_1)>0,Re(a_3)>0,Re(a_5)>0\right),\qquad(13)
\end{aligned}
$$

$$
\begin{aligned}
&X_{94}^{(4)}\left(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;c,c,c,c;x,y,z,u\right)\\
&\quad=\frac{1}{\Gamma(a_1)\Gamma(a_3)\Gamma(a_5)}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)}s^{a_1-1}t^{a_3-1}v^{a_5-1}\\
&\qquad\times\Phi_3^{(3)}\left(a_2,a_4;c;sy+tz,vu,s^2x\right)\ dsdtdv,\\
&\qquad\left(Re(a_1)>0,Re(a_3)>0,Re(a_5)>0\right),\qquad(14)
\end{aligned}
$$

$$
\begin{aligned}
&X_{95}^{(4)}\left(a_1,a_1,a_3,a_5,a_1,a_2,a_4,a_6;c_2,c_1,c_1,c_1;x,y,z,u\right)\\
&\quad=\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}t^{a_2-1}{}_0F_1\left(-;c_2;s^2x\right)\\
&\qquad\times\Xi_2^{(3)}\left(a_3,a_5,a_4,a_6;c_1;z,u,sty\right)dsdt,\\
&\qquad\left(Re(a_1)>0,Re(a_2)>0\right),\qquad(15)
\end{aligned}
$$

$$
\begin{aligned}
&X_{96}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c_2,c_1,c_2;x,y,z,u\right)\\
&=\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}t^{a_2-1}{}_0F_1\left(-;c_1;s^2x+t^2z\right)\\
&\qquad\times\Xi_2\left(a_3,a_4;c_2;u,sty\right)\ dsdt,\\
&\qquad\left(Re(a_1)>0,Re(a_2)>0\right),\qquad(16)
\end{aligned}
$$

$$X_{97}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c_1,c_2,c_1;x,y,z,u\right)$$
$$=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v+w)}s^{a_1-1}$$
$$\times t^{a_2-1}v^{a_3-1}w^{a_4-1}{}_0F_1\left(-;c_1;s^2x+sty+vwu\right)$$
$$\times {}_0F_1\left(-;c_2;t^2z\right)dsdtdvdw,$$
$$\left(Re(a_1)>0,Re(a_2)>0,Re(a_3)>0,Re(a_4)>0\right), \quad (17)$$

$$X_{98}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c_2,c_1,c_1;x,y,z,u\right)$$
$$=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v+w)}s^{a_1-1}$$
$$\times t^{a_2-1}v^{a_3-1}w^{a_4-1}{}_0F_1\left(-;c_1;s^2x+t^2z+vwu\right)$$
$$\times {}_0F_1\left(-;c_2;sty\right)dsdtdvdw,$$
$$\left(Re(a_1)>0,Re(a_2)>0,Re(a_3)>0,Re(a_4)>0\right), \quad (18)$$

$$X_{99}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c,c,c,c;x,y,z,u\right)$$
$$=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v+w)}s^{a_1-1}$$
$$\times t^{a_2-1}v^{a_3-1}w^{a_4-1}{}_0F_1\left(-;c;s^2x+sty+t^2z+vwu\right)dsdtdvdw,$$
$$\left(Re(a_1)>0,Re(a_2)>0,Re(a_3)>0,Re(a_4)>0\right), \quad (19)$$

where ${}_0F_1, {}_1F_1, \phi_2, , \Xi_2, \phi_3^{(3)}$ and $\Xi_2^{(3)}$ are the confluent hypergeometric functions defined by

$${}_0F_1\left(-;c;x\right)=\sum_{m=0}^{\infty}\frac{1}{(c)_m}\frac{x^m}{m!},$$

$${}_1F_1\left(a;c;x\right)=\sum_{m=0}^{\infty}\frac{(a)_m}{(c)_m}\frac{x^m}{m!},$$

$$\Phi_2\left(a,b;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},$$

$$\Phi_3^{(3)}\left(a,b;c;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_m(b)_n}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$\Xi_2\left(a,b;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}$$

and

$$\Xi_2^{(3)}(a,b,c,d;e;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_m(b)_n(c)_m(d)_n}{(e)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}.$$

Proof. To prove each of the integral representations from (11) to (19), it is enough to consider the expressions of confluent hypergeometric functions given above and then using the following known definition of gamma function (see [26, 27]):

$$\Gamma(x)=\int_0^{\infty}e^{-t}\,t^{x-1}\,dt,\;(Re(x)>0).$$

□

3 Integrals of Euler-type

This section gives several integral representations of Euler-type for the quadruple functions $X_{91}^{(4)}, X_{92}^{(4)}, ..., X_{99}^{(4)}$.

Theorem 3.1. *The following integral representations holds true*

$$\begin{aligned}X_{91}^{(4)}&(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;c_1,c_1,c_2,c_1;x,y,z,u)\\&=\frac{\Gamma(c_2)}{\Gamma(a_2)\Gamma(c_2-a_2)(S-R)^{c_2-a_3-1}}\\&\times\int_R^S(\alpha-R)^{a_2-1}(S-\alpha)^{c_2-a_2-1}\left[(S-R)-(\alpha-R)z\right]^{-a_3}\\&\times X_{18}\left(a_1,1+a_2-c_2,a_4,a_5;c_1;x,\frac{(R-\alpha)y}{(S-\alpha)},u\right)d\alpha,\\&\left(Re(a_2)>0,Re(c_2-a_2)>0,R<S\right),\end{aligned}\tag{20}$$

$$\begin{aligned}X_{91}^{(4)}&(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;c_1,c_1,c_2,c_1;x,y,z,u)\\&=\frac{\Gamma(c_2)}{\Gamma(a_3)\Gamma(c_2-a_3)(S-R)^{c_2-a_2-1}}\\&\times\int_R^S(\alpha-R)^{a_3-1}(S-\alpha)^{c_2-a_3-1}\left[(S-R)-(\alpha-R)z\right]^{-a_2}\\&\times X_{18}\left(a_1,a_2,a_4,a_5;c_1;x,\frac{(S-R)y}{[(S-R)-(\alpha-R)z]},u\right)d\alpha,\\&\left(Re(a_3)>0,Re(c_2-a_3)>0,R<S\right).\end{aligned}\tag{21}$$

