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ON APPROXIMATION OF CONTINUOUS BIVARIATE PERIODIC FUNCTIONS BY DEFERRED GENERALIZED DE LA VALLÉE POUSSIN MEANS OF THEIR FOURIER SERIES

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ABSTRACT. In this paper we have introduced the deferred generalized de la Vallée Poussin means. Using such means of partial sums of the Fourier series of continuous and periodic functions, we have proved some theorems pertaining to upper bound of such means, to the upper bound of the their deviation from the considered functions, and to the degree of approximation when such functions belong to Lipschitz classes.

1. Introduction

Let $C_{2\pi}$ be the class of real-valued functions of two variables that are continuous on $Q := [-\pi, \pi] \times [-\pi, \pi]$ and 2π-periodic in each variable separately.

For $f \in C_{2\pi}$, its Fourier series is given by

$$f(x,y) \sim \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \lambda_{n,m}(a_{n,m} \cos nx \cos my + b_{n,m} \sin nx \cos my \\ + c_{n,m} \cos nx \sin my + d_{n,m} \sin nx \sin my)$$

where

$$\lambda_{n,m} = \begin{cases} \frac{1}{4}, & \text{if} \quad n = m = 0 \\ \frac{1}{2}, & \text{if} \quad n = 0,\, m > 0; \ \text{ or } \ n > 0,\, m = 0 \\ 1, & \text{if} \quad n,\, m > 0, \end{cases}$$

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$$a_{n,m}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\cos nu\cos mvdudv;\ n,\ m\in\mathbb{N}\cup\{0\},$$
$$b_{n,m}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\sin nu\cos mvdudv;\ n\in\mathbb{N},\ m\in\mathbb{N}\cup\{0\},$$
$$c_{n,m}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\cos nu\sin mvdudv;\ n\in\mathbb{N}\cup\{0\},\ m\in\mathbb{N},$$

$$d_{n,m}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\sin nu\sin mvdudv;\ n,\ m\in\mathbb{N},$$

and the partial sum is given by

$$S_{n,m}(f;x,y)=\sum_{k=0}^{n}\sum_{\ell=0}^{m}\lambda_{k,\ell}(a_{k,\ell}\cos kx\cos \ell y+b_{k,\ell}\sin kx\cos\ell y$$
$$+c_{k,\ell}\cos kx\sin\ell y+d_{k,\ell}\sin kx\sin\ell y).$$

The Fejér means of partial sums $S_{n,m}(f;x,y)$ are given by

$$\sigma_{n,m}(f;x,y)=\frac{1}{(n+1)(m+1)}\sum_{k=0}^{n}\sum_{\ell=0}^{m}S_{k\ell}(f;x,y),$$

while the de la Vallée-Poussin means of $S_{n,m}(f;x,y)$ are

$$V_{n,p}^{m,q}(f;x,y)=\frac{1}{(p+1)(q+1)}\sum_{k=n}^{n+p}\sum_{\ell=m}^{m+q}S_{k,\ell}(f;x,y),\ (p\geq 0,\ q\geq 0).$$

For a 2π-periodic continuous function f of two variables x and y, let $E_{n,m}(f)$ denote the degree of best approximation of f by trigonometric polynomials $T_{n,m}(x,y)$ of order less or equal to n in x, and less or equal to m in y, i.e.,

$$E_{n,m}:=E_{n,m}(f):=\inf_{T_{n,m}}\Big\{\max_{x,y}|f(x,y)-T_{n,m}(x,y)|\Big\}.$$

The de la Vallée Poussin means as well as their generalization are used widely in mathematical literature. For example one of them is used in [1], the following result was proved in it, and in a way is related to ours:

Theorem 1.1. *If $f(x,y)\in C_{2\pi}$, then the deviations of the de la Vallée Poussin sums $V_{n,p}^{m,q}(f;x,y)$ from f satisfy the inequality*

$$|f(x,y)-V_{n,p}^{m,q}(f;x,y)|$$
$$\leq\frac{C}{(p+1)(q+1)}\sum_{i=n}^{n+p}\sum_{j=m}^{m+q}(1+\ln\frac{i+1}{i-n+1})(1+\ln\frac{j+1}{j-m+1})E_{i,j},$$

where $C>0$ is an absolute constant.

Recent results regarding to the deferred de la Vallée Poussin means, the interested reader can fnd in [3]-[7]. Also several results, pertaining to the degree of approximation of the periodic integrable functions, can be found in [8]-[12].

For our further research we introduce here some new means. Namely, let $\mu := (\mu_n)$, $\nu := (\nu_m)$ be two monotone non-decreasing sequences of integers such that $\mu_1 = \nu_1 = 1$, $\mu_{n+1} - \mu_n \le 1$, and $\nu_{m+1} - \nu_m \le 1$. Let $a := (a_n)$, $b := (b_n)$, $c := (c_m)$, $d := (d_m)$, be sequences of integers with conditions

$$\mu_n \le a_n \le b_n, \quad \nu_m \le c_m \le d_m, \quad (n, m = 1, 2, \dots). \tag{1.1}$$

The means

$$V^{m,c,d,\nu}_{n,a,b,\mu}(f; x, y) = \frac{1}{(b_n - a_n + \mu_n)(d_m - c_m + \nu_m)} \sum_{k=a_n-\mu_n}^{b_n-1} \sum_{\ell=c_m-\nu_m}^{d_m-1} S_{k,\ell}(x, y), \tag{1.2}$$

we call the deferred generalized de la Vallée Poussin means of the sequence $(S_{k,\ell}(x, y))$ generated by sequences μ, ν, a, b, c and d (for single Fourier series such means are introduced in [3]).

