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COMBINATORIAL IDENTITIES DEGENERATE r-DOWLING-LAH POLYNOMIALS AND NUMBERS ARISING FROM DEGENERATE UMBRAL CALCULUS

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ABSTRACT. Whitney numbers and Dowling polynomials are applied in various fields as well as combinatorics and number theory, and many scholars have been studying them. Also, the study of degenerate special polynomials and numbers has been studied by many scholars in recent years. In this paper, we introduce the degenerate r-Dowling-Lah polynomials and numbers and derive various identities between the degenerate r-Dowling-Lah polynomials and special polynomials and numbers by using degenerate umbral calculus.

1. INTRODUCTION

The unsigned Lah number $L(n,k)$ counts the number of ways a set of n elements can be partitioned into k nonempty linearly ordered subsets and the r-Lah number $L_r(n,k)$ counts the number of partitions of a set with $n+r$ elements into $k+r$ ordered blocks such that r distinguished elements have to be in distinct ordered blocks and an explicit formula of $L_r(n,k)$ (see [7, 12, 13, 18, 29]).

$$L(n,k)=\sum_{j=k}^{n}S_1(n,j)S_2(j,k),\quad \text{and}\quad \sum_{i=0}^{n}S_1(n,i)S_2(i,k)=\delta_{n,k},$$

where $S_1(n,k)$ and $S_2(n,k)$ are the Stirling number of the first and second kind, respectively, and $\delta_{n,k}$ is the Kronecker's symbol.

Dowling [9] introduced the Whitney numbers $\omega_m(n,k)$ and $W_m(n,k)$ of the first and second kind associated with a finite geometric lattice $Q_n(G)$ out of a finite set of n elements and a finite group G of order m, depend only on the order m of G. We let the interested reader refer to [1, 2, 4, 5, 6, 9, 28] for the details on the construction of $Q_n(G)$ and its many properties.

As a generalization of the the Whitney numbers of the first and second kind associated with $Q_n(G)$, Mezö [28] introduced r-Whitey numbers of the first and second kind given by

$$m^n(x)_n=\sum_{k=0}^{n}w_{m,r}(n,k)(mx+r)^k,$$

and

$$(mx+r)^n=\sum_{k=0}^{n}W_{m,r}(n,k)m^k(x)_k,$$

respectively, where $(x)_0=1,\ (x)_n=x(x-1)(x-2)\cdots(x-n+1),\ (n\geq 1)$.

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Cheon-Jung [5] gave some combinatorial interpretations over the Dowling lattice for the r-Whitney numbers of the first kind and the second kind. From the point of view of

$$L(n,k)=\sum_{j=k}^{n} S_1(n,j)S_2(j,k),$$

for $n\geq k\geq 0$, the r-Whitney-Lah numbers $W^L_{m,r}(n,k)$ are given by

$$W^L_{m,r}(n,k)=\sum_{j=k}^{n} w_{m,r}(n,j)W_{m,r}(j,k), \quad \text{(see [5])}, \tag{1}$$

and the generating function of $W^L_{m,r}(n,k)$ is given by

$$\sum_{n=k}^{\infty} W^L_{m,r}(n,k)\frac{t^n}{n!}=(1-mt)^{-\frac{2r}{m}}\frac{1}{k!}\left(\frac{t}{1-mt}\right)^k, \quad \text{(see [5, 10, 11, 23])}. \tag{2}$$

Recently, Gyimesi-Nyule [10] gave new combinational interpretations for r-Whitney-Lah numbers $W^L_{m,r}(n,k)$ $(n\geq k\geq 0)$ which are the number of r-Whitney-Lah coloured partition of $\{1,2,\cdots,n+r\}$ with m colours into $k+r$ ordered block and $W^L_{m,0}(0,0)=1$, for $n\geq k\geq 0, r\geq 0, n+r\geq 1$ and $m\geq 1$.

For $n\geq k\geq 0$ and $m\geq 1$, it is easy to see that

$$m^n\left\langle\frac{x+2r}{m}\right\rangle_n=\sum_{k=0}^{n} W^L_{m,r}(n,k)m^k\left(\frac{x}{m}\right)_k, \quad \text{(see [10])},$$

where $<x>_0=1$, $<x>_n=x(x+1)(x+2)\cdots(x+n-1)$, $(n\geq 1)$.

In view of the Bell polynomials, the r-Dowling-Lah polynomials of the second kind are given by

$$D^L_{m,r}(n|x)=\sum_{k=0}^{n} W^L_{m,r}(n,k)x^k, \quad \text{(see [10, 11])}. \tag{3}$$

and $D^L_{m,r}(0|x)=1$.

When $x=1$, $D^L_{m,r}(n)=D^L_{m,r}(n|1)$ are called the r-Dowling-Lah numbers, $D^L_{m,r}(0)=1$ and $D^L_{m,r}(0|x)=1$.

From (2) and (3), the generating function of the r-Dowling-Lah polynomials

$$\sum_{n=0}^{\infty} D^L_{m,r}(n|x)\frac{t^n}{n!}=(1-mt)^{-\frac{2r}{m}}\exp\left(\frac{xt}{1-mt}\right), \quad \text{(see [11, 23])}. \tag{4}$$

Many mathematicians have been studied various degenerate versions of special polynomials and numbers not only in some arithmetic and combinatorial aspects but also in applications to differential equations, identities of symmetry and probability theory [4, 14-23, 25, 26], beginning with Carlitz's degenerate Bernoulli polynomials and the degenerate Euler polynomials. In addition, Rota [31] introduced umbral calculus which was based on modern concepts such as linear functionals, linear operators, adjoints, and so on. Umbral calculus is one of the important methods for obtaining the symmetric identities for the degenerate version of special numbers and polynomials [14, 17, 18, 19, 27, 30, 31]. Kim-Kim [23] introduced the degenerate Whitney numbers of the first kind and second kind. Recently, the author [18, 19] studied some identities of the degenerate r-Dowling polynomials of the first and second kind by using umbral calculus and the partial truncated polynomials of these two numbers respectively. In this paper, we introduce the degenerate r-Dowling-Lah polynomials and numbers and derive various identities between the degenerate r-Dowling-Lah polynomials $D^L_{m,r,\lambda}(n|x)$ and special polynomials and numbers by using degenerate umbral calculus. At the same time, we derived the inversion formulas of those identities. Some of them include the degenerate falling factorials, the degenerate Bernoulli polynomials and numbers, the degenerate Euler polynomials and numbers, the degenerate Daehee polynomials and numbers, the degenerate Bell polynomials, the degenerate r-Lah-Bell polynomials and numbers, etc. In particular, $\lim_{\lambda\to 0} D^{L}_{m,r,\lambda}(n|x) = D^{L}_{m,r}(n|x)$ are r-Dowling-Lah polynomials. We derive many combinatorial identities involving degenerate r-Dowling-Lah polynomials (r-Dowling-Lah polynomials).

Now, we introduce some definitions and properties needed in this paper.
We note that

$$(5) \qquad (1-t)^{-m} = \sum_{l=0}^{\infty} \binom{-m}{l} (-1)^l t^l = \sum_{l=0}^{\infty} < m >_l \frac{t^l}{l!}, \quad \text{(see [7])}.$$

The r-Lah numbers $L_r(n,k)$ are given by

$$(6) \qquad L_r(n,k) = \binom{n+2r-1}{k+2r-1} \frac{n!}{k!} \ (k \geq 0), \quad \text{(see [13, 19, 29])}.$$

When $r = 0$, $L(n,k) = L_0(n,k)$ are called the Lah numbers. From (6), the generating function of $L_r(n,k)$ is given by

$$(7) \qquad \frac{1}{k!} \left(\frac{1}{1-t} \right)^{2r} \left(\frac{t}{1-t} \right)^k = \sum_{n=k}^{\infty} L_r(n,k) \frac{t^n}{n!}, (k \geq 0), \quad \text{(see [13, 19, 29])}.$$

When $r = 0$, we have $\frac{1}{k!} \left(\frac{t}{1-t} \right)^k = \sum_{n=k}^{\infty} L(n,k) \frac{t^n}{n!}$.
Recently, Kim-Kim introduced

$$(8) \qquad \mathbf{B}^L_{r,n}(x) = \sum_{k=0}^{n} L_r(n,k) x^k, \quad \text{(see [13])},$$

and

$$(9) \qquad \left(\frac{1}{1-t} \right)^{2r} e^{x\left(\frac{1}{1-t}-1\right)} = \sum_{n=k}^{\infty} \mathbf{B}^L_{r,n}(x) \frac{t^n}{n!}, \ (k \geq 0), \quad \text{(see [13])}.$$