Proof. First of all, we recall the following integral representations of the beta function (see [16, 27]):

$$B(a,b)=(S-R)^{1-a-b}\int_R^S(\alpha-R)^{a-1}(S-\alpha)^{b-1}d\alpha,$$
$$\left(Re(a)>0,Re(b)>0,R<S\right),\tag{22}$$

$$
\begin{aligned}
B(a,b) &= \frac{(S-T)^a(R-T)^b}{(S-R)^{a+b-1}} \int_R^S \frac{(\alpha-R)^{a-1}(S-\alpha)^{b-1}}{(\alpha-T)^{a+b}}\, d\alpha \\
&= (M+1)^a \int_0^1 \frac{\alpha^{a-1}(1-\alpha)^{b-1}}{(1+M\alpha)}\, d\alpha, \\
&(T<R<S, M>-1, Re(a)>0, Re(b)>0), \\
B(a,b) &= \int_0^1 \alpha^{a-1}(1-\alpha)^{b-1} d\alpha \\
= 2\int_0^{\frac{\pi}{2}} &(\sin\alpha)^{2a-1}(\cos\alpha)^{2b-1} d\alpha = \int_0^\infty \frac{\alpha^{a-1}}{(1+\alpha)^{a+b}} d\alpha, \\
&(Re(a)>0, Re(b)>0), \\
B(a,b) &= 2^{1-a-b}\int_{-1}^{1}(1+\alpha)^{a-1}(1-\alpha)^{b-1}\, d\alpha \\
&= 2M^a \int_0^\infty \frac{\cosh\alpha\,(\sinh\alpha)^{2a-1}}{\left(1+M\sinh^2\alpha\right)^{a+b}} d\alpha, \\
&(Re(a)>0, Re(b)>0, M>0).
\end{aligned}
$$

To prove the result in equality (20) asserted in Theorem 3.1, let Δ denote the right-hand side of the equality (20). Then from the definition of Exton's function X_{18} in (1), we obtain

$$
\begin{aligned}
\Delta = &\frac{\Gamma(c_2)}{\Gamma(a_2)\Gamma(c_2-a_2)} \\
&\times \sum_{m,n,p,q=0}^{\infty} \frac{(-1)^n (a_1)_{2m+n}(1+a_2-c_2)_n (a_3)_p (a_4)_q (a_5)_q}{(c_1)_{m+n+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!} \\
&\times \frac{1}{(S-R)^{c_2+p-1}} \int_R^S (\alpha-R)^{a_2+n+p-1}(S-\alpha)^{c_2-a_2-n-1}\, d\alpha.
\end{aligned} \tag{23}
$$

Using the integral representation of the beta function (22), we have

$$
\begin{aligned}
\Delta = &\frac{\Gamma(c_2)}{\Gamma(a_2)\Gamma(c_2-a_2)} \\
&\times \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_3)_p(a_4)_q(a_5)_q B(a_2+n+p, c_2-a_2-n)}{(c_1)_{m+n+q}(c_2-a_2)_{-n}} \\
&\times \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}.
\end{aligned} \tag{24}
$$

From the following well known identity (see, e.g., [16, 27]):

$$
B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},
$$

we are led to the desired result. Then, using a similar way, we can easily get (21). □

The following theorems can be proved like Theorem 3.1. So the details are omitted.

Theorem 3.2. *The following integral representations holds true*

$$\begin{aligned}
&X_{92}^{(4)}\left(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;c_1,c_2,c_1,c_1;x,y,z,u\right)\\
&=\frac{2M^{a_1}\Gamma(c_2)}{\Gamma(a_1)\Gamma(c_2-a_1)}\int_0^{\frac{\pi}{2}}\frac{\left(sin^2\alpha\right)^{a_1-\frac{1}{2}}\left(cos^2\alpha\right)^{c_2-a_1-\frac{1}{2}}}{(cos^2\alpha+Msin^2\alpha)^{c_2-a_2}}\\
&\times\left[\left(cos^2\alpha+Msin^2\alpha\right)-Mysin^2\alpha\right]^{-a_2}\\
&\times F_B^{(3)}\left(\frac{1+a_1-c_2}{2},a_2,\frac{a_1-c_2}{2}+1,a_3,a_5,a_4;c_1;\right.\\
&\left.4M^2xtan^4\alpha,\frac{\left(cos^2\alpha+Msin^2\alpha\right)z}{[(cos^2\alpha+Msin^2\alpha)-Mysin^2\alpha]},u\right)d\alpha,\\
&\left(Re(a_1)>0,Re(c_2-a_1)>0,M>-1\right),
\end{aligned}\tag{25}$$

$$\begin{aligned}
&X_{92}^{(4)}\left(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;c_1,c_2,c_1,c_1;x,y,z,u\right)\\
&=\frac{2M^{c_2-a_2}\Gamma(c_2)}{\Gamma(a_2)\Gamma(c_2-a_2)}\int_0^{\frac{\pi}{2}}\frac{\left(sin^2\alpha\right)^{c_2-a_2-\frac{1}{2}}\left(cos^2\alpha\right)^{a_2-\frac{1}{2}}}{(cos^2\alpha+Msin^2\alpha)^{c_2-a_1}}\\
&\times\left[\left(cos^2\alpha+Msin^2\alpha\right)-ycos^2\alpha\right]^{-a_1}\\
&\times F_B^{(3)}\left(\frac{a_1}{2},1+a_2-c_2,a_4,\frac{a_1+1}{2},a_3,a_5;c_1;\right.\\
&\left.\frac{4\left(cos^2\alpha+Msin^2\alpha\right)^2x}{[(cos^2\alpha+Msin^2\alpha)-Mysin^2\alpha]^2},\frac{-z}{M}cot^2\alpha,u\right)d\alpha,\\
&\left(Re(a_2)>0,Re(c_2-a_2)>0,M>-1\right).
\end{aligned}\tag{26}$$