Using these new means, we aim to prove a version of Theorem 1.1. The new theorem will coincide with Theorem 1.1 in a particular case, and using it, we will prove some other theorems related to the summability of double Fourier series, which is the main purpose of the paper. The reason why we have named these new means as *deferred generalized de la Vallée Poussin means* will be clarified at almost the end of our paper.

To realize our aim, we need a helpful lemma, given in next section.

2. A Lemma

Now, we prove next helpful lemma.

Lemma 2.1. *Denote by*

$$\mathbb{K}^{m,c,d,\nu}_{n,a,b,\mu}(t, w) := \sum_{k=a_n-\mu_n}^{b_n-1} \sum_{\ell=c_m-\nu_m}^{d_m-1} D_k(t) D_\ell(w)$$

the deferred de la Vallée Poussin kernel, where $D_r(z) := \frac{\sin(2r+1)z}{\sin z}$, $r \in \mathbb{N}$. *Then,*

$$\mathbb{K}^{m,c,d,\nu}_{n,a,b,\mu}(t,w) = \frac{\sin(b_n - a_n + \mu_n)t \sin(b_n + a_n - \mu_n)t}{\sin^2 t} \times \frac{\sin(d_m - c_m + \nu_m)w \sin(d_m + c_m - \nu_m)w}{\sin^2 w}.$$

Proof. Since,

$$\begin{aligned}
&\sum_{m=a_n-\mu_n}^{b_n-1} \frac{\sin(2m+1)t}{\sin t} \\
&= \sum_{m=0}^{b_n-1} \frac{2\sin(2m+1)t \sin t}{2\sin^2 t} - \sum_{m=0}^{a_n-\mu_n-1} \frac{2\sin(2m+1)t\sin t}{2\sin^2 t} \\
&= \frac{1-\cos(2b_n t)}{2\sin^2 t} - \frac{1-\cos 2(a_n-\mu_n)t}{2\sin^2 t} \\
&= \frac{\sin^2(b_n t) - \sin^2(a_n-\mu_n)t}{\sin^2 t} \\
&= \frac{\sin(b_n - a_n + \mu_n)t \sin(b_n + a_n - \mu_n)t}{\sin^2 t},
\end{aligned}$$

then we immediately obtain the required equality.

The proof is completed. □

In the sequel we pass to the main result.

3. Main Results

At first, we prove the following.

Theorem 3.1. *If the function* $f(x,y)$ *is bounded, i.e.* $|f(x,y)| \le K$, *then the means* $V^{m,c,d,\nu}_{n,a,b,\mu}(f;x,y)$ *satisfy the inequality*

$$\begin{aligned}
&|V^{m,c,d,\nu}_{n,a,b,\mu}(f;x,y)| \\
&= \mathcal{O}\left(\left(1 + \ln\frac{b_n + a_n - \mu_n}{b_n - a_n + \mu_n}\right)\left(1 + \ln\frac{d_m + c_m - \nu_m}{d_m - c_m + \nu_m}\right)\right). \qquad (3.1)
\end{aligned}$$

Proof. After some calculation we obtain

$$\begin{aligned}
S_{k,\ell}(x,y) = \frac{1}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} &[f(x+2t,y+2w) + f(x-2t,y+2w) \\
&+ f(x+2t,y-2w) + f(x-2t,y-2w)]D_k(t)D_\ell(w)dtdw,
\end{aligned}$$

where $D_m(z) = \frac{\sin(2m+1)z}{\sin z}$.

For the sake of brevity, we denote $M_n^{(1)} := b_n - a_n + \mu_n$, $M_n^{(2)} := b_n + a_n - \mu_n$, $N_m^{(1)} := d_m - c_m + \nu_m$, $N_m^{(2)} := d_m + c_m - \nu_m$, and we will use them whenever we consider it as reasonable.

Using the deferred generalized de la Vallée Poussin means of $S_{k,\ell}(x, y)$, we get

$$V_{n,a,b,\mu}^{m,c,d,\nu}(f; x, y) = \frac{1}{M_n^{(1)} N_m^{(1)} \pi^2} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \phi_{x,y}(t, w) \mathbb{K}_{n,a,b,\mu}^{m,c,d,\nu}(t, w) dt dw, \tag{3.2}$$

where

$$\phi_{x,y}(t, w) := f(x + 2t, y + 2w) + f(x - 2t, y + 2w) \\ + f(x + 2t, y - 2w) + f(x - 2t, y - 2w).$$