When $x = 1$, $\mathbf{B}^L_{r,n} = \mathbf{B}^L_{r,n}(1)$ are called r-extended Lah-Bell numbers respectively.
When $r = 0$, $\mathbf{B}^L_n(x) = \mathbf{B}^L_{0,n}(x)$ and $\mathbf{B}^L_n = \mathbf{B}^L_{0,n}(1)$ are called the Lah-Bell polynomials and numbers respectively.
For any $\lambda \in \mathbb{R}$,

$$(10) \qquad e^x_\lambda(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad \text{(see [4, 14-23])}.$$

where $(x)_{0,\lambda} = 1$ and $(x)_{n,\lambda} = x(x-\lambda)\cdots\big(x-(n-1)\lambda\big)$, $(n \geq 1)$.
The Bell polynomials are defined by the generating function

$$(11) \qquad e^{x(e^t-1)} = \sum_{n=0}^{\infty} bel_n(x) \frac{t^n}{n!}, \quad \text{(see [7, 8, 24, 25])}.$$

where $bel_n(x) = \sum_{k=0}^{n} S_2(n,k) x^k$.
For $n \geq 0$, the Stirling numbers of the first and second kind are defined by respectively

$$(12) \qquad (x)_n = \sum_{l=0}^{n} S_1(n,l) x^l \quad \text{and} \quad \frac{1}{k!} (\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k) \frac{t^n}{n!}, \quad \text{(see [3, 7])}.$$

and

$$x^n=\sum_{l=0}^{n}S_2(n,l)(x)_l \quad \text{and} \quad (e^t-1)^k=\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!}, \quad \text{(see [3, 7]).,} \tag{13}$$

where $(x)_0=1$ and $(x)_n=x(x-1)(x-2)\cdots(x-n+1)$.

The degenerate Stirling numbers of the first kind are given by

$$(x)_n=\sum_{l=0}^{n}S_{1,\lambda}(n,l)(x)_{l,\lambda} \quad \text{and} \quad \frac{1}{k!}\big(\log_\lambda(1+t)\big)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!},\ (n\geq 0) \quad \text{(see [15, 21]),} \tag{14}$$

The degenerate Stirling numbers of the second kind are given by

$$(x)_{n,\lambda}=\sum_{l=0}^{n}S_{2,\lambda}(n,l)(x)_l \quad \text{and} \quad \frac{1}{k!}\big(e_\lambda(t)-1\big)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!}, \quad (n\geq 0), \quad \text{(see [15, 21]),} \tag{15}$$

where $\log_\lambda(t)=\frac{1}{\lambda}(t^\lambda-1)$ is the compositional inverse of $e_\lambda(t)$ satisfying

$$\log_\lambda\big(e_\lambda(t)\big)=e_\lambda\big(\log_\lambda(t)\big)=t.$$

Recently, Kim-Kim [14] considered λ-linear functionals and λ-differential operators as follows:

For $f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\in\mathscr{F}$ and a fixed nonzero real number λ, each λ gives rise to the linear functional $\langle f(t)\mid\cdot\rangle_\lambda$ on $\mathbb{P}$, called λ-linear functional given by $f(t)$, which is defined by

$$\langle f(t)\mid (x)_{n,\lambda}\rangle_\lambda=a_n, \quad \text{for all } n\geq 0 \quad \text{(see [14]),} \tag{16}$$

and by (16), $\langle t^k\mid (x)_{n,\lambda}\rangle_\lambda=n!\delta_{n,k}, \quad (n,\ k\geq 0)$, where $\delta_{n,k}$ is the Kronecker's symbol.

For each $\lambda\in\mathbb{R}$, and each nonnegative integer k, they also defined the λ-differential operator on $\mathbb{P}$ by

$$(t^k)_\lambda(x)_{n,\lambda}=\begin{cases}(n)_k(x)_{n-k,\lambda}, & \text{if } k\leq n,\\ 0 & \text{if } k\geq n, \quad \text{(see [14]).}\end{cases} \tag{17}$$

Extending this linearly, any power series $f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\in\mathscr{F}$ yields the differential operator on $\mathbb{P}$, called λ-differential operator given by $f(t)$, which is defined by

$$(f(t))_\lambda(x)_{n,\lambda}=\sum_{k=0}^{n}\binom{n}{k}a_k(x)_{n-k,\lambda}, \tag{18}$$

and by linear extension.

Note that we should note that different λ's give rise to different linear functionals on $\mathbb{P}$ (see [14]). We also observe that, for $\lambda=0$, the linear functional $\langle f(t)\mid\cdot\rangle$ agrees with the one in $\langle f(t)\mid x^n\rangle=a_k$, $(k\geq 0)$.

The order $o(f(t))$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. The series $f(t)$ is called invertible if $o(f(t))=0$ and such series has a multiplicative inverse $1/f(t)$ of $f(t)$. $f(t)$ is called a delta series if $o(f(t))=1$ and it has a compositional inverse $\overline{f}(t)$ of $f(t)$ with $\overline{f}(f(t))=f(\overline{f}(t))=t$.

Let $f(t)$ and $g(t)$ be a delta series and an invertible series, respectively. Then there exists a unique sequences $s_{n,\lambda}(x)$ such that the orthogonality conditions

$$\langle g(t)\big(f(t)\big)^k \mid s_{n,\lambda}(x)\rangle_\lambda = n!\delta_{n,k}, \quad (n,k\geq 0) \quad \text{(see [14]).} \tag{19}$$

The sequences $s_{n,\lambda}(x)$ are called the λ-Sheffer sequence for $(g(t),f(t))$, which are denoted by $s_{n,\lambda}(x)\sim(g(t),f(t))_\lambda$.

The sequence $s_{n,\lambda}(x) \sim (g(t),f(t))_\lambda$ if and only if

$$\frac{1}{g\big(\overline{f}(t)\big)}e_\lambda^x\big(\overline{f}(t)\big) = \sum_{k=0}^{\infty}\frac{s_{k,\lambda}(x)}{k!}t^k \quad (n,k\geq 0) \quad \text{(see [14]),} \tag{20}$$

for all $y\in\mathbb{C}$.

Assume that for each $\lambda\in\mathbb{R}^*$ of the set of nonzero real numbers, $s_{n,\lambda}(x)$ is λ-Sheffer for $(g_\lambda(t),f_\lambda(t))$. Assume also that $\lim_{\lambda\to 0}f_\lambda(t)=f(t)$ and $\lim_{\lambda\to 0}g_\lambda(t)=g(t)$, for some delta series $f(t)$ and an invertible series $g(t)$. Then $\lim_{\lambda\to 0}\overline{f}_\lambda(t)=\overline{f}(t)$, where $\overline{f}(t)$ is the compositional inverse of $f(t)$ with $\overline{f}(f(t))=f(\overline{f}(t))=t$. We note that $\lim_{\lambda\to 0}s_{k,\lambda}(x)=s_k(x)$.

In this case, Kim-Kim called that the family $\{s_{n,\lambda}(x)\}_{\lambda\in\mathscr{R}-\{0\}}$ of λ-Sheffer sequences $s_{n,\lambda}$ are the degenerate (Sheffer) sequences for the Sheffer polynomial $s_n(x)$.

Let $s_{n,\lambda}(x)\sim(g(t),f(t))_\lambda$ and $r_{n,\lambda}(x)\sim(h(t),l(t))_\lambda$, $(n\geq 0)$. Then

$$\begin{aligned} & s_{n,\lambda}(x)=\sum_{k=0}^{n}\alpha_{n,k}r_{k,\lambda}(x), \quad (n\geq 0), \\ & \text{where} \quad \alpha_{n,k}=\frac{1}{k!}\left\langle\frac{h(\overline{f}(t))}{g(\overline{f}(t))}\big(l(\overline{f}(t))\big)^k \mid (x)_{n,\lambda}\right\rangle_\lambda, \quad (n,k\geq 0), \quad \text{(see [14]).}\end{aligned} \tag{21}$$

2. Degenerate r-Dowling-Lah polynomials and numbers

In this section, we introduce the degenerate r-Dowling-Lah polynomials and numbers and derive various properties for them.

