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A NEW CLASS OF TRICOMI-LEGENDRE-HERMITE-BERNOULLI POLYNOMIALS

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Abstract. In this paper, we introduce a new class of generalized Tricomi-Legendre-Hermite-Bernoulli polynomials and consequently of Tricomi, Bernoulli, and Hermite polynomials and their generalizations start from suitable generating functions. These polynomials are used to connect Fubini-Legendre-Hermite and Bell-Legendre-Hermite polynomials and to find new representations. We derive some implicit summation formulae for these families of special functions by applying the generating functions.

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1. Introduction

It is known that there are a large number of researchers have studied to find computational formulas for finite sums and infinite sums. Because it is often not easy to find computational formulas for any finite sum, involving special functions, special numbers, special polynomials, sums of higher powers of binomial coefficients. In order to find any computational formula for finite sums, still many new methods and techniques have been developed, investigated in mathematics, and also in other applied sciences. We know that finite sums and their computational formulas are special important mathematical tools most used by mathematicians, physicists, engineers and other scientists. Applications of the generating functions for special numbers and polynomials, and finite sums with their computational formulas have also been given by many different methods (see [1-18]). With this motivation, by an approach to generating functions arising from p-adic integrals and special functions, our purpose and motivations in this paper are to develop a computational methodology by deriving computational formulas for certain class of finite sums. The provided computational methodology provides the researchers a variety of methods that they can use in different fields and many situations. In this study, we will construct generating functions that include special numbers and polynomials, and special finite sums. With the help of these generating functions and their functional equation, some new computational formulas will be given for these special finite sums. On the other hand, the main motivation of this paper is to construct and

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investigate generating functions, given by Theorem 2.1, Theorem 2.2 and Theorem 2.6, with a great deal distinct applications for the following numbers ${}_{s}{}^{c}{}_{H}B_n^{[\alpha,p,m-1]}(x,y,z)$, represented with certain finite sum.

The Tricomi functions of order n (for $t \neq 0$ and for all finite x) are defined by means of the generating function (see [18]):

$$e^{(t-\frac{x}{t})} = \sum_{n=-\infty}^{\infty} C_n(x)t^n. \tag{1.1}$$

The Tricomi functions are also defined by

$$C_n(x) = n! \sum_{n=-\infty}^{\infty} \frac{(-1)^r x^r}{r!(\nu+r+1)}, n = 0, 1, 2\cdots. \tag{1.2}$$

Dattoli *et al.* [7] introduced the 2-variable Legendre polynomials $S_n(x,y)$ and $R_n(x,y)$ as follows:

$$S_n(x,y) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{x^k y^{n-2k}}{(n-2k)!(k!)^2}, \tag{1.3}$$

and

$$R_n(x,y) = (n!)^2 \sum_{k=0}^{\infty} \frac{(-1)^{n-k} x^{n-k} y^k}{((n-2k)!)^2(k!)^2}, \tag{1.4}$$

respectively, and are related with the ordinary Legendre polynomials $P_n(x)$ (see [12]) as follows:

$$P_n(x) = S_n\left(-\frac{1-x^2}{4}, x\right) = R_n\left(\frac{1-x}{2}, \frac{1+x}{2}\right). \tag{1.5}$$

From (1.3) and (1.4), we have

$$S_n(x,0) = n! \frac{x^{[\frac{n}{2}]}}{((\frac{n}{2})!)^2},\ S_n(0,y) = y^n, \tag{1.6}$$

$$R_n(x,0) = (-x)^n,\ R_n(0,y) = y^n. \tag{1.7}$$

The 2-variable Legendre polynomials $S_n(x,y)$ and $R_n(x,y)$ are given by the generating functions (see [7]):

$$e^{yt}C_0(-xt^2) = \sum_{n=0}^{\infty} S_n(x,y)\frac{t^n}{n!}, \tag{1.8}$$

and

$$C_0(xt)C_0(-yt) = \sum_{n=0}^{\infty} R_n(x,y)\frac{t^n}{(n!)^2}, \tag{1.9}$$

respectively.

Analogous to the generalization of Legendre polynomials by Dattoli *et al.* [8] extensions of the family of the generalized 2-variable Legendre polynomials $S_n^p(x,y)$ and $R_n^p(x,y)$ are also presented for which the usual properties and representations are naturally and simply extended. Analogous to these extensions, the extended Legendre polynomials are defined by means of the generating functions:

$$e^{yt^p}C_0(-xt^2)=\sum_{n=0}^{\infty}S_n^p(x,y)\frac{t^n}{n!}, \tag{1.10}$$

and

$$C_0(xt)C_0(-yt^p)=\sum_{n=0}^{\infty}R_n^p(x,y)\frac{t^n}{(n!)^2}, \tag{1.11}$$

respectively.

The even and odd generalized Legendre polynomials of 2-variables from the corresponding formulae can be determined. For example

$$\frac{S_{2n}^2(x,y)}{(2n!)}=\sum_{k=0}^{n}\frac{x^{n-k}y^k}{[(n-k)!]^2k!},\ S_{2n+1}^2(x,y)=0,\ S_2^2(x,y)=2!(x+y)$$

and for $p=1$, (1.10) and (1.11) reduce to (1.8) and (1.9), respectively.

On expanding exponential function and using (1.2) in (1.10), we can write

$$\sum_{n=0}^{\infty}S_n^p(x,y)\frac{t^n}{n!}=\sum_{m=0}^{\infty}\frac{y^mt^{pm}}{m!}\sum_{k=0}^{\infty}\frac{(-1)^k(-xt^2)^k}{k!k!}.$$

On setting $2k+mp = n$ and comparing the coefficients of like powers of t, we find

$$S_n^p(x,y)=n!\sum_{k=0}^{[\frac{n}{2}]}\frac{x^ky^{(n-2k)/p}}{[\frac{(n-2k)}{p}]!(k!)^2}. \tag{1.12}$$

Similarly by exploiting the same procedure leading to equations (1.11) and (1.10) allows us to derive

$$R_n^p(x,y)=(n!)^2\sum_{k=0}^{[\frac{n}{p}]}\frac{(-x)^{n-kp}y^k}{[(n-kp)!]^2(k!)^2}. \tag{1.13}$$

For $p = 1$, (1.13) and (1.12) reduces to (1.8) and (1.9), respectively.