Theorem 3.3. *The following integral representations holds true*

$$\begin{aligned}
&X_{93}^{(4)}\left(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;c_2,c_1,c_1,c_1;x,y,z,u\right)\\
&=\frac{\Gamma(c_1)}{\Gamma(a_3)\Gamma(a_4)\Gamma(c_1-a_3-a_4)}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}+\alpha\right)^{a_3-1}\left(\frac{1}{2}-\alpha\right)^{c_1-a_3-1}\\
&\times\left(\frac{1}{2}+\beta\right)^{a_4-1}\left(\frac{1}{2}-\beta\right)^{c_1-a_3-a_4-1}\left[1-\left(\frac{1}{2}+\alpha\right)z\right]^{-a_2}\\
&\times\left[1-\left(\frac{1}{2}-\alpha\right)\left(\frac{1}{2}+\beta\right)u\right]^{-a_5}\\
&\times H_4\left(a_1,a_2;c_2,c_1-a_3-a_4;x,\frac{\left(\frac{1}{2}-\alpha\right)\left(\frac{1}{2}-\beta\right)y}{\left[1-\left(\frac{1}{2}+\alpha\right)z\right]}\right)d\alpha d\beta,\\
&\left(Re(a_3)>0,Re(a_4)>0,Re(c_1-a_3-a_4)>0\right),
\end{aligned}\tag{27}$$

$$
\begin{aligned}
&X_{93}^{(4)}\left(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;c_2,c_1,c_1,c_1;x,y,z,u\right)\\
&\quad=\frac{\Gamma(c_1)\Gamma(a_1+a_3)}{\Gamma(a_1)\Gamma(a_3)\Gamma(a_4)\Gamma(c_1-a_4)}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}+\alpha\right)^{a_4-1}\\
&\times\left(\frac{1}{2}-\alpha\right)^{c_1-a_4-1}\left(\frac{1}{2}+\beta\right)^{a_3-1}\left(\frac{1}{2}-\beta\right)^{a_1-1}\left[1-\left(\frac{1}{2}+\alpha\right)u\right]^{-a_5}\\
&\times H_4\left(a_1+a_3,a_2;c_2,c_1-a_4;\left(\frac{1}{2}-\beta\right)^2x,\right.\\
&\left.\left(\frac{1}{2}-\alpha\right)\left(\frac{1}{2}-\beta\right)y+\left(\frac{1}{2}-\alpha\right)\left(\frac{1}{2}+\beta\right)z\right)d\alpha d\beta,\\
&\qquad(Re(a_i)>0,(i=1,,3,4),Re(c_1-a_4)>0)\,. \qquad (28)
\end{aligned}
$$

Theorem 3.4. *The following integral representations holds true*

$$
\begin{aligned}
&X_{94}^{(4)}\left(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;c,c,c,c;x,y,z,u\right)\\
&\quad=\frac{4M_1^{a_1}M_2^{a_1+a_2}\Gamma(a_1+a_2+a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}\int_0^\infty\int_0^\infty\\
&\times\frac{\cosh\alpha\left(\sinh^2\alpha\right)^{a_1-\frac{1}{2}}}{\left(1+M_1\sinh^2\alpha\right)^{a_1+a_2}}\frac{\cosh\beta\left(\sinh^2\beta\right)^{a_1+a_2-\frac{1}{2}}}{\left(1+M_2\sinh^2\beta\right)^{a_1+a_2+a_3}}\\
&\times F_3\left(\frac{a_1+a_2+a_3}{2},a_4,\frac{a_1+a_2+a_3+1}{2},a_5;c;\right.\\
&\frac{4M_1^2M_2^2x\sinh^4\alpha\sinh^4\beta}{\left(1+M_1\sinh^2\alpha\right)^2\left(1+M_2sinh^2\beta\right)^2}+\\
&\frac{4M_1M_2^2y\sinh^2\alpha\sinh^4\beta}{\left(1+M_1\sinh^2\alpha\right)^2\left(1+M_2\sinh^2\beta\right)^2}+\\
&\left.\frac{4M_2z\sinh^2\beta}{\left(1+M_1\sinh^2\alpha\right)\left(1+M_2\sinh^2\beta\right)^2},u\right)d\alpha d\beta,\\
&\qquad(Re(a_i)>0,(i=1,2,3),M_1>0,M_2>0)\,, \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
&X_{94}^{(4)}\left(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;c,c,c,c;x,y,z,u\right)\\
&\quad=\frac{2M^{c-a_2}\Gamma(c)}{\Gamma(a_2)\Gamma(c-a_2)}\int_0^\infty\frac{cosh\alpha\left(sinh^2\alpha\right)^{c-a_2-\frac{1}{2}}}{(1+Msinh^2\alpha)^{c-a_1-a_3}}\\
&\times\left[\left(1+Msinh^2\alpha\right)-y\right]^{-a_1}\left[\left(1+Msinh^2\alpha\right)-z\right]^{-a_3}\\
&\times F_3\left(\frac{a_1}{2},a_4,\frac{a_1+1}{2},a_5;c-a_2;\frac{4Mx\left(1+Msinh^2\alpha\right)sinh^2\alpha}{[(1+Msinh^2\alpha)-y]^2},\right.\\
&\left.\frac{Musinh^2\alpha}{(1+Msinh^2\alpha)}\right)d\alpha,\\
&\qquad(Re(a_2)>0,Re(c-a_2)>0,M>0)\,. \qquad (30)
\end{aligned}
$$