From (4), we naturally consider a type of degenerate r-Dowling-Lah polynomials as following:

$$\sum_{n=0}^{\infty}D_{m,r,\lambda}^L(n|x)\frac{t^n}{n!}=(1-mt)^{-\frac{2r}{m}}e_\lambda^x\left(\frac{t}{1-mt}\right). \tag{22}$$

When $x=1$, $D_{m,r,\lambda}^L(n)=D_{m,r,\lambda}^L(n|1)$ are called the degenerate r-Dowling-Lah numbers.
When $\lim_{\lambda\to 0}D_{m,r,\lambda}^L(n|x)=D_{m,r}^L(n|x)$ are r-Dowling-Lah polynomials.

We observe that

$$\begin{aligned}\sum_{n=0}^{\infty}D_{m,r,\lambda}^L(n|x)\frac{t^n}{n!} &= e_\lambda^x\left(\frac{t}{1-mt}\right)(1-mt)^{-\frac{2r}{m}} \\ &= \sum_{k=0}^{\infty}\frac{(x)_{k,\lambda}}{k!}\left(\frac{t}{1-mt}\right)^k(1-mt)^{-\frac{2r}{m}} \\ &= \sum_{k=0}^{\infty}(x)_{k,\lambda}\sum_{n=k}^{\infty}W_{m,r}^L(n,k)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}W_{m,r}^L(n,k)(x)_{k,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{23}$$

By comparing the coefficients of the both side of (23), we get

$$D_{m,r,\lambda}^L(n|x)=\sum_{k=0}^{n}W_{m,r}^L(n,k)(x)_{k,\lambda} \quad \text{and} \quad D_{m,r,\lambda}^L(0|x)=1. \tag{24}$$

When $x=1$, we have

$$D^L_{m,r,\lambda}(n)=\sum_{k=0}^{n}W^L_{m,r}(n,k)(1)_{k,\lambda}. \tag{25}$$

From (25), we obtain the degenerate r-Whitney numbers $W^L_{m,r}(n,k)(1)_{k,\lambda}$.

First, we have the recurrence relation as following theorem.

Theorem 1. *For $n,r\in\mathbb{N}\cup\{0\}, m\in\mathbb{N}$ and $\lambda\in\mathbb{R}$, we have*

$D^L_{m,r,\lambda}(1|x)=2r+x$ and

$D_{m,r,\lambda}(n+1|x)=(mn+2r)D^L_{m,r,\lambda}(n|x)+x\sum_{l=0}^{n}\binom{n}{l}(n-l)!(m-\lambda x)^{n-l}D^L_{m,r,\lambda}(l|x)$ if $n\geq 1$.

Proof. Differentiating with respect to t in (22), the left side of (22) is

$$\begin{aligned}\frac{d}{dt}(1-mt)^{-\frac{2r}{m}}e^x_\lambda\left(\frac{t}{1-mt}\right)&=\frac{2r}{1-mt}(1-mt)^{-\frac{2r}{m}}e^x_\lambda\left(\frac{t}{1-mt}\right)\\&\quad+(1-mt)^{-\frac{2r}{m}}e^x_\lambda\left(\frac{t}{1-mt}\right)\frac{x}{(1-mt+\lambda t)(1-mt)}.\end{aligned} \tag{26}$$

On the other hand, the right side of (22) is

$$\frac{d}{dt}\sum_{n=0}^{\infty}D^L_{m,r,\lambda}(n|x)\frac{t^n}{n!}=\sum_{n=1}^{\infty}D^L_{m,r,\lambda}(n|x)\frac{t^{n-1}}{(n-1)!} \tag{27}$$

By (26) and (27), we have

$$\begin{aligned}(1-mt)\sum_{n=1}^{\infty}D^L_{m,r,\lambda}(n|x)\frac{t^{n-1}}{(n-1)!}&=2r\sum_{n=0}^{\infty}D^L_{m,r,\lambda}(n|x)\frac{t^n}{n!}\\&\quad+\frac{x}{1-(m-\lambda)t}\sum_{n=0}^{\infty}D^L_{m,r,\lambda}(n|x)\frac{t^n}{n!}.\end{aligned} \tag{28}$$

From (28), we get

$$\begin{aligned}&\sum_{n=0}^{\infty}D^L_{m,r,\lambda}(n+1|x)\frac{t^n}{n!}-m\sum_{n=1}^{\infty}nD^L_{m,r,\lambda}(n|x)\frac{t^n}{n!}\\&\quad=2r\sum_{n=0}^{\infty}D^L_{m,r,\lambda}(n|x)\frac{t^n}{n!}+x\sum_{j=0}^{\infty}(m-\lambda)^j j!\frac{t^j}{j!}\sum_{l=0}^{\infty}D_{m,r,\lambda}(l|x)\frac{t^l}{l!}.\end{aligned} \tag{29}$$

From (29), we have

$$\begin{aligned}\sum_{n=0}^{\infty}D^L_{m,r,\lambda}(n+1|x)\frac{t^n}{n!}&=m\sum_{n=1}^{\infty}nD^L_{m,r,\lambda}(n|x)\frac{t^n}{n!}+2r\sum_{n=0}^{\infty}D^L_{m,r,\lambda}(n|x)\frac{t^n}{n!}\\&\quad+x\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}(n-l)!(m-\lambda)^{n-l}D^L_{m,r,\lambda}(l|x)\frac{t^n}{n!}.\end{aligned} \tag{30}$$

By comparing the coefficients of both sides of (30), we have the desired result. □

From (22), we observe that the compositional inverse function of $f(t)=\frac{t}{1+mt}$ is $\bar{f}(t)=\frac{t}{1-mt}=\frac{1}{m}\left(\frac{1}{1-mt}-1\right)$.

Combining with (20) and (22), we have the degenerate Sheffer sequence

$$D^L_{m,r,\lambda}(n|x) \sim \left((1+mt)^{-\frac{2r}{m}},\ \frac{t}{1+mt} \right)_\lambda . \tag{31}$$

For $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$, $(n \geq 0)$, by (20), we have

$$\left\langle \frac{1}{g(\bar{f}(t))} e^x_\lambda(\bar{f}(t)) \middle| (x)_{n,\lambda} \right\rangle_\lambda = \sum_{k=0}^{\infty} s_{k,\lambda}(x) \frac{1}{k!} \langle t^k | (x)_{n,\lambda} \rangle_\lambda = s_{n,\lambda}(x), \quad (n \geq 0). \tag{32}$$

Theorem 2. *For $n, r \in \mathbb{N} \cup \{0\}, m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, we have*

$$D^L_{m,r,\lambda}(x) = \sum_{j=0}^{n} \sum_{l=0}^{n-j} \binom{n}{j} m^{n-l} \left\langle \frac{2r}{m} \right\rangle_j L(n-j,l)(x)_{l,\lambda}.$$

Proof. From (31) and (32), we observe that

$$\begin{aligned} D^L_{m,r,\lambda}(z) &= \left\langle (1-mt)^{-\frac{2r}{m}} e^z_\lambda \left(\frac{t}{1-mt} \right) \middle| (x)_{n,\lambda} \right\rangle_\lambda \\ &= \sum_{j=0}^{n} \left\langle \frac{2r}{m} \right\rangle_j m^j \binom{n}{j} \left\langle e^z_\lambda \left(\frac{t}{1-mt} \right) \middle| (x)_{n-j,\lambda} \right\rangle_\lambda \\ &= \sum_{j=0}^{n} \left\langle \frac{2r}{m} \right\rangle_j m^j \binom{n}{j} \left\langle \sum_{l=0}^{\infty} (z)_{l,\lambda} m^{-l} \sum_{s=l}^{\infty} L(s,l) m^s \frac{t^s}{s!} \middle| (x)_{n-j,\lambda} \right\rangle_\lambda \\ &= \sum_{j=0}^{n} \binom{n}{j} m^j \left\langle \frac{2r}{m} \right\rangle_j \left\langle \sum_{s=0}^{\infty} \sum_{l=0}^{s} (z)_{l,\lambda} m^{-l+s} L(s,l) \frac{t^s}{s!} \middle| (x)_{n-j,\lambda} \right\rangle_\lambda \\ &= \sum_{j=0}^{n} \sum_{l=0}^{n-j} \binom{n}{j} m^{n-l} \left\langle \frac{2r}{m} \right\rangle_j L(n-j,l)(z)_{l,\lambda}. \end{aligned} \tag{33}$$