The exponential generating function for the geometric polynomials (also known as Fubini polynomials) $F_n(x)$ is given by [3] (see also [2]):

$$\frac{1}{1-x(e^t-1)}=\sum_{n=0}^{\infty}F_n(x)\frac{t^n}{n!}. \tag{1.14}$$

Geometric polynomials also have close relationship with Apostol-Bernoulli numbers $\beta_n(\lambda)$ and Euler numbers E_n as [3]:

$$\beta_n(\lambda)=\frac{n}{\lambda-1}F_n\left(\frac{\lambda}{1-\lambda}\right),\ \lambda\neq 1$$

$$E_n=F_n\left(\frac{-1}{2}\right),$$

where Apostol-Bernoulli numbers $\beta_n(\lambda)$ are defined by

$$\left(\frac{t}{\lambda e^t - 1}\right) = \sum_{n=0}^{\infty} \beta_n(\lambda)\frac{t^n}{n!}. \tag{1.15}$$

Ramanujan obtained the exponential generating function of the exponential (one variable Bell)polynomials $\phi_n(x)$ [see Berndt [1],Part 1,Chapter 3] as:

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} \phi_n(x)\frac{t^n}{n!}, \tag{1.16}$$

and proved the recurrence relation

$$\phi_{n+1}(x) = x(\phi_n(x) + \frac{d}{dx}\phi_n(x)).$$

Geometric and exponential polynomials are connected by the relation (see [3]):

$$F_n(x) = \int_0^{\infty} \phi_n(x)e^{-\lambda}d\lambda. \tag{1.17}$$

Next, we recall the higher-order Hermite polynomials, some times called the Kampé de Fériet polynomials or Gould-Hooper polynomials $H_n^{(m)}(x, y)$ are defined as (see [5, 6])

$$g_n^m(x, y) = H_n^{(m)}(x, y) = n!\sum_{k=0}^{[\frac{n}{m}]} \frac{y^k x^{n-mk}}{k!(n-mk)!}, \tag{1.18}$$

where m is a positive integer. These polynomials are specified by the generating function

$$e^{xt+yt^m} = \sum_{n=0}^{\infty} H_n^{(m)}(x, y)\frac{t^n}{n!}. \tag{1.19}$$

In particular, we note that

$$H_n^{(1)}(x =, y) = (x + y)^n,$$

$$H_n^{(2)}(x, y) = H_n(x, y),$$

where $H_n(x, y)$ denotes the 2-variables Hermite-Kampé de Fériet polynomials defined by the generating function

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y)\frac{t^n}{n!}. \tag{1.20}$$

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ (real or complex) of order α are usually defined by means of the following generating function (see [14-17]):

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \mid t \mid < 2\pi. \tag{1.21}$$

When $x = 0$, $B_n^{(\alpha)} = B_n^{(\alpha)}(0)$ are called the Bernoulli numbers of order α.

Pathan and Khan [17] introduced the generalized Bernoulli polynomials $B_n^{[\alpha,m-1]}, m \geq 1$, of order α are defined by means of the generating function defined in a suitable neighborhood of $t = 0$:

$$G^{[\alpha,m-1]}(x,t) = \left(\frac{t^m}{e^t - \sum_{m=0}^{h-1} \frac{t^h}{h!}}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{[\alpha,m-1]}(x)\frac{t^n}{n!}. \quad (1.22)$$

It may be remarked that since $G^{[\alpha,m-1]}(x,t) = A(t)e^{xt}$, the generalized polynomials $B_n^{[\alpha,m-1]}(x)$ belong to the class of Appell polynomials. For $m = 1$, (1.22) reduces to the generating function (1.21) of generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$. Generalized Bernoulli numbers $B_n^{[\alpha,m-1]}(x)$ are defined by setting $x = 0$ in (1.22) and assuming

$$B_n^{[\alpha,m-1]} = B_n^{[\alpha,m-1]}(0).$$

A generalization of Hermite and Bernoulli polynomials namely generalized Hermite-Bernoulli polynomials ${}_HB_n^{[\alpha,m-1]}, m \geq 1$ is recently given by Pathan and Khan (see [1]):

$$\left(\frac{t^m}{e^t - \sum\limits_{h=0}^{m-1} \frac{t^h}{h!}}\right)^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{[\alpha,m-1]}(x,y)\frac{t^n}{n!}. \quad (1.23)$$

For $\alpha = 1$, (1.23) reduces to another result of Pathan [14]. Further by taking $m = 1$, the result reduces to the known result of Dattoli *et al.* (see [6,p.386,Eq.(1.6)]):

$$\left(\frac{t}{e^t - 1}\right) e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n(x,y)\frac{t^n}{n!}. \quad (1.24)$$

For $j \geq 0$, the Stirling numbers of the second kind are defined by

$$x^j = \sum_{l=0}^{j} S_2(j,l)(x)_l, \text{(see [8-15])}. \quad (1.25)$$

From (1.25), we see that

$$\frac{1}{k!}(e^t - 1)^k = \sum_{j=k}^{\infty} S_2(j,k)\frac{t^j}{j!}. \quad (1.26)$$

The r-Stirling numbers of the second kind are defined by

$$e^{rt}\frac{1}{k!}(e^t - 1)^k = \sum_{j=k}^{\infty} S_r(j+r,k+r)\frac{t^j}{j!}, \text{(see [4])}. \quad (1.27)$$

Motivated by the work under progress in this direction, we introduce in this paper a new class of generalized Tricomi-Legendre-Hermite-Bernoulli polynomials ${}_{s^c H}B_n^{[\alpha,p,m-1]}(x,y,z)$. These polynomials generalize all the Bernoulli polynomials and their generalizations (1.19) to (1.22), Hermite polynomials of 2-variables Hermite polynomials.