Theorem 3.5. *The following integral representations holds true*

$$\begin{aligned}
&X_{95}^{(4)}\left(a_1,a_1,a_3,a_5,a_1,a_2,a_4,a_6;c_2,c_1,c_1,c_1;x,y,z,u\right)\\
&\quad=\frac{\Gamma(a_1+a_2)}{2^{a_1+a_2-1}\Gamma(a_1)\Gamma(a_2)}\int_{-1}^{1}(1+\alpha)^{a_1-1}(1-\alpha)^{a_2-1}\\
&\quad\times F_{73}^{(4)}\Big(\frac{a_1+a_2}{2},\frac{a_1+a_2}{2},a_3,a_5,\frac{a_1+a_2+1}{2},\frac{a_1+a_2+1}{2},\\
&\qquad a_4,a_6;c_1,c_2,c_1,c_1;(1+\alpha)^2x,(1+\alpha)(1-\alpha)y,z,u\Big)\,d\alpha,\\
&\qquad\qquad\left(Re(a_1)>0,Re(a_2)>0\right),
\end{aligned}\tag{31}$$

$$\begin{aligned}
&X_{95}^{(4)}\left(a_1,a_1,a_3,a_5,a_1,a_2,a_4,a_6;c_2,c_1,c_1,c_1;x,y,z,u\right)\\
&\quad=\frac{\Gamma(a_1+a_3)\Gamma(c_1)}{2^{a_1+a_3+c_1-4}\Gamma(a_1)\Gamma(a_3)\Gamma(a_5)\Gamma(c_1-a_5)}\\
&\quad\times\int_{-1}^{1}\int_{-1}^{1}\frac{\left[(1+\alpha)^2\right]^{a_1-\frac{1}{2}}\left[(1-\alpha)^2\right]^{a_3-\frac{1}{2}}}{(1+\alpha^2)^{a_1+a_3}}\\
&\times\frac{\left[(1+\beta)^2\right]^{a_5-\frac{1}{2}}\left[(1-\beta)^2\right]^{c_1-a_5-\frac{1}{2}}}{(1+\beta^2)^{c_1-a_6}}\left[\left(1+\beta^2\right)-\frac{1}{2}(1+\beta)^2u\right]^{-a_6}\\
&\times X_7\Bigg(a_1+a_3,a_2,a_4;c_2,c_1-a_5;\frac{(1+\alpha)^4x}{4(1+\alpha^2)^2},\frac{(1+\alpha)^2(1-\beta)^2y}{4(1+\alpha^2)(1+\beta^2)},\\
&\qquad\frac{(1-\alpha)^2(1-\beta)^2z}{4(1+\alpha^2)(1+\beta^2)}\Bigg)\,d\alpha d\beta,\\
&\qquad\left(Re(a_i)>0,(i=1,3,5),Re(c_1-a_5)>0\right).
\end{aligned}\tag{32}$$

Theorem 3.6. *The following integral representations holds true*

$$\begin{aligned}
&X_{96}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c_2,c_1,c_2;x,y,z,u\right)\\
&\quad=\frac{\Gamma(a_1+a_2)\Gamma(c_2)}{2^{a_1+a_2+c_2-2}\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(c_2-a_3)}\int_{-1}^{1}\int_{-1}^{1}(1+\alpha)^{a_3-1}\\
&\quad\times(1-\alpha)^{c_2-a_3-1}(1+\beta)^{a_2-1}(1-\beta)^{a_1-1}\left[1-\frac{1}{2}(1+\alpha)u\right]^{-a_4}\\
&\quad\times F_4\Bigg(\frac{a_1+a_2}{2},\frac{a_1+a_2+1}{2};c_1,c_2-a_3;(1-\beta)^2x+(1+\beta)^2z,\\
&\qquad\frac{1}{2}(1-\alpha)(1+\beta)(1-\beta)y\Bigg)\,d\alpha d\beta,\\
&\qquad\left(Re(a_i)>0,(i=1,2,3),Re(c_2-a_3)>0\right),
\end{aligned}\tag{33}$$

$$
\begin{aligned}
&X_{96}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c_2,c_1,c_2;x,y,z,u\right)\\
&=\frac{4M_1^{a}M_2^{c_2-a'}\Gamma(c_1)\Gamma(c_2)}{\Gamma(a)\Gamma(a')\Gamma(c_1-a)\Gamma(c_2-a')}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\left(\sin^2\alpha\right)^{a-\frac{1}{2}}\left(\cos^2\alpha\right)^{c_1-a-\frac{1}{2}}}{\left(\cos^2\alpha+M_1\sin^2\alpha\right)^{c_1}}\\
&\times\frac{\left(\sin^2\beta\right)^{c_2-a'-\frac{1}{2}}\left(\cos^2\beta\right)^{a'-\frac{1}{2}}}{\left(\cos^2\beta+M_2\sin^2\beta\right)^{c_2}}X_{12}\left(a_1,a_2;a,c_2-a',c_1-a;\right.\\
&\left.\frac{M_1x\sin^2\alpha}{\left(\cos^2\alpha+M_1\sin^2\alpha\right)},\frac{M_2y\sin^2\beta}{\left(\cos^2\beta+M_2\sin^2\beta\right)},\frac{z\cos^2\alpha}{\left(\cos^2\alpha+M_1\sin^2\alpha\right)}\right)\\
&\times{}_2F_1\left(a_3,a_4;a';\frac{u\cos^2\beta}{\left(\cos^2\beta+M_2\sin^2\beta\right)}\right)d\alpha d\beta,\\
&\left(Re(a)>0,Re(a')>0,Re(c_1-a)>0,Re(c_2-a')>0,\right.\\
&\left.M_1>0,M_2>0\right). \qquad (34)
\end{aligned}
$$