Then, for $\delta_1, \delta_1 \in (0, \pi/2]$, we put

$$\beta_1 := \beta_1(n) := \min\left(\frac{2}{\pi M_n^{(2)}}, \delta_1\right), \ \beta_2 := \beta_2(n) := \min\left(\frac{2}{\pi M_n^{(1)}}, \delta_1\right),$$

$$\beta_3 := \beta_3(m) := \min\left(\frac{2}{\pi N_m^{(2)}}, \delta_2\right), \text{ and } \beta_4 := \beta_3(m) := \min\left(\frac{2}{\pi N_m^{(1)}}, \delta_2\right).$$

Therefore, from (3.2) we write

$$|V_{n,a,b,\mu}^{m,c,d,\nu}(f;x,y)| \leq \frac{1}{M_n^{(1)}N_m^{(1)}\pi^2}$$
$$\times\Bigg(\int_0^{\beta_1}\int_0^{\beta_3}|\phi_{x,y}(t,w)|(\cdot)dtdw+\int_0^{\beta_1}\int_{\beta_3}^{\beta_4}|\phi_{x,y}(t,w)|(\cdot)dtdw$$
$$+\int_0^{\beta_1}\int_{\beta_4}^{\delta_2}|\phi_{x,y}(t,w)|(\cdot)dtdw+\int_0^{\beta_1}\int_{\delta_2}^{\frac{\pi}{2}}|\phi_{x,y}(t,w)|(\cdot)dtdw$$
$$+\int_{\beta_1}^{\beta_2}\int_0^{\beta_3}|\phi_{x,y}(t,w)|(\cdot)dtdw+\int_{\beta_1}^{\beta_2}\int_{\beta_3}^{\beta_4}|\phi_{x,y}(t,w)|(\cdot)dtdw$$
$$+\int_{\beta_1}^{\beta_2}\int_{\beta_4}^{\delta_2}|\phi_{x,y}(t,w)|(\cdot)dtdw+\int_{\beta_1}^{\beta_2}\int_{\delta_2}^{\frac{\pi}{2}}|\phi_{x,y}(t,w)|(\cdot)dtdw$$
$$+\int_{\beta_2}^{\delta_1}\int_0^{\beta_3}|\phi_{x,y}(t,w)|(\cdot)dtdw+\int_{\beta_2}^{\delta_1}\int_{\beta_3}^{\beta_4}|\phi_{x,y}(t,w)|(\cdot)dtdw$$
$$+\int_{\beta_2}^{\delta_1}\int_{\beta_4}^{\delta_2}|\phi_{x,y}(t,w)|(\cdot)dtdw+\int_{\beta_2}^{\delta_1}\int_{\delta_2}^{\frac{\pi}{2}}|\phi_{x,y}(t,w)|(\cdot)dtdw$$
$$+\int_{\delta_1}^{\frac{\pi}{2}}\int_0^{\beta_3}|\phi_{x,y}(t,w)|(\cdot)dtdw+\int_{\delta_1}^{\frac{\pi}{2}}\int_{\beta_3}^{\beta_4}|\phi_{x,y}(t,w)|(\cdot)dtdw$$
$$+\int_{\delta_1}^{\frac{\pi}{2}}\int_{\beta_4}^{\delta_2}|\phi_{x,y}(t,w)|(\cdot)dtdw+\int_{\delta_1}^{\frac{\pi}{2}}\int_{\delta_2}^{\frac{\pi}{2}}|\phi_{x,y}(t,w)|(\cdot)dtdw\Bigg):=\sum_{j=1}^{16}\mathbb{P}_j. \tag{3.3}$$

First of all, it is obvious that

$$\mathbb{P}_4=\mathbb{P}_8=\mathbb{P}_{12}=\mathbb{P}_{13}=\mathbb{P}_{14}=\mathbb{P}_{15}=\mathbb{P}_{16}=0$$

for $\delta_1=\delta_2=\frac{\pi}{2}$.

Moreover, using the well-known inequalities $|\sin(rt)|\leq r|\sin t|$, $|\sin(rt)|\leq 1$, $(r\in\mathbb{N})$, Jordan's inequality $\pi|\sin(t)|\geq 2t$, $t\in[0,\pi/2]$, and Lemma 2.1 we get

$$\mathbb{P}_1\leq C\int_0^{\beta_1}\int_0^{\beta_3}\frac{M_n^{(1)}tM_n^{(2)}tN_m^{(1)}wN_m^{(2)}w}{(tw)^2}dtdw\leq CM_n^{(1)}N_m^{(1)}, \tag{3.4}$$

$$\mathbb{P}_2\leq C\int_0^{\beta_1}\int_{\beta_3}^{\beta_4}\frac{M_n^{(1)}tM_n^{(2)}tN_m^{(1)}w}{(tw)^2}dtdw\leq CM_n^{(1)}N_m^{(1)}\log\frac{N_m^{(2)}}{N_m^{(1)}}, \tag{3.5}$$

$$\mathbb{P}_3\leq C\int_0^{\beta_1}\int_{\beta_4}^{\delta_2}\frac{M_n^{(1)}tM_n^{(2)}t}{(tw)^2}dtdw\leq CM_n^{(1)}N_m^{(1)}, \tag{3.6}$$