The plan of this paper is as follows. In section 2, we introduce Tricomi-Legendre-Hermite-Bernoulli polynomials and explore their properties. In section 3, several summation formulas involving Legendre polynomials of 2-variables and Tricomi-Legendre-Hermite-Bernoulli polynomials are established. Finally, section 4 gives the connection between Bell and Fubini polynomials associated with Tricomi-Legendre-Hermite polynomials.

2. A new class of Tricomi-Legendre-Hermite-Bernoulli polynomials

The generalized Tricomi-Legendre-Hermite polynomials $s^cH_n^p(x,y,z)$ and $R^CH_n^p(x,y,z)$ defined by the following generating functions (see [15]):

$$e^{zt+yt^p}C_0(-xt^2)=\sum_{n=0}^{\infty}s^cH_n^p(x,y,z)\frac{t^n}{n!}, \tag{2.1}$$

and

$$e^{zt^p}C_0(xt)C_0(-yt)=\sum_{n=0}^{\infty}R^CH_n^p(x,y,z)\frac{t^n}{n!}, \tag{2.2}$$

respectively.

Definition 2.1. We introduce the generalized Tricomi-Legendre-Hermite-Bernoulli polynomials $s^c{}_HB_n^{[\alpha,p,m-1]}(x,y,z)$, for a real or complex parameter α defined in a suitable neighborhood of $t=0$ by means of the following generating function:

$$\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}e^{xt+yt^p}C_0(-zt^2)=\sum_{n=0}^{\infty}s^c{}_HB_n^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!}. \tag{2.3}$$

Remark 2.1. On taking $\alpha=0$ in (2.3), the generalized Tricomi-Legendre-Hermite-Bernoulli polynomials $s^c{}_HB_n^{[\alpha,p,m-1]}(x,y,z)$ of order α reduces to familiar generalized Tricomi-Legendre-Hermite polynomials $s^cH_n^p(x,y,z)$ in (2.1).

Corollary 2.1. On taking $m=\alpha=1$ and $p=2$ in (2.3), we get

$$\left(\frac{t}{e^t-1}\right)^{\alpha}e^{xt+yt^2}C_0(-zt^2)=\sum_{n=0}^{\infty}s^c{}_HB_n^{(2)}(x,y,z)\frac{t^n}{n!}. \tag{2.4}$$

Corollary 2.2. Letting $z=0$ and $p=2$ in (2.3), the result reduces to (see [16])

$$\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HB_n^{[\alpha,m-1]}(x,y)\frac{t^n}{n!}.$$

Theorem 2.1. The following formula holds true:

$$
{}_{s^c H}B_n^{[\alpha,p,m-1]}(x,y,z) = \sum_{r=0}^{n} \binom{n}{r} B_n^{[\alpha,m-1]} {}_{s^c}H_n^p(x,y,z). \tag{2.5}
$$

Proof. By using (2.1) and (2.3), we can easily obtain (2.5). So, we omit the proof of theorem. □

Theorem 2.2. The following explicit summation formula for Tricomi-Legendre-Hermite-Bernoulli polynomials ${}_{s^c H}B_n^{[\alpha,p,m-1]}(x,y,z)$ holds true:

$$
{}_{s^c H}B_n^{[\alpha,p,m-1]}(x,y,z) = \sum_{r=0}^{n} \binom{n}{r} B_{n-r}^{[\alpha,m-1]}(x) S_r^p(y,z). \tag{2.6}
$$

Proof. By using (1.10) and (2.3), we have

$$
\sum_{n=0}^{\infty} {}_{s^c H}B_n^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!} = \left(\frac{t^m}{e^t - \sum\limits_{h=0}^{m-1} \frac{t^h}{h!}} \right)^{\alpha} e^{xt+yt^p} C_0(-zt^2), \tag{2.7}
$$

$$
= \sum_{n=0}^{\infty} B_n^{[\alpha,m-1]}(x)\frac{t^n}{n!} \sum_{r=0}^{\infty} S_r^p(y,z)\frac{t^r}{r!}.
$$

Replacing n by $n-r$ in the r.h.s. of above equation, we get

$$
\sum_{n=0}^{\infty} {}_{s^c H}B_n^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{r=0}^{n} \binom{n}{r} B_{n-r}^{[\alpha,m-1]}(x) S_r^p(y,z)\frac{t^n}{n!}. \tag{2.8}
$$

Equating the coefficients of $\frac{t^n}{n!}$, we required at the desired result (2.6). □

Theorem 2.3. The following explicit summation formula for Tricomi-Legendre-Hermite-Bernoulli polynomials ${}_{s^c H}B_n^{[\alpha,p,m-1]}(x,y,z)$ holds true:

$$
{}_{s^c H}B_n^{[\alpha,p,m-1]}(x,y,z) = \sum_{r=0}^{n} \binom{n}{r} B_{n-r}^{[\alpha,m-1]} \sum_{l=0}^{[\frac{r}{2}]} g_{r-2l}^p(x,y)\frac{z^l}{(r-2l)!l!l!}. \tag{2.8}
$$

Proof. From equation (2.3), we have

$$
\sum_{n=0}^{\infty} {}_{s^c H}B_n^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!} = \left(\frac{t^m}{e^t - \sum\limits_{h=0}^{m-1} \frac{t^h}{h!}} \right)^{\alpha} e^{xt+yt^p} C_0(-zt^2), \tag{2.9}
$$

$$
= \sum_{n=0}^{\infty} B_n^{[\alpha,m-1]}\frac{t^n}{n!} \sum_{r=0}^{\infty} g_r^p(x,y)\frac{t^r}{r!} \sum_{l=0}^{\infty} \frac{z^l t^{2l}}{l!l!}
$$

$$
= \sum_{n=0}^{\infty} B_n^{[\alpha,m-1]}\frac{t^n}{n!} \sum_{r=0}^{\infty}\sum_{l=0}^{[\frac{r}{2}]} g_{r-2l}^p(x,y)\frac{z^l t^r}{(r-2l)!l!l!}.
$$