Theorem 3.7. *The following integral representations holds true*

$$
\begin{aligned}
&X_{97}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c_1,c_2,c_1;x,y,z,u\right)\\
&=\frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_2)\Gamma(a_3)\Gamma(c_1-a_3)\Gamma(c_2-a_2)}\int_0^{\infty}\int_0^{\infty}(1+\alpha)^{1+a_2-2c_2}\\
&\times\beta^{c_1-a_3-1}(1+\beta)^{a_4-c_1}\left[\alpha(1+\alpha)+z\right]^{c_2-a_2-1}\left[(1+\beta)-u\right]^{-a_4}\\
&\times H_3\left(a_1,1+a_2-c_2;c_1-a_3;\frac{\beta x}{(1+\beta)},\frac{-(1+\alpha)\beta y}{(1+\beta)\left[\alpha(1+\alpha)+z\right]}\right)d\alpha d\beta,\\
&\left(Re(a_2)>0,Re(a_3)>0,Re(c_1-a_3)>0,Re(c_2-a_2)>0\right), \qquad (35)
\end{aligned}
$$

$$
\begin{aligned}
&X_{97}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c_1,c_2,c_1;x,y,z,u\right)\\
&=\frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\int_0^1\alpha^{a_1-1}(1-\alpha)^{a_2-1}F_R\left(\frac{a_1+a_2}{2},a_3,\frac{a_1+a_2}{2},\right.\\
&\left.\frac{a_1+a_2+1}{2},a_4,\frac{a_1+a_2+1}{2};c_2,c_1,c_1;4(1-\alpha)^2z,u,4\alpha^2x+4\alpha(1-\alpha)y\right)d\alpha,\\
&\left(Re(a_1)>0,Re(a_2)>0\right). \qquad (36)
\end{aligned}
$$

Theorem 3.8. *The following integral representations holds true*

$$\begin{aligned}
&X_{98}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c_2,c_1,c_1;x,y,z,u\right)\\
&=\frac{(1+M_1)^{a_1}(1+M_2)^{a_2}\Gamma(a_1+a_3)\Gamma(a_2+a_4)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}\int_0^1\int_0^1\frac{\alpha^{a_1-1}\left(1-\alpha\right)^{a_3-1}}{(1+M_1\alpha)^{a_1+a_3}}\\
&\times\frac{\beta^{a_2-1}\left(1-\beta\right)^{a_4-1}}{(1+M_2\beta)^{a_2+a_4}}X_5^{(4)}\left(a_1+a_3,a_1+a_3,a_2+a_4,a_1+a_3,a_1+a_3,\right.\\
&\qquad a_2+a_4,a_2+a_4,a_2+a_4;c_1,c_1,c_1,c_2;\frac{(1+M_1)^2\alpha^2x}{(1+M_1\alpha)^2},\\
&\left.\frac{(1-\alpha)\left(1-\beta\right)u}{(1+M_1\alpha)\left(1+M_2\beta\right)},\frac{(1+M_2)^2\beta^2z}{(1+M_2\beta)^2},\frac{(1+M_1)(1+M_2)\alpha\beta y}{(1+M_1\alpha)\left(1+M_2\beta\right)}\right)d\alpha d\beta,\\
&\qquad\qquad(Re(a_i)>0,(i=1,2,3,4),M_1>-1,M_2>-1)\,, \qquad (37)
\end{aligned}$$

$$\begin{aligned}
&X_{98}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c_2,c_1,c_1;x,y,z,u\right)\\
&\qquad=\frac{2\Gamma(c_1)}{\Gamma(a_3)\Gamma(c_1-a_3)}\int_0^{\frac{\pi}{2}}(sin^2\alpha)^{c_1-a_3-\frac12}(cos^2\alpha)^{a_3-\frac12}\\
&\quad\times(1-ucos^2\alpha)^{-a_4}X_{11}\left(a_1,a_2;c_1-a_3,c_2;xsin^2\alpha,y,zsin^2\alpha\right)d\alpha,\\
&\qquad\qquad(Re(a_3)>0,Re(c_1-a_3)>0)\,. \qquad (38)
\end{aligned}$$

Theorem 3.9. *The following integral representations holds true*

$$\begin{aligned}
&X_{99}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c,c,c,c;x,y,z,u\right)\\
&\qquad=\frac{\Gamma(c)(S-T)^{c-a}(R-T)^a}{\Gamma(a)\Gamma(c-a)(S-R)^{c-1}}\int_R^S\frac{(\alpha-R)^{c-a-1}(S-\alpha)^{a-1}}{(\alpha-T)^c}\\
&\qquad\times X_{96}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c-a,a,c-a,a;\right.\\
&\qquad\frac{(S-T)(\alpha-R)x}{(S-R)(\alpha-T)},\frac{(R-T)(S-\alpha)y}{(S-R)(\alpha-T)},\frac{(S-T)(\alpha-R)z}{(S-R)(\alpha-T)},\\
&\qquad\left.\frac{(R-T)(S-\alpha)u}{(S-R)(\alpha-T)}\right)d\alpha,\\
&\qquad\qquad(Re(a)>0,Re(c-a)>0,T<R<S)\,, \qquad (39)
\end{aligned}$$