From (33), we get the desire identity. □

From (19) and (31), we recall

$$D^L_{m,r,\lambda}(n|x) \sim \left((1+mt)^{-\frac{2r}{m}}, \frac{t}{1+mt} \right)_\lambda, \quad \text{and} \quad \left\langle g(t)(f(t))^k | s_{n,\lambda} \right\rangle_\lambda = n! \delta_{n,k}, \quad (n,k \geq 0).$$

For $n \in \mathbb{N} \cup \{0\}$ and $p(x) = \sum_{k=0}^{n} A_k D^L_{m,r,\lambda}(k|x) \in \mathbb{P}_m$.

$$\begin{aligned} \left\langle (1+mt)^{-\frac{2r}{m}} \left(\frac{t}{1+mt} \right)^k \middle| p(x) \right\rangle_\lambda &= \sum_{j=0}^{n} A_j \left\langle (1+mt)^{-\frac{2r}{m}} \left(\frac{t}{1+mt} \right)^k \middle| D^L_{m,r,\lambda}(k|x) \right\rangle_\lambda \\ &= \sum_{j=0}^{n} A_j j! \delta_{j,k} = k! A_k. \end{aligned} \tag{34}$$

From (34), we have

$$A_k = \frac{1}{k!} \left\langle (1+mt)^{-\frac{2r}{m}} \left(\frac{t}{1+mt} \right)^k \middle| p(x) \right\rangle_\lambda \tag{35}$$

By (35), we have

$$p(x) = \sum_{k=0}^{n} A_k D^L_{m,r,\lambda}(k|x), \quad \text{where} \quad A_k = \frac{1}{k!} \left\langle (1+mt)^{-\frac{2r}{m}} \left(\frac{t}{1+mt} \right)^k \middle| p(x) \right\rangle_\lambda . \tag{36}$$

Theorem 3. *For $n, r \in \mathbb{N} \cup \{0\}, m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, we have*

$$(x)_{n,\lambda} = \sum_{k=0}^{n} (-1)^{k+n} W_{m,r}^{L}(n,k) D_{m,r,\lambda}^{L}(k|x).$$

When $\lambda \to 0$, we have

$$x^n = \sum_{k=0}^{n} (-1)^{k+n} W_{m,r}^{L}(n,k) D_{m,r}^{L}(k|x).$$

Proof. To find the inversion formula of Theorem 2, let $p(x) = (x)_{n,\lambda}$. By (36),

$$(x)_{n,\lambda} = \sum_{k=0}^{n} A_k D_{m,r,\lambda}^{L}(k|x), \tag{37}$$

where, by (2),

$$\begin{aligned} A_k &= \frac{1}{k!}\left\langle (1+mt)^{-\frac{2r}{m}} \left(\frac{t}{1+mt}\right)^k \middle| (x)_{n,\lambda} \right\rangle_{\lambda} \\ &= \left\langle (1-m(-t))^{-\frac{2r}{m}} \frac{1}{k!} \left(\frac{-t}{1-m(-t)}\right)^k (-1)^k \middle| (x)_{n,\lambda} \right\rangle_{\lambda} \\ &= (-1)^k \left\langle \sum_{l=k}^{\infty} W_{m,r}^{L}(l.k) \frac{(-t)^l}{l!} \middle| (x)_{n,\lambda} \right\rangle_{\lambda} = (-1)^{k+n} W_{m,r}^{L}(n,k). \end{aligned} \tag{38}$$

Combining with (37) and (38), we have the desire result. □

Theorem 4. *For $n, r \in \mathbb{N} \cup \{0\}, m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, we have*

$$D_{m,r,\lambda}^{L}(n|x) = \sum_{k=0}^{n} \sum_{l=k}^{n} S_{2,\lambda}(l,k) W_{m,r}^{L}(n,l)(x)_k.$$

When $\lambda \to 0$, we have

$$D_{m,r}^{L}(n|x) = \sum_{k=0}^{n} \sum_{l=k}^{n} S_2(l,k) W_{m,r}^{L}(n,l)(x)_k.$$

Proof. From $e_{\lambda}^{x}(\log_{\lambda}(1+t)) = (1+t)^x = \sum_{l=0}^{\infty} (x)_l \frac{t^l}{l!}$, we have the degenerate Sheffer sequence

$$(x)_n \sim (1, e_{\lambda}(t) - 1)_{\lambda}. \tag{39}$$

From (21), (31) and (39), we get

$$D_{m,r,\lambda}^{L}(n|x) = \sum_{k=0}^{n} \alpha_{n,k} (x)_k, \tag{40}$$

where, by (2), (10)

$$\begin{aligned}
\alpha_{n,k} &= \frac{1}{k!}\left\langle (1-mt)^{-\frac{2r}{m}} \left(e_\lambda\left(\frac{t}{1-mt}\right)-1\right)^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\
&= \sum_{l=k}^{n} S_{2,\lambda}(l,k) \left\langle (1-mt)^{-\frac{2r}{m}} \frac{1}{l!}\left(\frac{t}{1-mt}\right)^l \middle| (x)_{n,\lambda} \right\rangle_\lambda \qquad (41)\\
&= \sum_{l=k}^{n} S_{2,\lambda}(l,k) \left\langle \sum_{j=l}^{\infty} W_{m,r}^{L}(j,l)\frac{t^j}{j!} \middle| (x)_{n,\lambda} \right\rangle_\lambda = \sum_{l=k}^{n} S_{2,\lambda}(l,k) W_{m,r}^{L}(n,l).
\end{aligned}$$

By (40) and (41), we obtain the desired identity. □

Theorem 5. *For $n,r \in \mathbb{N}\cup\{0\}, m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, we have*

$$(x)_n = \sum_{k=0}^{n}\sum_{j=0}^{n} (-1)^{k+j} S_{1,\lambda}(n,j) W_{m,r}^{L} D_{m,r,\lambda}^{L}(k|x)$$

When $\lambda \to 0$, we have

$$(x)_n = \sum_{k=0}^{n}\sum_{j=0}^{n} (-1)^{k+j} S_1(n,k) W_{m,r}^{L}(j,k) D_{m,r}^{L}(k|x).$$

Proof. To find the inversion formula of Theorem 4, let $p(x) = (x)_n$. By (36), we get

$$(x)_n = \sum_{k=0}^{n} A_k D_{m,r,\lambda}^{L}(k|x), \qquad (42)$$

where, From (2), (12) and (14),

$$\begin{aligned}
A_k &= \frac{1}{k!}\left\langle (1+mt)^{-\frac{2r}{m}} \left(\frac{t}{1+mt}\right)^k \middle| (x)_n \right\rangle_\lambda \\
&= (-1)^k \left\langle \sum_{l=k}^{\infty} W_{m,r}^{L}(l,k)\frac{(-1)^l}{l!} t^l \middle| (x)_n \right\rangle_\lambda \\
&= \sum_{l=k}^{n} \frac{(-1)^{k+l}}{l!} W_{m,r}^{L}(l,k) \left\langle t^l \middle| \sum_{j=0}^{n} S_{1,\lambda}(n,j)(x)_{j,\lambda} \right\rangle_\lambda \qquad (43)\\
&= \sum_{l=k}^{n} \frac{(-1)^{k+l}}{l!} W_{m,r}^{L}(l,k) \sum_{j=0}^{n} S_{1,\lambda}(n,j) \left\langle t^l \middle| (x)_{j,\lambda} \right\rangle_\lambda \\
&= \sum_{j=0}^{n} S_{1,\lambda}(n,j) \frac{(-1)^{k+j}}{j!} W_{m,r}^{L}(j,k) j! = \sum_{j=0}^{n} (-1)^{k+j} S_{1,\lambda}(n,j) W_{m,r}^{L}(j,k).
\end{aligned}$$

Combining with (42) and (43), we have the desired identity. □

Naturally, from (6), Kim-Lee defined a degenerate r-extended Lah-Bell polynomials by

$$\left(\frac{1}{1-t}\right)^{2r} e_\lambda^x\left(\frac{t}{1-t}\right) = \sum_{n=0}^{\infty} \mathbf{B}_{r,n,\lambda}^{L}(x)\frac{t^n}{n!} \quad \text{(see [17])}. \qquad (44)$$

When $x = 1$, $\mathbf{B}_{r,n,\lambda}^{L} := \mathbf{B}_{r,n,\lambda}^{L}(1)$ is called the n-th degenerate r-extended Lah-Bell number. When $\lambda \to 0$, $\lim_{\lambda\to 0} \mathbf{B}_{r,n,\lambda}^{L} = \mathbf{B}_{r,n}^{L}$ is the n-th r-extended Lah-Bell number.