$$(3.7)\quad \mathbb{P}_5 \le C\int_{\beta_1}^{\beta_2}\int_0^{\beta_3} \frac{M_n^{(1)}tN_m^{(1)}wN_m^{(2)}w}{(tw)^2}dtdw \le CM_n^{(1)}N_m^{(1)}\ln\frac{M_n^{(2)}}{M_n^{(1)}},$$

$$(3.8)\quad \mathbb{P}_6 \le C\int_{\beta_1}^{\beta_2}\int_{\beta_3}^{\beta_3} \frac{M_n^{(1)}tN_m^{(1)}w}{(tw)^2}dtdw \le CM_n^{(1)}N_m^{(1)}\ln\frac{M_n^{(2)}}{M_n^{(1)}}\ln\frac{N_m^{(2)}}{N_m^{(1)}},$$

$$(3.9)\qquad \mathbb{P}_7 \le C\int_{\beta_1}^{\beta_2}\int_{\beta_4}^{\delta_2} \frac{M_n^{(1)}t}{(tw)^2}dtdw \le CM_n^{(1)}N_m^{(1)}\ln\frac{M_n^{(2)}}{M_n^{(1)}},$$

$$(3.10)\qquad \mathbb{P}_9 \le C\int_{\beta_2}^{\delta_1}\int_0^{\beta_3} \frac{N_m^{(1)}wN_m^{(2)}w}{(tw)^2}dtdw \le CM_n^{(1)}N_m^{(1)},$$

$$(3.11)\qquad \mathbb{P}_{10} \le C\int_{\beta_2}^{\delta_1}\int_{\beta_3}^{\beta_4} \frac{N_m^{(1)}w}{(tw)^2}dtdw \le CM_n^{(1)}N_m^{(1)}\ln\frac{N_m^{(2)}}{N_m^{(1)}},$$

and

$$(3.12)\qquad \mathbb{P}_{11} \le C\int_{\beta_2}^{\delta_1}\int_{\beta_4}^{\delta_2} \frac{dtdw}{(tw)^2} \le CM_n^{(1)}N_m^{(1)}.$$

Finally, inserting (3.4)–(3.12) (with $\delta_1 = \delta_2 = \frac{\pi}{2}$) into (3.3), we immediately obtain (3.1).

The proof is completed. □

Theorem 3.2. *If the function $f(x,y)$ is continuous, then the estimate*

$$|V_{n,a,b,\mu}^{m,c,d,\nu}(f;x,y) - f(x,y)|$$
$$= \mathcal{O}\left(\left(1+\ln\frac{b_n+a_n-\mu_n}{b_n-a_n+\mu_n}\right)\left(1+\ln\frac{d_m+c_m-\nu_m}{d_m-c_m+\nu_m}\right)\right)E_{a_n-\mu_n,c_m-\nu_m}$$

holds true uniformly in all (x,y).

Proof. Let $t_{n,m}^*(x,y)$ denote the trigonometric polynomial of best approximation of $f(x,y)$ whose degree is not higher then m with respect to x and not higher then m with respect to y. It is clear that

$$V_{n,a,b,\mu}^{m,c,d,\nu}(f-t_{n,m}^*;x,y) = V_{n,a,b,\mu}^{m,c,d,\nu}(f;x,y) - t_{n,m}^*(x,y),$$

whenever $a_n - \mu_n \ge n$ and $c_m - \nu_m \ge m$ hold true.

Whence, we can write

$$\begin{aligned}|V^{m,c,d,\nu}_{n,a,b,\mu}(f;x,y)-f(x,y)| &\le |V^{m,c,d,\nu}_{n,a,b,\mu}(f;x,y)-t^*_{a_n-\mu_n,c_m-\nu_m}(x,y)|\\ &\quad +|t^*_{a_n-\mu_n,c_m-\nu_m}(x,y)-f(x,y)|\\ &\le |V^{m,c,d,\nu}_{n,a,b,\mu}(f-t^*_{a_n-\mu_n,c_m-\nu_m};x,y)|\\ &\quad +E_{a_n-\mu_n,c_m-\nu_m}. \end{aligned}\tag{3.13}$$

By the definition of the deferred generalized de la Vallée Poussin means, from (3.2), we easy obtain

$$|V^{m,c,d,\nu}_{n,a,b,\mu}(f-t^*_{a_n-\mu_n,c_m-\nu_m};x,y)| \le \frac{4E_{a_n-\mu_n,c_m-\nu_m}}{M^{(1)}_n N^{(1)}_m \pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} |\mathbb{K}^{m,c,d,\nu}_{n,a,b,\mu}(t,w)|dtdw$$

Thus, with same reasoning as in the proof of Theorem 3.1 for $\delta_1 = \delta_2 = \frac{\pi}{2}$ and using the constant function $g(t,w) := E_{a_n-\mu_n,c_m-\nu_m}$ instead of $|\phi_{x,y}(t,w)|$, we get

$$\begin{aligned}|V^{m,c,d,\nu}_{n,a,b,\mu}(f-t^*_{a_n-\mu_n,c_m-\nu_m};x,y)| = \mathcal{O}\Bigg(\left(1+\ln\frac{b_n+a_n-\mu_n}{b_n-a_n+\mu_n}\right)\times\\ \left(1+\ln\frac{d_m+c_m-\nu_m}{d_m-c_m+\nu_m}\right)\Bigg)E_{a_n-\mu_n,c_m-\nu_m}.\end{aligned}\tag{3.14}$$