$$X_{99}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c,c,c,c;x,y,z,u\right)$$
$$= \frac{4(1+M_1)^a(1+M_2)^{a_1}\Gamma(c)\Gamma(a_1+a_2)}{\Gamma(a)\Gamma(a_1)\Gamma(a_2)\Gamma(c-a)} \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}$$
$$\times \frac{\left(sin^2\alpha\right)^{a-\frac{1}{2}}\left(cos^2\alpha\right)^{c-a-\frac{1}{2}}}{(1+M_1sin^2\alpha)^c}\frac{\left(sin^2\beta\right)^{a_1-\frac{1}{2}}\left(cos^2\beta\right)^{a_2-\frac{1}{2}}}{(1+M_2sin^2\beta)^{a_1+a_2}}$$
$$\times {}_2F_1\left(\frac{a_1+a_2}{2},\frac{a_1+a_2+1}{2};a;\right.$$
$$\frac{4(1+M_1)(1+M_2)^2xsin^2\alpha sin^4\beta+(1+M_1)(1+M_2)}{(1+M_1sin^2\alpha)}$$
$$\left.\frac{ysin^2\alpha sin^22\beta+4(1+M_1)zsin^2\alpha cos^4\beta}{(1+M_2sin^2\beta)^2}\right)$$
$$\times {}_2F_1\left(a_3,a_4;c-a;\frac{ucos^2\alpha}{(1+M_1sin^2\alpha)}\right)d\alpha d\beta,$$
$$(Re(a)>0, Re(a_1)>0, Re(a_2)>0,$$
$$Re(c-a)>0, M_1>-1, M_2>-1). \quad (40)$$

4 Operational Formulas

In this section, we present some operational representations with help of the following inverse pairs of symbolic operators defined by Choi and Hasanov (see [14]):

$$H_{t_1,\ldots,t_i}(a,b) := \frac{\Gamma(b)\Gamma(a+\delta_1+\cdots+\delta_i)}{\Gamma(a)\Gamma(b+\delta_1+\cdots+\delta_i)}$$
$$= \sum_{k_1,\ldots,k_i=0}^{\infty}\frac{(b-a)_{k_1+\cdots+k_i}(-\delta_1)_{k_1}\ldots(-\delta_i)_{k_i}}{(b)_{k_1+\cdots+k_i}k_1!\ldots k_i!} \quad (41)$$

and

$$\bar{H}_{t_1,\ldots,t_i}(a,b) := \frac{\Gamma(a)\Gamma(b+\delta_1+\cdots+\delta_i)}{\Gamma(b)\Gamma(a+\delta_1+\cdots+\delta_i)}$$
$$= \sum_{k_1,\ldots,k_i=0}^{\infty}\frac{(b-a)_{k_1+\cdots+k_i}(-\delta_1)_{k_1}\ldots(-\delta_i)_{k_i}}{(1-a-\delta_1-\cdots-\delta_i)_{k_1+\cdots+k_i}k_1!\ldots k_i!}, \quad (42)$$

where $\delta_j := t_j\frac{\partial}{\partial t_j}, j=1,\ldots,i; i\in\mathbb{N} := \{1,2,3,\ldots\}$.

Theorem 4.1. *Each of the following formulas holds true:*

$$X_{91}^{(4)}\left(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;c_1,c_1,c_2,c_1;x,y,z,u\right)$$
$$= H_z(a_3,c_2)(1-z)^{-a_2}X_{18}\left(a_1,a_2,a_4,a_5;c_1;x,\frac{y}{1-z},u\right), \quad (43)$$

$$(1-z)^{-a_2} X_{18}\left(a_1, a_2, a_4, a_5; c_1; x, \frac{y}{1-z}, u\right)$$
$$= \bar{H}_z(a_3, c_2) X_{91}^{(4)}\left(a_1, a_1, a_2, a_4, a_1, a_2, a_3, a_5; c_1, c_1, c_2, c_1; x, y, z, u\right), \tag{44}$$

Theorem 4.2. *Each of the following formulas holds true:*

$$X_{92}^{(4)}\left(a_1, a_1, a_2, a_4, a_1, a_2, a_3, a_5; c_1, c_2, c_1, c_1; x, y, z, u\right)$$
$$= H_u(a_5, a) X_{92}^{(4)}\left(a_1, a_1, a_2, a_4, a_1, a_2, a_3, a; c_1, c_2, c_1, c_1; x, y, z, u\right), \tag{45}$$

$$X_{92}^{(4)}\left(a_1, a_1, a_2, a_4, a_1, a_2, a_3, a_5; c_1, c_2, c_1, c_1; x, y, z, u\right)$$
$$= \bar{H}_u(a, a_5) X_{92}^{(4)}\left(a_1, a_1, a_2, a_4, a_1, a_2, a_3, a; c_1, c_2, c_1, c_1; x, y, z, u\right), \tag{46}$$

Theorem 4.3. *Each of the following formulas holds true:*

$$X_{93}^{(4)}\left(a_1, a_1, a_2, a_4, a_1, a_2, a_3, a_5; c_2, c_1, c_1, c_1; x, y, z, u\right)$$
$$= H_{y,z}(a_2, a) X_{93}^{(4)}\left(a_1, a_1, a, a_4, a_1, a, a_3, a_5; c_2, c_1, c_1, c_1; x, y, z, u\right), \tag{47}$$

$$X_{93}^{(4)}\left(a_1, a_1, a_2, a_4, a_1, a_2, a_3, a_5; c_2, c_1, c_1, c_1; x, y, z, u\right)$$
$$= H_x(c, c_2) X_{93}^{(4)}\left(a_1, a_1, a_2, a_4, a_1, a_2, a_3, a_5; c, c_1, c_1, c_1; x, y, z, u\right), \tag{48}$$