Theorem 6. *For $n,r \in \mathbb{N}\cup\{0\}, m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, we have*

$$D^{L}_{m,r,\lambda}(n|x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\binom{n}{l}(m-1)^{l-k}m^{m-l}\left\langle\frac{2r}{m}-2r\right\rangle_{n-l}L_r(l,k)\right)\mathbf{B}^{L}_{r,k,\lambda}(x).$$

When $\lambda \to 0$, we have

$$D^{L}_{m,r}(n|x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\binom{n}{l}(m-1)^{l-k}m^{m-l}\left\langle\frac{2r}{m}-2r\right\rangle_{n-l}L_r(l,k)\right)\mathbf{B}^{L}_{r,k}(x).$$

Proof. From (20) and (44), we observe the degenerate Sheffer sequences of $B^{L}_{r,n,\lambda}(x)$ as follows:

$$\mathbf{B}^{L}_{r,n,\lambda}(x) \sim \left(\left(\frac{1}{1+t}\right)^{2r},\ \frac{t}{1+t}\right)_{\lambda}. \tag{45}$$

From (21), (31) and (45), we get

$$D^{L}_{m,r,\lambda}(n|x) = \sum_{k=0}^{n}\alpha_{n,k}\mathbf{B}^{L}_{r,k,\lambda}(x), \tag{46}$$

where, by (5) and (7),

$$\begin{aligned}
\alpha_{n,k} &= \frac{1}{k!}\left\langle(1-mt)^{-\frac{2r}{m}}\left(\frac{1-mt}{1-(m-1)t}\right)^{2r}\left(\frac{t}{1-(m-1)t}\right)^{k}\middle|(x)_{n,\lambda}\right\rangle_{\lambda}\\
&= \left\langle(1-mt)^{-\frac{2r}{m}+2r}\left(\frac{1}{1-(m-1)t}\right)^{2r}\left(\frac{1}{m-1}\right)^{k}\left(\frac{(m-1)t}{1-(m-1)t}\right)^{k}\middle|(x)_{n,\lambda}\right\rangle_{\lambda}\\
&= (m-1)^{-k}\sum_{l=k}^{n}\binom{n}{l}(m-1)^{l}L_r(l,k)\left\langle(1-mt)^{-\frac{2r}{m}+2r}\middle|(x)_{n-l,\lambda}\right\rangle_{\lambda}\\
&= \sum_{l=k}^{n}\binom{n}{l}(m-1)^{l-k}L_r(l,k)m^{m-l}\left\langle\frac{2r}{m}-2r\right\rangle_{n-l}.
\end{aligned} \tag{47}$$

Combining with (46) and (47), we get the desired result. □

Theorem 7. *For $n,r \in \mathbb{N}\cup\{0\}, m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, we have*

$$\mathbf{B}^{L}_{r,n,\lambda}(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\frac{(-1)^{l+k}}{l!}W^{L}_{m,r}(l,k)L_r(n,l)\right)D^{L}_{m,r,\lambda}(k|x).$$

When $\lambda \to 0$, we have

$$\mathbf{B}^{L}_{r,n}(x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\frac{(-1)^{l+k}}{l!}W^{L}_{m,r}(l,k)L_r(n,l)\right)D^{L}_{m,r}(k|x).$$

Proof. To find the inverse formula of Theorem 6, let $p(x) = \mathbf{B}^{L}_{r,n,\lambda}(x)$.

From $\sum_{n=0}^{\infty}\mathbf{B}^{L}_{r,n,\lambda}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{n}L_r(n,r)(x)_{j,\lambda}\frac{t^n}{n!}$, we have $\mathbf{B}^{L}_{r,n,\lambda}(x) = \sum_{j=0}^{n}L_r(n,r)(x)_{j,\lambda}$.
By (36) and (44), we get

$$\mathbf{B}^{L}_{r,n,\lambda}(x) = \sum_{k=0}^{n}A_kD^{L}_{m,r,\lambda}(k|x). \tag{48}$$

where, from (2) and (7),

$$
\begin{aligned}
A_k &= \frac{1}{k!}\left\langle (1+mt)^{-\frac{2r}{m}}\left(\frac{t}{1+mt}\right)^k \middle| \mathbf{B}^L_{r,n,\lambda}(x)\right\rangle_\lambda \\
&= \sum_{l=k}^{n} \frac{(-1)^{l+k}}{l!} W^L_{m,r}(l,k)\left\langle t^l \middle| \sum_{j=0}^{n} L_r(n,j)(x)_{j,\lambda}\right\rangle_\lambda \\
&= \sum_{l=k}^{n} \frac{(-1)^{l+k}}{l!} W^L_{m,r}(l,k)\sum_{j=0}^{n} L_r(n,j)\left\langle t^l | (x)_{j,\lambda}\right\rangle_\lambda \\
&= \sum_{l=k}^{n} \frac{(-1)^{l+k}}{l!} W^L_{m,r}(l,k)L_r(n,l).
\end{aligned} \tag{49}
$$

Combining with (48) and (49), we have the desired result. □

The degenerate Bernoulli polynomials are given by the generating function to be

$$
\frac{t}{e_\lambda(t)-1}e^x_\lambda(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [14, 15, 17])}, \tag{50}
$$

When $x=0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ $(n \geq 0)$, are called the degenerate Bernoulli numbers.
From (50), we obtain

$$
\beta_{n,\lambda}(x) = \sum_{l=0}^{n} \binom{n}{l} \beta_{l,\lambda}(x)_{n-l,\lambda}, \quad \text{(see [14, 15, 17])}, \tag{51}
$$

Theorem 8. *For $n, r \in \mathbb{N}\cup\{0\}, m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, we have*

$$
\begin{aligned}
D^L_{m,r,\lambda}(m|x) &= \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\sum_{j=0}^{n-l}\sum_{d=0}^{n-l-j}\binom{n}{l}\binom{n}{j}\frac{(1)_{d+1,\lambda}}{d+1}m^{n-k-d}\left\langle\frac{2r}{m}\right\rangle_j L(l.k)L(n-l-j,d)\right)B_{k,\lambda}(x) \\
&= \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\sum_{d=0}^{n-l}\binom{n}{l}\frac{(1)_{d+1,\lambda}}{d+1}m^{l-k}L(l,k)W^L_{m,r}(n-l,d)\right)B_{k,\lambda}(x).
\end{aligned}
$$

When $\lambda \to 0$, we have

$$
D^L_{m,r,\lambda}(m|x) = \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\sum_{j=0}^{n-l}\sum_{d=0}^{n-l-j}\binom{n}{l}\binom{n}{j}\frac{(1)_{d+1,\lambda}}{d+1}m^{n-k-d}\left\langle\frac{2r}{m}\right\rangle_j L(l.k)L(n-l-j,d)\right)B_k(x).
$$

Proof. Form (50), we have

$$
B_{n,\lambda}(x) \sim \left(\frac{e_\lambda(t)-1}{t},\, t\right)_\lambda. \tag{52}
$$