Replacing n by $n-r$ in the r.h.s. of above equation, we get

$$\sum_{n=0}^{\infty} {}_{s^c}{}_H B_n^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r} B_{n-r}^{[\alpha,m-1]}\sum_{l=0}^{[\frac{r}{2}]} g_{r-2l}^{p}(x,y)\frac{z^l}{(r-2l)!l!l!}\frac{t^n}{n!}. \tag{2.10}$$

On comparing the coefficients of $\frac{t^n}{n!}$, we arrive at the desired result (2.8). □

Theorem 2.4. The following explicit summation formula for Tricomi-Legendre-Hermite-Bernoulli polynomials ${}_{s^c}{}_H B_n^{[\alpha,p,m-1]}(x,y,z)$ holds true:

$${}_{s^c}{}_H B_n^{[\alpha,p,m-1]}(x,y,z) = \sum_{r=0}^{n}\binom{n}{r} B_{n-r}^{[\alpha,m-1]}\sum_{l=0}^{[\frac{r}{p}]} S_{r-pl}(x,y)\frac{y^l}{(r-pl)!l!}. \tag{2.11}$$

Proof. Using (1.8) and (2.3), we have

$$\sum_{n=0}^{\infty} {}_{s^c}{}_H B_n^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!} = \left(\frac{t^m}{e^t - \sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} e^{xt+yt^p} C_0(-zt^2), \tag{2.12}$$

$$= \sum_{n=0}^{\infty} B_n^{[\alpha,m-1]}\frac{t^n}{n!}\sum_{r=0}^{\infty} S_r(x,z)\frac{t^r}{r!}\sum_{l=0}^{\infty}\frac{y^l t^{pl}}{l!}$$

$$= \sum_{n=0}^{\infty} B_n^{[\alpha,m-1]}\frac{t^n}{n!}\sum_{r=0}^{\infty}\sum_{l=0}^{[\frac{r}{p}]} S_{r-pl}(x,z)\frac{y^l t^r}{(r-pl)!l!}.$$

Replacing n by $n-r$ in the r.h.s. of above equation, we get

$$\sum_{n=0}^{\infty} {}_{s^c}{}_H B_n^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r} B_{n-r}^{[\alpha,m-1]}\sum_{l=0}^{[\frac{r}{p}]} S_{r-pl}(x,y)\frac{y^l}{(r-pl)!l!}\frac{t^n}{n!}. \tag{2.13}$$

On comparing the coefficients of $\frac{t^n}{n!}$, we acquire the result (2.11). □

Theorem 2.5. The following explicit summation formula for Tricomi-Legendre-Hermite-Bernoulli polynomials ${}_{s^c}{}_H B_n^{[\alpha,p,m-1]}(x,y,z)$ holds true:

$$\sum_{r=0}^{n}\binom{n}{r} B_{n-r}^{[\alpha,m-1]} S_r(x,z) = \sum_{l=0}^{[\frac{l}{p}]}(-y)^l {}_{s^c}{}_H B_{n-pl}^{[\alpha,p,m-1]}(x,y,z)\frac{n!}{(n-pl)!l!}. \tag{2.14}$$

Proof. By (2.3), we have

$$\sum_{n=0}^{\infty} {}_{s^c}{}_H B_n^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!} = \left(\frac{t^m}{e^t - \sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} e^{xt+yt^p} C_0(-zt^2), \tag{2.15}$$

$$\sum_{n=0}^{\infty} B_n^{[\alpha,m-1]}\frac{t^n}{n!}\sum_{r=0}^{\infty} S_r(x,z)\frac{t^r}{r!} = e^{-yt^p}\sum_{n=0}^{\infty} s^c{}_H B_n^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!}.$$

By expanding the exponential function and using definition (1.8), we get

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r} B_{n-r}^{[\alpha,m-1]} S_r(x,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\frac{(-y)^l t^{pl}}{l!}\sum_{n=0}^{\infty} s^c{}_H B_n^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!}.$$

Replacing n by $n-pl$ in the r.h.s. of above equation, we have

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r} B_{n-r}^{[\alpha,m-1]} S_r(x,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{l=0}^{[\frac{l}{p}]}(-y)^l s^c{}_H B_{n-pl}^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{(n-pl)!l!}.$$

Finally, equating the coefficients of t^n, we find (2.14). □

Similarly, we consider another generalization of the family of Tricomi-Legendre-Hermite-Bernoulli polynomials $R^c{}_H B_n^{[\alpha,p,m-1]}(x,y,z)$:

$$\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} e^{zt^p} C_0(xt)C_0(-yt) = \sum_{n=0}^{\infty} R^c{}_H B_n^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!}. \quad (2.16)$$

Theorem 2.6. The following explicit summation formula for Tricomi-Legendre-Hermite-Bernoulli polynomials $R^c{}_H B_n^{[\alpha,p,m-1]}(x,y,z)$ holds true:

$$R^c{}_H B_n^{[\alpha,p,m-1]}(x,y,z) = \sum_{k=0}^{n}\binom{n}{r} B_{n-k}^{[\alpha,m-1]}\sum_{l=0}^{[\frac{k}{p}]} R_{k-pl}(x,y)\frac{z^l}{[(k-pl)!]^2 l!}. \quad (2.17)$$

Proof. Using (1.9) and (2.3), we have

$$\sum_{n=0}^{\infty} R^c{}_H B_n^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!} = \left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} e^{zt^p} C_0(xt)C_0(-yt) \quad (2.18)$$

$$= \sum_{n=0}^{\infty} B_n^{[\alpha,m-1]}\frac{t^n}{n!}\sum_{k=0}^{\infty} R_k(x,y)\frac{t^k}{k!k!}\sum_{l=0}^{\infty}\frac{z^l t^{pl}}{l!}$$

$$= \sum_{n=0}^{\infty} B_n^{[\alpha,m-1]}\frac{t^n}{n!}\sum_{k=0}^{\infty}\sum_{l=0}^{[\frac{k}{p}]} R_{k-pl}(x,y)\frac{z^l t^k}{[(k-pl)!]^2 l!}.$$