Theorem 4.4. *Each of the following formulas holds true:*

$$X_{94}^{(4)}\left(a_1, a_1, a_2, a_4, a_1, a_2, a_3, a_5; c, c, c, c; x, y, z, u\right)$$
$$= H_z(a_3, a) H_u(a_4, a') X_{94}^{(4)}\left(a_1, a_1, a_2, a', a_1, a_2, a, a_5; c, c, c, c; x, y, z, u\right), \tag{49}$$

$$X_{94}^{(4)}\left(a_1, a_1, a_2, a_4, a_1, a_2, a_3, a_5; c, c, c, c; x, y, z, u\right)$$
$$= \bar{H}_{x,y,z,u}(c, c') X_{94}^{(4)}\left(a_1, a_1, a_2, a_4, a_1, a_2, a_3, a_5; c', c', c', c'; x, y, z, u\right), \tag{50}$$

Theorem 4.5. *Each of the following formulas holds true:*

$$X_{95}^{(4)}\left(a_1, a_1, a_3, a_5, a_1, a_2, a_4, a_6; c_2, c_1, c_1, c_1; x, y, z, u\right)$$
$$= H_y(a_2, a) H_z(a_3, a') H_u(a_5, a'') X_{95}^{(4)}\left(a_1, a_1, a', a'', a_1, a, a_4, a_6; c_2, c_1, c_1, c_1; x, y, z, u\right), \tag{51}$$

$$X_{95}^{(4)}\left(a_1, a_1, a_3, a_5, a_1, a_2, a_4, a_6; c_2, c_1, c_1, c_1; x, y, z, u\right)$$
$$= \bar{H}_u(a, a_6) X_{95}^{(4)}\left(a_1, a_1, a_3, a_5, a_1, a_2, a_4, a; c_2, c_1, c_1, c_1; x, y, z, u\right), \tag{52}$$

Theorem 4.6. *Each of the following formulas holds true:*

$$X_{96}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c_2,c_1,c_2;x,y,z,u\right)$$
$$= H_{y,u}(c,c_2)X_{96}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c,c_1,c;x,y,z,u\right), \quad (53)$$

$$X_{96}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c_2,c_1,c_2;x,y,z,u\right)$$
$$= \bar{H}_{x,z}(c_1,c)X_{96}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c,c_2,c,c_2;x,y,z,u\right), \quad (54)$$

Theorem 4.7. *Each of the following formulas holds true:*

$$X_{97}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c_1,c_2,c_1;x,y,z,u\right)$$
$$= H_{x,y,u}(c,c_1)X_{97}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c,c,c_2,c;x,y,z,u\right), \quad (55)$$

$$X_{97}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c_1,c_2,c_1;x,y,z,u\right)$$
$$= \bar{H}_{z}(c_2,c)X_{97}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c_1,c,c_1;x,y,z,u\right), \quad (56)$$

Theorem 4.8. *Each of the following formulas holds true:*

$$X_{98}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c_2,c_1,c_1;x,y,z,u\right)$$
$$= \bar{H}_{u}(a,a_3)X_{98}^{(4)}\left(a_1,a_1,a_2,a,a_1,a_2,a_2,a_4;c_1,c_2,c_1,c_1;x,y,z,u\right), \quad (57)$$

$$X_{98}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c_2,c_1,c_1;x,y,z,u\right)$$
$$= \bar{H}_{y}(c_2,c)X_{98}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c_1,c,c_1,c_1;x,y,z,u\right), \quad (58)$$

Theorem 4.9. *Each of the following formulas holds true:*

$$X_{99}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c,c,c,c;x,y,z,u\right)$$
$$= H_{u}(a_4,a)X_{99}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a;c,c,c,c;x,y,z,u\right), \quad (59)$$

$$X_{99}^{(4)}\left(a_1,a_1,a_2,a_3,a_1,a_2,a_2,a_4;c,c,c,c;x,y,z,u\right)$$
$$= \bar{H}_{u}(a,a_3)X_{99}^{(4)}\left(a_1,a_1,a_2,a,a_1,a_2,a_2,a_4;c,c,c,c;x,y,z,u\right). \quad (60)$$

The proofs of the above theorems are based upon application of Mellin and Mellin-Barnes integral representation methods for hypergeometric functions (see [22]). The details of the proofs are omitted.

5 Generating Functions

Here, we establish generating functions involving our hypergeometric functions in four variables. Because the proofs of the following equalities are similar to the proofs of results in Mellin-Barnes [7, 23, 24, 27], we omit these proofs.

Theorem 5.1. *The following generating functions holds:*

$$\sum_{i,j=0}^{\infty}\frac{(a_2)_i(a_3)_j}{i!j!}X_{91}^{(4)}\left(a_1,a_1,a_2+i,a_4,a_1,a_2+i,a_3+j,a_5;c_1,c_1,c_2,c_1\right.$$
$$;x,y,z,u)\,t^iv^j=(1-t)^{-a_2}(1-v)^{-a_3}X_{91}^{(4)}\left(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;\right.$$
$$\left.c_1,c_1,c_2,c_1;x,\frac{y}{1-t},\frac{z}{(1-t)(1-v)},u\right),\quad (61)$$