Consequently, relations (3.13) and (3.14) imply the required inequality. The proof is completed. □

There are several Lipschitz classes defined for functions in two variables. We will use one given in [2], which is appropriate for our purpose. It is said that $f(x,y)\in \mathrm{Lip}(\alpha_1,\alpha_2)$, $0<\alpha_1,\alpha_2\le 1$, if

$$\sup_{|h_1|\le\delta_1,|h_2|\le\delta_2}|f(x+h_1,y+h_2)-f(x,y)| = \mathcal{O}\left(\delta_1^{\alpha_1}\delta_2^{\alpha_2}\right),$$

where $\delta_1,\delta_2>0$ and $h_1,h_2>0$.

Theorem 3.3. *If $f(x,y)\in Lip(\alpha_1,\alpha_2)$, then for $0<\alpha_1,\alpha_2<1$:*

$$\begin{aligned}&|V^{m,c,d,\nu}_{n,a,b,\mu}(f;x,y)-f(x,y)|\\ &\quad = \mathcal{O}\left(\left(\frac{1}{\left(M^{(1)}_n\right)^{\alpha_1}}+\frac{1}{\left(M^{(2)}_n\right)^{\alpha_1}}\right)\left(\frac{1}{\left(N^{(1)}_m\right)^{\alpha_2}}+\frac{1}{\left(N^{(2)}_m\right)^{\alpha_2}}\right)\right),\end{aligned}$$

while for $\alpha_1 = \alpha_2 = 1$*:*

$$
\begin{aligned}
|V^{m,c,d,\nu}_{n,a,b,\mu}(f;x,y) - f(x,y)| &= \mathcal{O}\left(\left(\frac{1}{M^{(1)}_n} + \frac{1}{M^{(2)}_n}\right)\frac{1+\ln\left(N^{(1)}_m\right)}{N^{(1)}_m}\right) \\
&+\mathcal{O}\left(\left(\frac{1}{N^{(1)}_m} + \frac{1}{N^{(2)}_m}\right)\frac{1+\ln\left(M^{(1)}_n\right)}{M^{(1)}_n}\right) \\
&+\mathcal{O}\left(\frac{\left(1+\ln\left(M^{(1)}_n\right)\right)\left(1+\ln\left(N^{(1)}_m\right)\right)}{M^{(1)}_n N^{(1)}_m}\right),
\end{aligned}
$$

hold true uniformly in all (x, y)*.*

Proof. By definition of the deferred generalized de la Vallée Poussin means of $s_m(f;x,y)$, we easily can obtain the equality

$$
\begin{aligned}
&V^{m,c,d,\nu}_{n,a,b,\mu}(f;x,y) - f(x,y) \\
&\qquad = \frac{1}{M^{(1)}_n N^{(1)}_m \pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \Phi_{x,y}(t,w)\mathbb{K}^{m,c,d,\nu}_{n,a,b,\mu}(t,w)dtdw,
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_{x,y}(t,w) := {}& f(x+2t, y+2w) + f(x-2t, y+2w) \\
&+ f(x+2t, y-2w) + f(x-2t, y-2w) - 4f(x,y).
\end{aligned}
$$

Since $f(x,y) \in \mathrm{Lip}(\alpha_1, \alpha_2)$, then

$$\begin{aligned}
&|V^{m,c,d,\nu}_{n,a,b,\mu}(f;x,y) - f(x,y)| \\
&= \frac{\mathcal{O}(1)}{M^{(1)}_n N^{(1)}_m \pi^2} \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} t^{\alpha_1} w^{\alpha_2} |\mathbb{K}^{m,c,d,\nu}_{n,a,b,\mu}(t,w)| dtdw \\
&= \frac{1}{M^{(1)}_n N^{(1)}_m \pi^2} \left(\int_0^{\frac{1}{M^{(2)}_n}}\int_0^{\frac{1}{N^{(2)}_m}} (\cdot)dtdw + \int_0^{\frac{1}{M^{(2)}_n}}\int_{\frac{1}{N^{(2)}_m}}^{\frac{1}{N^{(1)}_m}} (\cdot)dtdw \right. \\
&\quad + \int_0^{\frac{1}{M^{(2)}_n}}\int_{\frac{1}{N^{(1)}_m}}^{\frac{\pi}{2}} (\cdot)dtdw + \int_{\frac{1}{M^{(2)}_n}}^{\frac{1}{M^{(1)}_n}}\int_0^{\frac{1}{N^{(2)}_m}} (\cdot)dtdw \\
&\quad + \int_{\frac{1}{M^{(2)}_n}}^{\frac{1}{M^{(1)}_n}}\int_{\frac{1}{N^{(2)}_m}}^{\frac{1}{N^{(1)}_m}} (\cdot)dtdw + \int_{\frac{1}{M^{(2)}_n}}^{\frac{1}{M^{(1)}_n}}\int_{\frac{1}{N^{(1)}_m}}^{\frac{\pi}{2}} (\cdot)dtdw \\
&\quad \left. + \int_{\frac{1}{M^{(1)}_n}}^{\frac{\pi}{2}}\int_0^{\frac{1}{N^{(2)}_m}} (\cdot)dtdw + \int_{\frac{1}{M^{(1)}_n}}^{\frac{\pi}{2}}\int_{\frac{1}{N^{(2)}_m}}^{\frac{1}{N^{(1)}_m}} (\cdot)dtdw \right) \\
&\quad + \int_{\frac{1}{M^{(1)}_n}}^{\frac{\pi}{2}}\int_{\frac{1}{N^{(1)}_m}}^{\frac{\pi}{2}} (\cdot)dtdw := \sum_{i=1}^{9} \mathbb{B}_i. \qquad (3.15)
\end{aligned}$$