Replacing n by $n-k$ in the r.h.s of above equation, we have

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} B_{n-k}^{[\alpha,m-1]}\sum_{l=0}^{[\frac{k}{p}]} R_{k-pl}(x,y)\frac{z^l}{[(k-pl)!]^2 l!}\right)\frac{t^n}{n!}. \quad (2.19)$$

Therefore, by (2.18) and (2.19), we arrive at the desired result (2.17). □

Theorem 2.7. The following explicit summation formula for Tricomi-Legendre-Hermite-Bernoulli polynomials ${R^c}_H B_n^{[\alpha,p,m-1]}(x,y,z)$ holds true:

$$\sum_{l=0}^{n} B_{n-l}^{[\alpha,m-1]} \sum_{s=0}^{[\frac{l}{2}]} g_{l-2s}^{p}(z,y) R_s(-x,u) \frac{1}{(l-2s)!s!s!(n-l)!}$$

$$= \sum_{l=0}^{n} \sum_{r=0}^{l} B_{n-l}^{[\alpha,m-1]} S_{l-r}^{p}(x,z) S_r(u,y) \frac{1}{(l-r)!(n-l)!r!}. \tag{2.20}$$

Proof. Equation (2.3), we have

$$A(t) = \left(\frac{t^m}{e^t - \sum\limits_{h=0}^{m-1} \frac{t^h}{h!}} \right)^{\alpha} e^{yt+zt^p} C_0(-xt^2) C_0(-ut^2)$$

$$= \sum_{n=0}^{\infty} B_n^{[\alpha,m-1]} \frac{t^n}{n!} \sum_{l=0}^{\infty} g_l^p(z,y) \frac{t^l}{l!} \sum_{s=0}^{\infty} R_s(-x,u) \frac{t^{2s}}{s!s!}$$

$$= \sum_{n=0}^{\infty} B_n^{[\alpha,m-1]} \frac{t^n}{n!} \sum_{l=0}^{\infty} \sum_{s=0}^{[\frac{l}{2}]} g_{l-2s}^p(z,y) R_s(-x,u) \frac{t^l}{(l-2s)!s!s!}$$

$$A(t) = \sum_{n=0}^{\infty} \sum_{l=0}^{n} B_{n-l}^{[\alpha,m-1]} \sum_{s=0}^{[\frac{l}{2}]} g_{l-2s}^p(z,y) R_s(-x,u) \frac{t^n}{(l-2s)!s!s!(n-l)!}. \tag{2.21}$$

Similarly, $A(t)$ can be written as

$$A(t) = \left(\frac{t^m}{e^t - \sum\limits_{h=0}^{m-1} \frac{t^h}{h!}} \right)^{\alpha} e^{yt+zt^p} C_0(-xt^2) C_0(-ut^2)$$

$$= \sum_{n=0}^{\infty} B_n^{[\alpha,m-1]} \frac{t^n}{n!} \sum_{l=0}^{\infty} S_l^p(x,z) \frac{t^l}{l!} \sum_{r=0}^{\infty} S_r(u,y) \frac{t^r}{r!}$$

$$= \sum_{n=0}^{\infty} B_n^{[\alpha,m-1]} \frac{t^n}{n!} \sum_{l=0}^{\infty} \sum_{r=0}^{\infty} S_{l-r}^p(x,z) S_r(u,y) \frac{t^l}{(l-r)!r!}$$

$$A(t) = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \sum_{r=0}^{l} B_{n-l}^{[\alpha,m-1]} S_{l-r}^p(x,z) S_r(u,y) \frac{t^n}{(l-r)!(n-l)!r!}. \tag{2.22}$$

Therefore, by (2.21) and (2.22), we get assertion (2.20) of Theorem 2.7. □

3. Summation formula for the Tricomi-Legendre-Hermite-Bernoulli polynomials

In this section, we prove the following result involving Tricomi-Legendre-Hermite-Bernoulli polynomials ${}_{s^c H}B_n^{[\alpha,p,m-1]}(x,y,z)$ by using series rearrangement techniques. We begin the following theorem as follows.

Theorem 3.1. The following summation formula for Tricomi-Legendre-Hermite-Bernoulli polynomials ${}_{s^c H}B_n^{[\alpha,p,m-1]}(x,y,z)$ holds true:

$$ {}_{s^c H}B_{k+l}^{[\alpha,p,m-1]}(w,y,z) = \sum_{m,n=0}^{k,l} \begin{pmatrix} k \\ m \end{pmatrix} \begin{pmatrix} l \\ n \end{pmatrix} (w-x)^{m+n} {}_{s^c H}B_{k+l-m-n}^{[\alpha,p,m-1]}(x,y,z). \tag{3.1} $$

Proof. Replacing t by $t+u$ in (2.3) and using the formula (see [18]):

$$ \sum_{n=0}^{\infty} f(n)\frac{(x+y)^n}{n!} = \sum_{m,n=0}^{\infty} f(n+m)\frac{x^n y^m}{n!m!}, \tag{3.2} $$

in the resultant equation, we find the following generating function for the Tricomi-Legendre-Hermite-Bernoulli ${}_{s^c H}B_n^{[\alpha,p,m-1]}(x,y,z)$

$$ \left(\frac{(t+u)^m}{e^{t+u} - \sum\limits_{h=0}^{m-1} \frac{(t+u)^h}{h!}}\right)^{\alpha} e^{x(t+u)+y(t+u)^p} C_0(-z(t+u)^2) = \sum_{k,l=0}^{\infty} {}_{s^c H}B_{k+l}^{[\alpha,p,m-1]}(x,y,z)\frac{t^k}{k!}\frac{u^l}{l!}, $$

which can be written as

$$ \left(\frac{(t+u)^m}{e^{t+u} - \sum\limits_{h=0}^{m-1} \frac{(t+u)^h}{h!}}\right)^{\alpha} e^{y(t+u)^p} C_0(-z(t+u)^2) = e^{-x(t+u)} \sum_{k,l=0}^{\infty} {}_{s^c H}B_{k+l}^{[\alpha,p,m-1]}(x,y,z)\frac{t^k}{k!}\frac{u^l}{l!}. \tag{3.3} $$