From (21), (31) and (52), we have

$$
D^L_{m,r,\lambda}(n|x) = \sum_{k=0}^{n} \alpha_{n,k}B_{k,\lambda}(x), \tag{53}
$$

where, by (5), (7) and (10),

$$\begin{aligned}
\alpha_{n,k} &= \frac{1}{k!}\left\langle (1-mt)^{-\frac{2r}{m}} \frac{e_\lambda\left(\frac{t}{1-mt}\right)-1}{\frac{t}{1-mt}}\left(\frac{t}{1-mt}\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
&= \left\langle (1-mt)^{-\frac{2r}{m}}\left(e_\lambda\left(\frac{t}{1-mt}\right)-1\right)\left(\frac{t}{1-mt}\right)^{-1} \middle| \left(\frac{1}{k!m^k}\left(\frac{mt}{1-mt}\right)^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
&= \sum_{l=k}^{n} L(l,k)m^{l-k}\binom{n}{l}\left\langle \left(e_\lambda\left(\frac{t}{1-mt}\right)-1\right)\left(\frac{t}{1-mt}\right)^{-1} \middle| \left((1-mt)^{-\frac{2r}{m}}\right)_\lambda (x)_{n-l,\lambda}\right\rangle_\lambda \qquad (54)\\
&= \sum_{l=k}^{n}\binom{n}{l}m^{l-k}L(l,k)\sum_{j=0}^{n-l}\left\langle\frac{2r}{m}\right\rangle_j m^j\binom{n}{j}\left\langle \sum_{s=0}^{\infty}\sum_{d=0}^{s}\frac{(1)_{d+1,\lambda}}{d+1}m^{s-d}L(s,d)\frac{t^s}{s!} \middle| (x)_{n-l-j,\lambda}\right\rangle_\lambda \\
&= \sum_{l=k}^{n}\sum_{j=0}^{n-l}\sum_{d=0}^{n-l-j}\binom{n}{l}\binom{n}{j}\frac{(1)_{d+1,\lambda}}{d+1}m^{n-k-d}\left\langle\frac{2r}{m}\right\rangle_j L(l,k)L(n-l-j,d).
\end{aligned}$$

Combining with (53) and (54), we get the first equality.

In another way of (54). From (2), (7) and (10), we have

$$\begin{aligned}
\alpha_{n,k} &= \frac{1}{k!}\left\langle (1-mt)^{-\frac{2r}{m}}\left(e_\lambda\left(\frac{t}{1-mt}\right)-1\right)\left(\frac{t}{1-mt}\right)^{-1}\left(\frac{t}{1-mt}\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
&= \sum_{l=k}^{n}\binom{n}{l}m^{l-k}L(l,k)\left\langle (1-mt)^{-\frac{2r}{m}}\sum_{d=0}^{\infty}(1)_{d+1,\lambda}\frac{1}{(d+1)!}\left(\frac{t}{1-mt}\right)^d \middle| (x)_{n-l,\lambda}\right\rangle_\lambda \\
&= \sum_{l=k}^{n}\binom{n}{l}m^{l-k}L(l,k)\left\langle \sum_{d=0}^{\infty}\frac{(1)_{d+1,\lambda}}{d+1}\sum_{s=d}^{\infty}W_{m,r}^L(s,d)\frac{t^s}{s!} \middle| (x)_{n-l,\lambda}\right\rangle_\lambda \qquad (55)\\
&= \sum_{l=k}^{n}\binom{n}{l}m^{l-k}L(l,k)\left\langle \sum_{s=0}^{\infty}\sum_{d=0}^{s}\frac{(1)_{d+1,\lambda}}{d+1}W_{m,r}^L(s,d)\frac{t^s}{s!} \middle| (x)_{n-l,\lambda}\right\rangle_\lambda \\
&= \sum_{l=k}^{n}\sum_{d=0}^{n-l}\binom{n}{l}\frac{(1)_{d+1,\lambda}}{d+1}m^{l-k}L(l,k)W_{m,r}^L(n-l,d).
\end{aligned}$$

From (53) and (55), we have the second equality.

□

Theorem 9. *For $n,r\in\mathbb{N}\cup\{0\}, m\in\mathbb{N}$ and $\lambda\in\mathbb{R}$, we have*

$$B_{n,\lambda}(x) = \sum_{k=0}^{n}\sum_{j=0}^{n}\binom{n}{j}(-1)^{k+n-j}W_{m,r}^L(n-j,k)B_{j,\lambda}D_{m,r,\lambda}^L(k|x).$$

When $\lambda\to 0$, we have

$$B_n(x) = \sum_{k=0}^{n}\sum_{j=0}^{n}\binom{n}{j}(-1)^{k+n-j}W_{m,r}^L(n-j,k)B_{j,\lambda}D_{m,r}^L(k|x).$$

Proof. To find the inversion formula of Theorem 8, let $p(x)=B_{n,\lambda}(x)$. By (36), we have

$$B_{n,\lambda}(x) = \sum_{k=0}^{n}A_k D_{m,r,\lambda}^L(k|x), \quad (n\geq 0), \qquad (56)$$

where, from (2) and (51),

$$(57)\quad \begin{aligned} A_k &= \frac{1}{k!}\left\langle (1+mt)^{-\frac{2r}{m}}\left(\frac{t}{1+mt}\right)^k \middle| B_{n,\lambda}(x)\right\rangle_\lambda \\ &= (-1)^k\left\langle \sum_{l=k}^{\infty}(-1)^l W^L_{m,r}(l,k)\frac{1}{l!}t^l \middle| B_{n,\lambda}(x)\right\rangle_\lambda \\ &= \sum_{l=k}^{n}(-1)^{k_l} W^L_{m,r}(l,k)\frac{1}{l!}\left\langle t^l \middle| \sum_{j=0}^{n}\binom{n}{j}B_{j,\lambda}(x)_{n-j,\lambda}\right\rangle_\lambda \\ &= \sum_{l=k}^{n}(-1)^{k+l} W^L_{m,r}(l,k)\frac{1}{l!}\sum_{j=0}^{n}\binom{n}{j}B_{j,\lambda}\langle t^l|(x)_{n-j,\lambda}\rangle_\lambda \\ &= (-1)^{k+n-j}W^L_{m,r}(n-j,k)\frac{1}{(n-j)!}\sum_{j=0}^{n}\binom{n}{j}B_{j,\lambda}(n-j)! \\ &= \sum_{j=0}^{n}\binom{n}{j}(-1)^{k+n-j}W^L_{m,r}(n-j,k)B_{j,\lambda}. \end{aligned}$$

Combining with (56) and (57), we have the desired identity. □

The degenerate Euler polynomials are given by the generating function to be

$$(58)\quad \frac{2}{e_\lambda(t)+1}e^x_\lambda(t) = \sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [17, 18])}.$$

When$x=0$, $E_{n,\lambda}=E_{n,\lambda}(0)$ $(n\geq 0)$, are called the degenerate Euler numbers.
From (58), we obtain

$$(59)\quad E_{n,\lambda}(x) = \sum_{l=0}^{n}\binom{n}{l}E_{l,\lambda}(x)_{n-l,\lambda}, \quad \text{(see [17, 18])},$$

Theorem 10. *For $n,r\in\mathbb{N}\cup\{0\}, m\in\mathbb{N}$ and $\lambda\in\mathbb{R}$, we have*

$$D^L_{m,r,\lambda}(n|x) = \sum_{k=0}^{n}\left(\frac{1}{2}\sum_{l=k}^{n}\binom{n}{l}m^{l-k}L(l,k)\left\{D^L_{m,r,\lambda}(n-l)+m^{n-l}\left\langle\frac{2r}{m}\right\rangle_{n-k}\right\}\right)E_{k,\lambda}(x).$$

When $\lambda\to 0$, we have

$$D^L_{m,r}(n|x) = \sum_{k=0}^{n}\left(\frac{1}{2}\sum_{l=k}^{n}\binom{n}{l}m^{l-k}L(l,k)\left\{D^L_{m,r,\lambda}(n-l)+m^{n-l}\left\langle\frac{2r}{m}\right\rangle_{n-k}\right\}\right)E_{k,\lambda}(x).$$

Proof. From (58), we have the Sheffer sequence

$$(60)\quad E_{n,\lambda}(x) \sim \left(\frac{e_\lambda(t)+1}{2}, t\right)_\lambda.$$

From (21), (31) and (60), we have

$$(61)\quad D^L_{m,r,\lambda}(n|x) = \sum_{k=0}^{n}\alpha_{n,k}E_{k,\lambda}(x),$$

where, by (5), (7) and (10)

$$\begin{aligned}
\alpha_{n,k} &= \frac{1}{k!2}\left\langle (1-mt)^{-\frac{2r}{m}}\left(e_\lambda\left(\frac{t}{1-mt}\right)+1\right)\left(\frac{t}{1-mt}\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
&= \frac{1}{2m^k}\left\langle (1-mt)^{-\frac{2r}{m}}\left(e_\lambda\left(\frac{t}{1-mt}\right)+1\right)\sum_{l=k}^{\infty} L(l,k)m^l\frac{1}{l!}t^l \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
&= \frac{1}{2m^k}\sum_{l=k}^{n}\binom{n}{l}m^l L(l,k)\left\langle (1-mt)^{-\frac{2r}{m}}e_\lambda\left(\frac{t}{1-mt}\right)+(1-mt)^{-\frac{2r}{m}} \middle| (x)_{n-l,\lambda}\right\rangle_\lambda \\
&= \frac{1}{2}\sum_{l=k}^{n} m^{l-k}L(l,k)\left\{D^L_{m,r,\lambda}(n-l)+m^{n-l}\left\langle\frac{2r}{m}\right\rangle_{n-l}\right\}.
\end{aligned} \tag{62}$$

Combining with (61) and (62), we get the desired identity.