Firstly, let $0 < \alpha_1 \leq 1$ and $0 < \alpha_2 \leq 1$. Then we have

$$\begin{aligned}
\mathbb{B}_1 &\leq C \int_0^{\frac{1}{M^{(2)}_n}}\int_0^{\frac{1}{N^{(2)}_m}} t^{\alpha_1} w^{\alpha_2} \frac{M^{(1)}_n t M^{(2)}_n t N^{(1)}_m w N^{(2)}_m w}{(tw)^2} dtdw \\
&\leq C \frac{M^{(1)}_n N^{(1)}_m}{\left(M^{(2)}_n\right)^{\alpha_1} \left(N^{(2)}_m\right)^{\alpha_2}}, \qquad (3.16)
\end{aligned}$$

$$\begin{aligned}
\mathbb{B}_2 &\leq C \int_0^{\frac{1}{M^{(2)}_n}}\int_{\frac{1}{N^{(2)}_m}}^{\frac{1}{N^{(1)}_m}} t^{\alpha_1} w^{\alpha_2} \frac{M^{(1)}_n t M^{(2)}_n t N^{(1)}_m w}{(tw)^2} dtdw \\
&\leq C \frac{M^{(1)}_n}{\left(M^{(2)}_n\right)^{\alpha_1} \left(N^{(1)}_m\right)^{\alpha_2}}. \qquad (3.17)
\end{aligned}$$

Let $0 < \alpha_1 \leq 1$, but we need to distinguish whether $0 < \alpha_2 < 1$ or $\alpha_2 = 1$. Having this in our mind we obtain

$$\begin{aligned}\mathbb{B}_3 &\leq C\int_0^{\frac{1}{M_n^{(2)}}}\int_{\frac{1}{N_m^{(1)}}}^{\frac{\pi}{2}} t^{\alpha_1}w^{\alpha_2}\frac{M_n^{(1)}tM_n^{(2)}t}{(tw)^2}dtdw\\
&\leq C\frac{M_n^{(1)}}{\left(M_n^{(2)}\right)^{\alpha_1}}\times\begin{cases}\dfrac{N_m^{(1)}}{(1-\alpha_2)\left(N_m^{(1)}\right)^{\alpha_2}}, & \alpha_2<1;\\ 1+\log\left(N_m^{(1)}\right), & \alpha_2=1.\end{cases}\end{aligned}\tag{3.18}$$

Then,

$$\begin{aligned}\mathbb{B}_4 &\leq C\int_{\frac{1}{M_n^{(2)}}}^{\frac{1}{M_n^{(1)}}}\int_0^{\frac{1}{N_m^{(2)}}} t^{\alpha_1}w^{\alpha_2}\frac{M_n^{(1)}tN_m^{(1)}wN_m^{(2)}w}{(tw)^2}dtdw\\
&\leq C\frac{N_m^{(1)}}{\left(M_n^{(1)}\right)^{\alpha_1}\left(N_m^{(2)}\right)^{\alpha_2}},\end{aligned}\tag{3.19}$$

$$\begin{aligned}\mathbb{B}_5 &\leq C\int_{\frac{1}{M_n^{(2)}}}^{\frac{1}{M_n^{(1)}}}\int_{\frac{1}{N_m^{(2)}}}^{\frac{1}{N_m^{(1)}}} t^{\alpha_1}w^{\alpha_2}\frac{M_n^{(1)}tN_m^{(1)}w}{(tw)^2}dtdw\\
&\leq C\frac{1}{\left(M_n^{(1)}\right)^{\alpha_1}\left(N_m^{(1)}\right)^{\alpha_2}}.\end{aligned}\tag{3.20}$$

Moreover, for $0<\alpha_1\leq 1$, we have

$$\begin{aligned}\mathbb{B}_6 &\leq C\int_{\frac{1}{M_n^{(2)}}}^{\frac{1}{M_n^{(1)}}}\int_{\frac{1}{N_m^{(1)}}}^{\frac{\pi}{2}} t^{\alpha_1}w^{\alpha_2}\frac{M_n^{(1)}t}{(tw)^2}dtdw\\
&\leq C\frac{1}{\left(M_n^{(1)}\right)^{\alpha_1}}\times\begin{cases}\dfrac{N_m^{(1)}}{(1-\alpha_2)\left(N_m^{(1)}\right)^{\alpha_2}}, & \alpha_2<1;\\ 1+\log\left(N_m^{(1)}\right), & \alpha_2=1.\end{cases}\end{aligned}\tag{3.21}$$