$$\sum_{i,j=0}^{\infty}\frac{(a_1)_i(a_5)_j}{i!j!}X_{92}^{(4)}\left(a_1+i,a_1+i,a_2,a_4,a_1+i,a_2,a_3,a_5+j;c_1,c_2,c_1,\right.$$
$$c_1;x,y,z,u)\,t^iv^j=(1-t)^{-a_1}(1-v)^{-a_5}X_{92}^{(4)}\left(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;\right.$$
$$\left.c_1,c_2,c_1,c_1;\frac{x}{(1-t)^2},\frac{y}{1-t},z,\frac{u}{1-v}\right),\quad (62)$$

$$\sum_{i=0}^{\infty}\frac{(a_2)_i}{i!}X_{93}^{(4)}\left(a_1,a_1,a_2+i,a_4,a_1,a_2+i,a_3,a_5;c_2,c_1,c_1,c_1;x,y,z,u\right)t^i$$
$$=(1-t)^{-a_2}X_{93}^{(4)}\left(a_1,a_1,a_2,a_4,a_1,a_2,a_3,a_5;c_2,c_1,c_1,c_1;x,\right.$$
$$\left.\frac{y}{1-t},\frac{z}{1-t},u\right),\quad (63)$$

$$\sum_{i,j=0}^{\infty}\frac{(a_1)_i(a_2)_j}{i!j!}X_{94}^{(4)}\left(a_1+i,a_1+i,a_2+j,a_4,a_1+i,a_2+j,a_3,a_5;\right.$$
$$c,c,c,c;x,y,z,u)\,t^iv^j=(1-t)^{-a_1}(1-v)^{-a_2}X_{94}^{(4)}\left(a_1,a_1,a_2,a_4,a_1,a_2,\right.$$
$$\left.a_3,a_5;c,c,c,c;\frac{x}{(1-t)^2},\frac{y}{(1-t)(1-v)},\frac{z}{1-v},u\right),\quad (64)$$

$$\sum_{i,j,k=0}^{\infty}\frac{(a_2)_i(a_4)_j(a_6)_k}{i!j!k!}X_{95}^{(4)}\left(a_1,a_1,a_3,a_5,a_1,a_2+i,a_4+j,a_6+k;\right.$$
$$c_2,c_1,c_1,c_1;x,y,z,u)\,t^iv^jw^k=(1-t)^{-a_2}(1-v)^{-a_4}(1-w)^{-a_6}$$
$$\times X_{95}^{(4)}\left(a_1,a_1,a_3,a_5,a_1,a_2,a_4,a_6;c_2,c_1,c_1,c_1;x,\frac{y}{1-t},\frac{z}{1-v},\frac{u}{1-w}\right),\quad (65)$$

$$\sum_{i,j=0}^{\infty}\frac{(a_1)_i(a_2)_j}{i!j!}X_{96}^{(4)}\left(a_1+i,a_1+i,a_2+j,a_3,a_1+i,a_2+j,a_2+j,\right.$$
$$a_4;c_1,c_2,c_1,c_2;x,y,z,u)\,t^iv^j=(1-t)^{-a_1}(1-v)^{-a_2}X_{96}^{(4)}\left(a_1,a_1,a_2,\right.$$
$$\left.a_3,a_1,a_2,a_2,a_4;c_1,c_2,c_1,c_2;\frac{x}{(1-t)^2},\frac{y}{(1-t)(1-v)},\frac{z}{(1-v)^2},u\right),\quad (66)$$

$$\sum_{i=0}^{\infty} \frac{(a_3)_i}{i!} X_{97}^{(4)} (a_1, a_1, a_2, a_3 + i, a_1, a_2, a_2, a_4; c_1, c_1, c_2, c_1; x, y, z, u)\, t^i = (1-t)^{-a_3} X_{97}^{(4)} \left(a_1, a_1, a_2, a_3, a_1, a_2, a_2, a_4; c_1, c_1, c_2, c_1; x, y, z, \frac{u}{1-t}\right), \quad (67)$$

$$\sum_{i,j,k=0}^{\infty} \frac{(a_1)_i (a_2)_j (a_3)_k}{i!j!k!} X_{98}^{(4)} (a_1 + i, a_1 + i, a_2 + j, a_3 + k, a_1 + i, a_2 + j, a_2 + j, a_4; c_1, c_2, c_1, c_1; x, y, z, u)\, t^i v^j w^k = (1-t)^{-a_1} (1-v)^{-a_2} (1-w)^{-a_3} \times X_{98}^{(4)} \left(a_1, a_1, a_2, a_3, a_1, a_2, a_2, a_4; c_1, c_2, c_1, c_1; \frac{x}{(1-t)^2}, \frac{y}{(1-t)(1-v)}, \frac{z}{(1-v)^2}, \frac{u}{1-w}\right), \quad (68)$$

$$\sum_{i,j=0}^{\infty} \frac{(a_2)_i (a_3)_j}{i!j!} X_{99}^{(4)} (a_1, a_1, a_2 + i, a_3 + j, a_1, a_2 + i, a_2 + i, a_4; c, c, c, c; x, y, z, u)\, t^i v^j = (1-t)^{-a_2} (1-v)^{-a_3} X_{99}^{(4)} \left(a_1, a_1, a_2, a_3, a_1, a_2, a_2, a_4; c, c, c, c; x, \frac{y}{1-t}, \frac{z}{(1-t)^2}, \frac{u}{1-v}\right). \quad (69)$$

Availability of supporting data

Data will be provided on request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Funding

This research received no external funding.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements

The authors would like to thanks the worthy referees and editor for their valuable suggestions for our paper. Praveen Agarwal was very thankful to the SERB (project TAR/2018/000001), DST(project DST/INT/DAAD/P-21/2019, and INT/RUS/RFBR/308) and NBHM (DAE)(project 02011/12/2020 NBHM(R.P)/R&D II/7867) for their necessary support.

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