□

Theorem 11. *For $n, r \in \mathbb{N}\cup\{0\}, m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, we have*

$$E_{n,\lambda}(x) = \sum_{k=0}^{n}\sum_{j=0}^{n}\binom{n}{j}(-1)^{k+n-j}W^L_{m,r}(n-j,k)E_{j,\lambda}D^L_{m,r,\lambda}(k|x),$$

When $\lambda \to 0$, we have

$$E_n(x) = \sum_{k=0}^{n}\sum_{j=0}^{n}\binom{n}{j}(-1)^{k+n-j}W^L_{m,r}(n-j,k)E_j\cdot D^L_{m,r}(k|x).$$

Proof. To find the inversion formula of Theorem 10, let $p(x) = E_{n,\lambda}(x)$.
From (36), we have

$$E_{n,\lambda}(x) = \sum_{k=0}^{n} A_k D^L_{m,r,\lambda}(k|x), \tag{63}$$

where from (2),(9) and (59)

$$\begin{aligned}
A_k &= \frac{1}{k!}\left\langle (1+mt)^{-\frac{2r}{m}}\left(\frac{t}{1+mt}\right)^k \middle| E_{n,\lambda}(x)\right\rangle_\lambda \\
&= \sum_{l=k}^{n}\frac{(-1)^{k+l}}{l!}W^L_{m,r}(l,k)\left\langle t^l \middle| \sum_{j=0}^{n}\binom{n}{j}E_{j,\lambda}(x)_{n-j,\lambda}\right\rangle_\lambda \\
&= \sum_{l=k}^{n}\frac{(-1)^{k+l}}{l!}W^L_{m,r}(l,k)\sum_{j=0}^{n}\binom{n}{j}E_{j,\lambda}\langle t^l|(x)_{n-j,\lambda}\rangle_\lambda \\
&= \sum_{j=0}^{n}\binom{n}{j}(-1)^{k+n-j}W^L_{m,r}(n-j,k)E_{j,\lambda}.
\end{aligned} \tag{64}$$

Combining with (63) and (64), we have the desired identity.

□

The fully degenerate Bell polynomials are given by

$$e^x_\lambda\left(e_\lambda(t)-1\right) = \sum_{n=0}^{\infty} bel_{n,\lambda}(x)\frac{t^l}{l!} \quad \text{(see [17, 18]).} \tag{65}$$

From (65), we observe that

$$\sum_{n=0}^{\infty}\mathrm{bel}_{n,\lambda}(x)\frac{t^n}{n!}=e_\lambda^x\big(e_\lambda(t)-1\big)=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}S_{2,\lambda}(n,j)(x)_{j,\lambda}\right)\frac{t^n}{n!}. \tag{66}$$

By comparing with the coefficients of both side of (66), we get

$$bel_{n,\lambda}(x)=\sum_{l=0}^{n}S_{2,\lambda}(n,l)(x)_{l,\lambda}. \tag{67}$$

Theorem 12. *For $n,r\in\mathbb{N}\cup\{0\},m\in\mathbb{N}$ and $\lambda\in\mathbb{R}$, we have*

$$\begin{aligned}D^L_{m,r,\lambda}(x|n)&=\sum_{k=0}^{n}\left(\sum_{j=k}^{n}\sum_{l=k}^{j}\binom{n}{j}m^{n-l}\left\langle\frac{2r}{m}\right\rangle_{n-j}S_{1,\lambda}(l,k)L(j,l)\right)bel_{k,\lambda}(x)\\&=\sum_{k=0}^{n}\left(\sum_{l=k}^{n}S_{1,\lambda}(l,k)W^L_{m,r}(n,l)\right)bel_{k,\lambda}(x).\end{aligned}$$

When $\lambda\to 0$, we have

$$\begin{aligned}D^L_{m,r}(x|n)&=\sum_{k=0}^{n}\left(\sum_{j=k}^{n}\sum_{l=k}^{j}\binom{n}{j}m^{n-l}\left\langle\frac{2r}{m}\right\rangle_{n-j}S_{1}(l,k)L(j,l)\right)bel_{k}(x)\\&=\sum_{k=0}^{n}\left(\sum_{l=k}^{n}S_{1}(l,k)W^L_{m,r}(n,l)\right)bel_{k}(x).\end{aligned}$$

Proof. From (65), we have the degenerate Sheffer sequence

$$bel_{n,\lambda}(x)\sim\big(1,\log_\lambda(1+t)\big)_\lambda. \tag{68}$$

By (21),(31) and (68), we have

$$D^L_{m,r,\lambda}(n|x)=\sum_{k=0}^{n}\alpha_{n,k}bel_{k,\lambda}(x), \tag{69}$$

where, from (5), (7) and (14),

$$\begin{aligned}\alpha_{n,k}&=\frac{1}{k!}\left\langle(1-mt)^{-\frac{2r}{m}}\left(\log_\lambda\left(1+\frac{t}{1-mt})\right)\right)^k\middle|(x)_{n,\lambda}\right\rangle_\lambda\\&=\left\langle(1-mt)^{-\frac{2r}{m}}\sum_{l=k}^{\infty}S_{1,\lambda}(l,k)\frac{1}{l!}\left(\frac{t}{1-mt}\right)^l\middle|(x)_{n,\lambda}\right\rangle_\lambda\\&=\left\langle(1-mt)^{-\frac{2r}{m}}\sum_{l=k}^{\infty}S_{1,\lambda}(l,k)m^{-k}\sum_{j=l}^{\infty}L(j,l)m^j\frac{t^j}{j!}\middle|(x)_{n,\lambda}\right\rangle_\lambda\\&=\sum_{j=k}^{n}\sum_{l=k}^{j}m^{j-l}S_{1,\lambda}(l,k)L(j,l)\binom{n}{j}\langle(1-mt)^{-\frac{2r}{m}}|(x)_{n-j,\lambda}\rangle_\lambda\\&=\sum_{j=k}^{n}\sum_{l=k}^{j}m^{j-l}S_{1,\lambda}(l,k)L(j,l)\binom{n}{j}\left\langle\frac{2r}{m}\right\rangle_{n-j}m^{n-j}\\&=\sum_{j=k}^{n}\sum_{l=k}^{j}\binom{n}{j}m^{n-l}\left\langle\frac{2r}{m}\right\rangle_{n-j}S_{1,\lambda}(l,k)L(j,l).\end{aligned} \tag{70}$$

Combining with (69) and (70), we have the first equality.

In another way of (70), we observe that

$$(71)\quad \begin{aligned}\alpha_{n,k} &= \frac{1}{k!}\left\langle (1-mt)^{-\frac{r}{2m}}\left(\log_\lambda\left(1+\frac{t}{1-mt}\right)\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \left\langle \sum_{l=k}^{\infty} S_{1,\lambda}(l,k)\frac{1}{l!}\left(\frac{t}{1-mt}\right)^l (1-mt)^{-\frac{2r}{m}} \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \left\langle \sum_{l=k}^{\infty} S_{1,\lambda}(l,k)\sum_{j=l}^{\infty} W_{m,r}^L(j,l)\frac{t^j}{j!} \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \left\langle \sum_{j=k}^{\infty}\sum_{l=k}^{j} S_{1,\lambda}(l,k)W_{m,r}^L(j,l)\frac{t^j}{j!} \middle| (x)_{n,\lambda}\right\rangle_\lambda = \sum_{l=k}^{n} S_{1,\lambda}(l,k)W_{m,r}^L(n,l).\end{aligned}$$

Combining with (69) and (71), we get the second equality.