Similarly to (3.17), for $0<\alpha_2\leq 1$ and $0<\alpha_1<1$ or $\alpha_1=1$, we get

$$\begin{aligned}\mathbb{B}_7 &\leq C\int_{\frac{1}{M_n^{(1)}}}^{\frac{\pi}{2}}\int_0^{\frac{1}{N_m^{(2)}}} t^{\alpha_1}w^{\alpha_2}\frac{N_m^{(1)}wN_m^{(2)}w}{(tw)^2}dtdw\\
&\leq C\frac{N_m^{(1)}}{\left(N_m^{(2)}\right)^{\alpha_2}}\times\begin{cases}\dfrac{M_n^{(1)}}{(1-\alpha_1)\left(M_n^{(1)}\right)^{\alpha_1}}, & 0<\alpha_1<1;\\ 1+\log\left(M_n^{(1)}\right), & \alpha_1=1.\end{cases}\end{aligned}\tag{3.22}$$

Also, for $0 < \alpha_2 \leq 1$, we have

$$\begin{aligned} \mathbb{B}_8 &\leq C\int_{\frac{1}{M_n^{(1)}}}^{\frac{\pi}{2}}\int_{\frac{1}{N_m^{(2)}}}^{\frac{1}{N_m^{(1)}}} t^{\alpha_1}w^{\alpha_2}\frac{N_m^{(1)}w}{(tw)^2}dtdw \\ &\leq C\frac{1}{\left(N_m^{(1)}\right)^{\alpha_2}}\times\begin{cases}\dfrac{M_n^{(1)}}{(1-\alpha_1)\left(M_n^{(1)}\right)^{\alpha_1}}, & 0<\alpha_1<1;\\ 1+\log\left(M_n^{(1)}\right), & \alpha_1 = 1.\end{cases} \qquad (3.23)\end{aligned}$$

Finally,

$$\begin{aligned} \mathbb{B}_9 &\leq C\int_{\frac{1}{M_n^{(1)}}}^{\frac{\pi}{2}}\int_{\frac{1}{N_m^{(1)}}}^{\frac{\pi}{2}} t^{\alpha_1-2}w^{\alpha_2-2}dtdw \\ &\leq C\begin{cases}\dfrac{M_n^{(1)}N_m^{(1)}}{(1-\alpha_1)(1-\alpha_2)\left(M_n^{(1)}\right)^{\alpha_1}\left(N_m^{(1)}\right)^{\alpha_2}}, & 0<\alpha_1,\alpha_2<1;\\ \left(1+\log\left(M_n^{(1)}\right)\right)\left(1+\log\left(N_m^{(1)}\right)\right), & \alpha_1=\alpha_2 = 1.\end{cases} \qquad (3.24)\end{aligned}$$

Hence, the conclusion of our theorem follows from (3.15) along with (3.16)–(3.24).

The proof is completed. □

4. Remarks and Corollaries

Let $a := (a_m)$, $b := (b_m)$, $c := (c_n)$, and $d := (d_n)$, considered so far, be sequences of non-negative integers with conditions

$$a_m < b_m, \quad c_n < d_n, \quad (m, n = 1, 2, \dots), \qquad (4.1)$$

and

$$\lim_{m\to\infty} b_m = +\infty, \quad \lim_{n\to\infty} d_n = +\infty. \qquad (4.2)$$

If $\mu_n = 1$ and $\nu_n = 1$ for all $n, m \geq 1$, then the double deferred de la Vallée Poussin mean $V_{n,a+2,b+1,\mu}^{m,c+2,d+1,\nu}(f;x,y)$ reduces to

$$D_{a,c}^{b,d}(f;x,y) := \frac{1}{(b_m-a_m)(d_n-c_n)}\sum_{k=a_m+1}^{b_m}\sum_{\ell=c_n+1}^{d_n} S_{k,\ell}(f;x,y),$$

which is the double deferred Cesàro mean of the sum $S_{k,\ell}(f;x,y)$ introduced implicitly in [15] (this is why we called our new means *deferred generalized Vallée Poussin means*). It was shown there, that (3.1) and

(4.2) are conditions of regularity for $D_{a,c}^{b,d}$. Therefore, if conditions (4.1) and (4.2) are satisfied, then Theorems 3.1–3.3 imply the following.

Corollary 4.1. *If the function $f(x,y)$ is bounded, i.e. $|f(x,y)| \leq K$, then the means $D_{a,c}^{b,d}(f;x,y)$ satisfy the inequality*

$$|D_{a,c}^{b,d}(f;x,y)| = \mathcal{O}\left(\left(1+\log\frac{b_n+a_n-1}{b_n-a_n+1}\right)\left(1+\log\frac{d_m+c_m-1}{d_m-c_m+1}\right)\right).$$

Corollary 4.2. *If the function $f(x,y)$ is continuous, then the estimate*

$$\begin{aligned}&|D_{a,c}^{b,d}(f;x,y)-f(x,y)|\\&=\mathcal{O}\left(\left(1+\log\frac{b_n+a_n-1}{b_n-a_n+1}\right)\left(1+\log\frac{d_m+c_m-1}{d_m-c_m+1}\right)\right)E_{a_n-1,c_m-1}\end{aligned}$$

holds true uniformly in all (x,y).