Replacing x by w in equation (3.3) and equating the resultant equation itself, we find

$$ \sum_{k,l=0}^{\infty} {}_{s^c H}B_{k+l}^{[\alpha,p,m-1]}(w,y,z)\frac{t^k}{k!}\frac{u^l}{l!} = \exp[(w-x)(t+u)] \sum_{k,l=0}^{\infty} {}_{s^c H}B_{k+l}^{[\alpha,p,m-1]}(x,y,z)\frac{t^k}{k!}\frac{u^l}{l!}, \tag{3.4} $$

which on using formula (3.2) in the first summation on the r.h.s. becomes

$$ \sum_{k,l=0}^{\infty} {}_{s^c H}B_{k+l}^{[\alpha,p,m-1]}(w,y,z)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{m,n=0}^{\infty} \frac{(w-x)^{m+n} t^m u^n}{m!n!} \sum_{k,l=0}^{\infty} {}_{s^c H}B_{k+l}^{[\alpha,p,m-1]}(x,y,z)\frac{t^k}{k!}\frac{u^l}{l!}. \tag{3.5} $$

Now replacing k by $k-m$ and l by $l-n$ in the r.h.s. of equation (3.5) and using the result (see [18]):

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A(n,k) = \sum_{k=0}^{\infty}\sum_{n=0}^{k} A(n,k-n), \tag{3.6}$$

in the r.h.s. of equation (3.5), we find

$$\sum_{k,l=0}^{\infty} {}_{s^c}{}_H B_{k+l}^{[\alpha,p,m-1]}(w,y,z)\frac{t^k}{k!}\frac{u^l}{l!}$$

$$= \sum_{k,l=0}^{\infty}\sum_{m,n=0}^{k,l} \binom{k}{m}\binom{l}{n}(w-x)^{m+n} {}_{s^c}{}_H B_{k+l-m-n}^{[\alpha,p,m-1]}(x,y,z)\frac{t^k}{k!}\frac{u^l}{l!}. \tag{3.7}$$

Finally, equating the coefficients of like powers of t and u in equation (3.7), we get the assertion (3.1) of Theorem 3.1. □

Remark 3.1. Taking $l=0$ in assertion (3.1) of Theorem 3.1, we deduce the following consequence of Theorem 3.1.

Corollary 3.1. The following summation formula for Tricomi-Legendre-Hermite-Bernoulli polynomials ${}_{s^c}{}_H B_n^{[\alpha,p,m-1]}(x,y,z)$ holds true:

$${}_{s^c}{}_H B_k^{[\alpha,p,m-1]}(w,y,z) = \sum_{m=0}^{k}\binom{k}{m}(w-x)^m {}_{s^c}{}_H B_{k-m}^{[\alpha,p,m-1]}(x,y,z). \tag{3.8}$$

Remark 3.2. Replacing w by $w+x$ in equation (3.8), we obtain

$${}_{s^c}{}_H B_k^{[\alpha,p,m-1]}(w+x,y,z) = \sum_{m=0}^{k}\binom{k}{m}(w)^m {}_{s^c}{}_H B_{k-m}^{[\alpha,p,m-1]}(x,y,z). \tag{3.9}$$

Theorem 3.2. The following summation formula for Tricomi-Legendre-Hermite-Bernoulli polynomials ${}_{s^c}{}_H B_n^{[\alpha,p,m-1]}(x,y,z)$ holds true:

$${}_{s^c}{}_H B_n^{[\alpha,p,m-1]}(x,y,z) = \sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n}{k}\binom{n-k}{r}(-w)^k {}_H B_{n-r-k}^{[\alpha,p,m-1]}(x,y)S_r(w,z). \tag{3.10}$$

Proof. Using (2.3) and (1.8), we have

$$\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}\exp(xt+yt^p)\exp(wt)C_0(-zt^2) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} {}_H B_n^{[\alpha,p,m-1]}(x,y)S_r(w,z)\frac{t^{n+r}}{n!r!} \tag{3.11}$$

Replacing n by $n-r$ in the r.h.s. of equation (3.11) and then by using equation (3.6) in the r.h.s. of the resultant equation, we find

$$\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}\exp(xt+yt^p)\exp(wt)C_0(-zt^2)=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\left(\begin{array}{c}n\\r\end{array}\right){}_HB_{n-r}^{[\alpha,p,m-1]}(x,y)S_r(w,z)\frac{t^n}{n!},\tag{3.12}$$

which on shifting the $\exp(wt)$ to the r.h.s. and using series definition of exponential on the r.h.s., gives

$$\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}\exp(xt+yt^p)C_0(-zt^2)$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n}\left(\begin{array}{c}n\\r\end{array}\right)(-w)^k{}_HB_{n-r}^{[\alpha,p,m-1]}(x,y)S_r(w,z)\frac{t^{n+k}}{n!k!}.\tag{3.13}$$

Again, replacing n by $n-k$ in the r.h.s. of equation (3.13), we get

$$\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}\exp(xt+yt^p)C_0(-zt^2)$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{r=0}^{n-k}\left(\begin{array}{c}n\\k\end{array}\right)\left(\begin{array}{c}n-k\\r\end{array}\right)(-w)^k{}_HB_{n-r-k}^{[\alpha,p,m-1]}(x,y)S_r(w,z)\frac{t^n}{n!}.\tag{3.14}$$

Finally, using generating function (2.3) in the l.h.s. of equation (3.14) and then equating the coefficients of like powers of t in the resultant equation, we get assertion (3.10) of Theorem 3.2. □

Theorem 3.3. The following summation formula for Tricomi-Legendre-Hermite-Bernoulli polynomials ${}_{s^c{}_H}B_n^{[\alpha,p,m-1]}(x,y,z)$ holds true:

$$\sum_{s=0}^{[\frac{n}{2}]}B_{n-2s}^{[\alpha,p,m-1]}(y)R_s(-z,u)\frac{n!}{(n-2s)!s!s!}$$
$$=\sum_{k=0}^{n}\sum_{r=0}^{n-k}\left(\begin{array}{c}n-k\\r\end{array}\right)\left(\begin{array}{c}n\\k\end{array}\right)H_k^{(q)}(-x,-w)\,{}_{s^c{}_H}B_{n-r}^{[\alpha,p,m-1]}(x,y,z)S_r^q(u,w).\tag{3.15}$$

Proof. From (2.3) and (1.10), we have

$$\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}\exp(xt+yt^p)C_0(-zt^2)\exp(wt^m)C_0(-ut^2)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} {}_{s^c}{}_{H}B_n^{[\alpha,p,m-1]}(x,y,z)S_r^p(u,w)\frac{t^{n+r}}{n!r!}. \tag{3.16}$$

Replacing n by $n-r$ in the r.h.s. of equation (3.17) and then using the lemma (3.6) in the r.h.s. of resultant equation, we find

$$\left(\frac{t^m}{e^t - \sum_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} \exp(xt+yt^p)C_0(-zt^2)\exp(wt^q)C_0(-ut^2)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r} {}_{s^c}{}_{H}B_{n-r}^{[\alpha,p,m-1]}(x,y,z)S_r^q(u,w)\frac{t^n}{n!}, \tag{3.17}$$

which on shifting the $\exp(xt+wt^q)$ to the r.h.s. and using generating function (1.19) on the r.h.s., gives

$$\left(\frac{t^m}{e^t - \sum_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} \exp(yt^p)C_0(-zt^2)C_0(-ut^2)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r} H_k^{(q)}(-x,-w) {}_{s^c}{}_{H}B_{n-r}^{[\alpha,p,m-1]}(x,y,z)S_r^q(u,w)\frac{t^{n+k}}{n!k!}. \tag{3.18}$$

Again, replacing n by $n-k$ in the r.h.s. of equation (3.18), we get

$$\sum_{n=0}^{\infty}\sum_{s=0}^{[\frac{n}{2}]} B_{n-2s}^{[\alpha,p,m-1]}(y)R_s(-z,u)\frac{t^n}{(n-2s)!s!s!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{r=0}^{n-k}\binom{n-k}{r}\binom{n}{k} H_k^{(q)}(-x,-w) {}_{s^c}{}_{H}B_{n-r}^{[\alpha,p,m-1]}(x,y,z)S_r^q(u,w)\frac{t^n}{n!}. \tag{3.19}$$

Equating the coefficients of like powers of t in the resultant equation, we get assertion (3.15) of Theorem 3.3. □

Theorem 3.4. The following summation formula for Tricomi-Legendre-Hermite-Bernoulli polynomials ${}_{s^c}{}_{H}B_n^{[\alpha,p,m-1]}(x,y,z)$ holds true:

$${}_{s^c}{}_{H}B_n^{[\alpha,p,m-1]}(x+u,y+w,z) = \sum_{r=0}^{n}\binom{n}{r} {}_{s^c}{}_{H}B_{n-r}^{[\alpha,p,m-1]}(x,y,z)H_r^{(p)}(u,w). \tag{3.20}$$

Proof. Replacing x by $x+u$ and y by $y+w$ in (2.3) and using (1.19), we have

$$\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}\exp((x+u)t+(y+w)t^p)C_0(-zt^2)=\sum_{n=0}^{\infty}{}_{s^c H}B_n^{[\alpha,p,m-1]}(x+u,y+w,z)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}{}_{s^c H}B_n^{[\alpha,p,m-1]}(x,y,z)H_r^{(p)}(u,w)\frac{t^{n+r}}{n!r!}. \quad (3.21)$$

Replacing n by $n-r$ in the r.h.s. of equation (3.21) and then using the lemma (3.6) in the r.h.s. of resultant equation, we find

$$\sum_{n=0}^{\infty}{}_{s^c H}B_n^{[\alpha,p,m-1]}(x+u,y+w,z)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}{}_{s^c H}B_{n-r}^{[\alpha,p,m-1]}(x,y,z)H_r^{(p)}(u,w)\frac{t^n}{n!}. \quad (3.22)$$

Equating the coefficients of t on both sides, we acquire the result (3.20). □

4. Connection between Bell and Fubini polynomials

In this section, we derive some theorem connection between Bell and Fubini polynomials associated with Tricomi-Legendre-Hermite polynomials. We start following theorem as follows.

Theorem 4.1. The following summation formula for Tricomi-Legendre-Hermite-Bernoulli polynomials ${}_{s^c H}B_n^{[\alpha,p,m-1]}(x,y,z)$ holds true:

$$_{s^c \Phi H}{B_n}^{[\alpha,p,m-1]}(x,y,z)=\sum_{r=0}^{n}\binom{n}{r}\Phi_r(x)\,{}_{s^c H}B_{n-r}^{[\alpha,p,m-1]}(x,y,z). \quad (4.1)$$

Proof. Using equation (1.16) and (2.3), we have

$$\sum_{n=0}^{\infty}{}_{s^c \Phi H}{B_n}^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!}=\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}e^{x(t+e^t-1)+yt^p}C_0(-zt^2)$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\Phi_r(x)\,{}_{s^c H}B_{n-r}^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!}.$$

Which on identifying the coefficients of t on both sides, yields (4.1). □

Theorem 4.2. The following summation formula for Tricomi-Legendre-Hermite-Bernoulli polynomials ${}_{s^c H}B_n^{[\alpha,p,m-1]}(x,y,z)$ holds true:

$$_{s^c F H}{B_n}^{[\alpha,p,m-1]}(x,y,z)=\sum_{r=0}^{n}\binom{n}{r}F_r(x)\,{}_{s^c H}B_{n-r}^{[\alpha,p,m-1]}(x,y,z). \quad (4.2)$$