Corollary 4.3. *If $f(x,y)\in Lip(\alpha_1,\alpha_2)$, then for $0<\alpha_1,\alpha_2<1$,*

$$\begin{aligned}|D_{a,c}^{b,d}(f;x,y)-f(x,y)| = \mathcal{O}\Bigg(&\left(\frac{1}{(b_n-a_n+1)^{\alpha_1}}+\frac{1}{(b_n+a_n-1)^{\alpha_1}}\right)\\&\times\left(\frac{1}{(d_m-c_m+1)^{\alpha_2}}+\frac{1}{(d_m+c_m-1)^{\alpha_2}}\right)\Bigg),\end{aligned}$$

while for $\alpha_1=\alpha_2=1$,

$$\begin{aligned}&|D_{a,c}^{b,d}(f;x,y)-f(x,y)|\\&=\mathcal{O}\left(\left(\frac{1}{b_n-a_n+1}+\frac{1}{b_n+a_n-1}\right)\frac{1+\log(d_m-c_m+1)}{d_m-c_m+1}\right)\\&\quad+\mathcal{O}\left(\left(\frac{1}{d_m-c_m+1}+\frac{1}{d_m+c_m-1}\right)\frac{1+\log(b_n-a_n+1)}{b_n-a_n+1}\right)\\&\quad+\mathcal{O}\left(\frac{(1+\log(b_n-a_n+1))(1+\log(d_m-c_m+1))}{(b_n-a_n+1)(d_m-c_m+1)}\right),\end{aligned}$$

hold true uniformly in all (x,y).

Note that for $a_n=b_n=n$ and $c_m=d_m=m$, for all $n,m\geq 1$, we obtain

$$V_{n,a,b,\mu}^{m,c,d,\nu}(f;x,y)\equiv V_{m,n}^{\mu,\nu}(f;x,y),$$

where

$$V_{n,m}^{\mu,\nu}(f;x,y)=\frac{1}{\mu_n\nu_m}\sum_{k=n-\mu_n}^{n-1}\sum_{\ell=m-\mu_m}^{m-1}s_{k,\ell}(x,y),\quad (n,m\geq 1),$$

which are generalized de la Vallée Poussin means of the sequence $(S_{k,\ell}(x,y))$ generated by sequences (μ_n) and (ν_m). Indeed, the means $V_{n,m}^{\mu,\nu}(f;x,y)$ are two-dimensional extension of the generalized de la Vallée Poussin means introduced in [14] for single Fourier series, therefore Theorems 3.1–3.3 also contain two-dimensional counterparts of some results obtained in [14]. For example, Theorem 3.3 implies next corollary.

Corollary 4.4. *If $f(x,y) \in Lip(\alpha_1,\alpha_2)$, then for $0<\alpha_1,\alpha_2<1$,*

$$|V_{n,m}^{\mu,\nu}(f;x,y)-f(x,y)| = \mathcal{O}\left(\frac{1}{\mu_n^{\alpha_1}\nu_m^{\alpha_2}}\right),$$

while for $\alpha_1=\alpha_2=1$,

$$|V_{n,m}^{\mu,\nu}(f;x,y)-f(x,y)| = \mathcal{O}\left(\frac{1+\log(\mu_n\nu_m)}{\mu_n\nu_m}\right) + \mathcal{O}\left(\frac{(1+\log(\mu_n))(1+\log(\nu_m))}{\mu_n\nu_m}\right),$$

hold true uniformly in all (x,y).

Remark 4.1. *As a particular case, for $a_n=n+1$, $b_n=n+p$, $\mu_n=1$, $d_m=m+1$, $c_m=m+q$, $\nu_m=1$, and $p,q>0$, Theorem 1.1 and Theorem 3.2 induce the same result.*

Remark 4.2. *We note that for $a_n=b_n=n$, $\mu_n=1$, $c_m=d_m=m$, and $\nu_m=1$, $(\forall n,m\geq 1)$*

$$V_{n,a+1,b+1,1}^{m,c+1,d+1,1}(f;x,y) \equiv S_{n,m}(f;x,y)$$

and for $a_n=1$, $b_n=n$, $\mu_n=1$, $d_m=m$, $c_m=1$, $\nu_m=1$, $(\forall n,m\geq 1)$

$$V_{n+1,a,b+1,1}^{m+1,c,d+1,1}(f;x,y) \equiv \sigma_{n,m}(f;x,y).$$

Based on the latest remark and Theorem 3.2 we deduce:

Corollary 4.5. *If the function $f(x,y)$ is continuous, then the estimate*

$$|S_{n,m}(f;x,y)-f(x,y)| = \mathcal{O}\left((4+\ln(n+1))(4+\ln(m+1))\right)E_{n,m}(f),$$

holds true uniformly in all (x,y).

Remark 4.3. *The estimate obtained in Corollary 4.5 is the two-dimensional version of a result obtained in [13].*

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