Proof. Using (2.3) and (1.14), we have

$$\sum_{n=0}^{\infty} {s^c}_F H^{B_n}{}^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!} = \left(\frac{t^m}{e^t - \sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} \frac{e^{xt+yt^p}}{1-w(e^t-1)}C_0(-zt^2)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\left(\begin{array}{c} n \\ r \end{array}\right) F_r(x) {s^c}_H B_{n-r}^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!}.$$

Which on equating the coefficients of t on both sides, gives the desired result (4.2). □

Theorem 4.3. The following summation formula for Tricomi-Legendre-Hermite-Bernoulli polynomials ${s^c}_H B_n^{[\alpha,p,m-1]}(x,y,z)$ holds true:

$${s^c}_F H^{B_n}{}^{[\alpha,p,m-1]}(x,y,z) = \sum_{r=0}^{n}\left(\begin{array}{c} n \\ r \end{array}\right) F^{B_{n-r}^{[\alpha,m-1]}}(w) s^c H_r^p(x,y,z). \qquad (4.3)$$

Proof. Using (1.14) and (2.3), we have

$$\sum_{n=0}^{\infty} {s^c}_F H^{B_n}{}^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!} = \left(\frac{t^m}{e^t - \sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} \frac{e^{xt+yt^p}}{1-w(e^t-1)}C_0(-zt^2)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\left(\begin{array}{c} n \\ r \end{array}\right) F^{B_{n-r}^{[\alpha,m-1]}}(w) s^c H_r^p(x,y,z)\frac{t^n}{n!}.$$

Equating the coefficients of t on both sides, we obtain (4.3). □

Theorem 4.4. The following summation formula for Tricomi-Legendre-Hermite-Bernoulli polynomials ${s^c}_H B_n^{[\alpha,p,m-1]}(x,y,z)$ holds true:

$${s^c}_F H^{B_n}{}^{[\alpha,p,m-1]}(x,y,z) = \sum_{l=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right) {s^c}_H B_{n-l}^{[\alpha,p,m-1]}(x,y,z)\sum_{k=0}^{l}\left\{\begin{array}{c} l \\ k \end{array}\right\} k! w^k. \qquad (4.4)$$

Proof. By (1.26) and (2.3), we have

$$\sum_{n=0}^{\infty} {s^c}_F H^{B_n}{}^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!} = \left(\frac{t^m}{e^t - \sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} \frac{e^{xt+yt^p}}{1-w(e^t-1)}C_0(-zt^2)$$

$$= \sum_{n=0}^{\infty} {s^c}_H B_n^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!}\sum_{k=0}^{\infty} w^k (e^t-1)^k$$

$$= \sum_{n=0}^{\infty} {s^c}_H B_n^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!}\sum_{l=0}^{\infty}\sum_{k=0}^{l}\left\{\begin{array}{c} l \\ k \end{array}\right\} k! w^k \frac{t^l}{l!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{k} {}_{s^c}{}_{H}B_{n-l}^{[\alpha,p,m-1]}(x,y,z)\sum_{k=0}^{l}\left\{\begin{matrix} l \\ k \end{matrix}\right\} k!w^k\right)\frac{t^n}{n!}.$$

On comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we get at the desired result (4.4). □

Theorem 4.5. The following summation formula for Tricomi-Legendre-Hermite-Bernoulli polynomials ${}_{s^c}{}_{H}B_n^{[\alpha,p,m-1]}(x,y,z)$ holds true:

$${}_{s^c}{}_{F}H^{B_n{}^{[\alpha,p,m-1]}}(x,y,z) = \sum_{l=0}^{n}\binom{n}{l} {}_{s^c}{}_{H}B_{n-l}^{[\alpha,p,m-1]}(x,y,z)\sum_{k=0}^{l} k!w^k S_r(l+r,k+r). \tag{4.5}$$

Proof. Replacing x by $x+r$ in (2.3) and using [4, p. 250, Theorem 16], we have

$$\sum_{n=0}^{\infty} {}_{s^c}{}_{F}H^{B_n{}^{[\alpha,p,m-1]}}(x,y,z)\frac{t^n}{n!} = \left(\frac{t^m}{e^t - \sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}\frac{e^{(x+r)t+yt^p}}{1-w(e^t-1)}C_0(-zt^2)$$

$$= \sum_{n=0}^{\infty} {}_{s^c}{}_{H}B_n^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!}e^{rt}\sum_{k=0}^{\infty} w^k(e^t-1)^k$$

$$= \sum_{n=0}^{\infty} {}_{s^c}{}_{H}B_n^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!}e^{rt}\sum_{k=0}^{\infty} w^k\sum_{l=k}^{\infty} k!S_2(l,k)\frac{t^l}{l!}$$

$$= \sum_{n=0}^{\infty} {}_{s^c}{}_{H}B_n^{[\alpha,p,m-1]}(x,y,z)\frac{t^n}{n!}\sum_{l=0}^{\infty}\sum_{k=0}^{l} k!w^k S_r(l+r,k+r)\frac{t^l}{l!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} {}_{s^c}{}_{H}B_{n-l}^{[\alpha,p,m-1]}(x,y,z)\sum_{k=0}^{l} k!w^k S_r(l+r,k+r)\right)\frac{t^n}{n!}.$$

On comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we get at the desired result (4.5). □

5. Conclusion.

6. Conclusion

Motivated by importance and potential for applications in certain problems in number theory, combinatorics, classical and numerical analysis and other fields of applied mathematics, various special numbers and polynomials, and their variants and generalizations have been extensively investigated (for example, see the references here and those cited therein). The results presented here, being very general, can be specialized to yield a large number of identities involving known or new simpler numbers and polynomials. For example, the case $\alpha = 0$ of the results presented here give the corresponding ones for the generalized Tricomi-Legendre-Hermite polynomials [15].

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