□

Theorem 13. *For $n,r \in \mathbb{N}\cup\{0\}, m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, we have*

$$bel_{n,\lambda}(x) = \sum_{k=0}^{n}\sum_{j=0}^{n}(-1)^{k+j}S_{2,\lambda}(n,j)W_{m,r}^L(j,k)D_{m,r,\lambda}^L(k|x)$$

When $\lambda \to 0$, we have

$$bel_n(x) = \sum_{k=0}^{n}\sum_{j=0}^{n}(-1)^{k+j}S_{2,\lambda}(n,j)W_{m,r}^L(j,k)D_{m,r}^L(k|x)$$

Proof. To find the inversion formula of Theorem 12, let $p(x) = bel_{n,\lambda}(x)$. By (36),

$$(72)\quad bel_{n,\lambda}(x) = \sum_{k=0}^{n} A_k D_{m,r,\lambda}^L(k|x),$$

where, from (2) and (67),

$$(73)\quad \begin{aligned}A_k &= \frac{1}{k!}\left\langle (1+mt)^{-\frac{2r}{m}}\left(\frac{t}{1+mt}\right)^k \middle| bel_{n,\lambda}(x)\right\rangle_\lambda \\ &= \frac{1}{k!}\left\langle (1+mt)^{-\frac{2r}{m}}\left(\frac{t}{1+mt}\right)^k \middle| \sum_{j=0}^{n} S_{2,\lambda}(n,j)(x)_{j,\lambda}\right\rangle_\lambda \\ &= \sum_{j=0}^{n} S_{2,\lambda}(n,j)\left\langle \sum_{l=k}^{\infty}(-1)^{k+l}W_{m,r}^L(l,k)\frac{t^l}{l!} \middle| (x)_{j,\lambda}\right\rangle_\lambda \\ &= \sum_{j=0}^{n} S_{2,\lambda}(n,j)(-1)^{k+j}W_{m,r}^L(j,k).\end{aligned}$$

Combining with (72) and (73), we get the desired result.

□

The degenerate Daehee polynomials $d_{n,\lambda}(x)$ are given by

$$(74)\quad \frac{\log_\lambda(1+t)}{t}e_\lambda^x\big(\log_\lambda(1+t)\big) = \sum_{n=0}^{\infty} d_{n,\lambda}(x)\frac{t^n}{n!} \quad \text{(see [16, 17])}.$$

When $x=0$, $d_{n,\lambda} = d_{n,\lambda}(0)$ are called the degenerate Daehee numbers.

From (74), we note that

$$d_{n,\lambda}(x)=\sum_{m=0}^{n}\binom{n}{m}d_{m,\lambda}(x)_{n-m,\lambda}. \tag{75}$$

Theorem 14. *For $n,r\in\mathbb{N}\cup\{0\},m\in\mathbb{N}$ and $\lambda\in\mathbb{R}$, we have*

$$D^{L}_{m,r,\lambda}(n|x)=\sum_{k=0}^{m}\left(\sum_{j=k}^{n}\sum_{l=k}^{j}\sum_{s=0}^{n-j}\binom{n}{j}\frac{(1)_{s+1,\lambda}m^{n-l-s}}{s+1}S_{2,\lambda}(l,k)W^{L}_{m,r}(j,l)L(d,s)\right)d_{k,\lambda}(x).$$

When $\lambda\to 0$, we have

$$D^{L}_{m,r}(n|x)=\sum_{k=0}^{m}\left(\sum_{j=k}^{n}\sum_{l=k}^{j}\sum_{s=0}^{n-j}\binom{n}{j}\frac{(1)_{s+1,\lambda}m^{n-l-s}}{s+1}S_{2,\lambda}(l,k)W^{L}_{m,r}(j,l)L(d,s)\right)d_{k}(x).$$

Proof. From (20) and (74) we have the degenerate Sheffer sequence.

$$d_{n,\lambda}(x)\sim\left(\frac{e_{\lambda}(t)-1}{t},e_{\lambda}(t)-1\right)_{\lambda} \tag{76}$$

By (21), (31) and (76), we have

$$D^{L}_{m,r,\lambda}(x)=\sum_{k=0}^{n}\alpha_{n,k}d_{k,\lambda}(x), \tag{77}$$

where, by (2), (7), (10) and (15),

$$\begin{aligned}\alpha_{n,k}&=\frac{1}{k!}\left\langle\frac{e_{\lambda}\left(\frac{t}{1-mt}\right)-1}{\frac{t}{1-mt}}(1-mt)^{-\frac{2r}{m}}\left(e_{\lambda}\left(\frac{t}{1-mt}-1\right)^{k}\right|(x)_{n,\lambda}\right\rangle_{\lambda}\\&=\left\langle\frac{e_{\lambda}\left(\frac{t}{1-mt}\right)-1}{\frac{t}{1-mt}}\sum_{j=k}^{\infty}\sum_{l=k}^{j}m^{j-l}S_{2,\lambda}(l,k)W^{L}_{m,r}(j,l)\frac{1}{j!}t^{j}\right|(x)_{n,\lambda}\right\rangle_{\lambda}\\&=\sum_{j=k}^{n}\sum_{l=k}^{j}\binom{n}{j}m^{j-l}S_{2,\lambda}(l.k)W^{L}_{m,r}(j,l)\left\langle\sum_{s=0}^{\infty}\frac{(1)_{s+1,\lambda}}{(s+1)m^{s}}\frac{1}{s!}\left(\frac{mt}{1-mt}\right)^{s}\right|(x)_{n-j,\lambda}\right\rangle_{\lambda}\\&=\sum_{j=k}^{n}\sum_{l=k}^{j}\binom{n}{j}m^{j-l}S_{2,\lambda}(l.k)W^{L}_{m,r}(j,l)\left\langle\sum_{d=0}^{\infty}\sum_{s=0}^{d}\frac{(1)_{s+1,\lambda}}{s+1}m^{d-s}L(d,s)\right|(x)_{n-j,\lambda}\right\rangle_{\lambda}\\&=\sum_{j=k}^{n}\sum_{l=k}^{j}\sum_{s=0}^{n-j}\binom{n}{j}\frac{(1)_{s+1,\lambda}m^{n-l-s}}{s+1}S_{2,\lambda}(l,k)W^{L}_{m,r}(j,l)L(d,s).\end{aligned} \tag{78}$$

From (77) and (78), we have the desired result. □

Theorem 15. *For $n,r\in\mathbb{N}\cup\{0\},m\in\mathbb{N}$ and $\lambda\in\mathbb{R}$, we have*

$$d_{n,\lambda}(x)=\sum_{k=0}^{n}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}d_{j,\lambda}W^{L}_{m,r}(n-j,k)D_{m,r,\lambda}{}^{L}(k|x).$$

When $\lambda\to 0$, we have

$$d_{n}(x)=\sum_{k=0}^{n}\sum_{j=0}^{n}\binom{n}{j}(-1)^{k+n-j}d_{j,\lambda}W^{L}_{m,r}(n-j,k)D^{L}_{m,r}(k|x).$$

Proof. To find the inversion formula of Theorem 14, let $p(x) = d_{n,\lambda}(x)$.
From (31), we have

$$d_{n,\lambda}(x) = \sum_{k=0}^{n} A_k D^L_{m,r,\lambda}(k|x), \tag{79}$$

where, by (2) and (75),

$$\begin{aligned} A_k &= \frac{1}{k!}\left\langle (1+mt)^{-\frac{2r}{m}}\left(\frac{t}{1+mt}\right)^k \middle| d_{n,\lambda}(x)\right\rangle_\lambda \\ &= \frac{1}{k!}\left\langle (1+mt)^{-\frac{2r}{m}}\left(\frac{t}{1+mt}\right)^k \middle| \sum_{j=0}^{n}\binom{n}{j} d_{j,\lambda}(x)_{n-j,\lambda}\right\rangle_\lambda \\ &= \sum_{j=0}^{n}\binom{n}{j} d_{j,\lambda}\left\langle (-1)^k \sum_{l=k}^{\infty} W^L_{m,r}(l,k)(-1)^l \frac{t^l}{l!} \middle| (x)_{n-j,\lambda}\right\rangle_\lambda \\ &= \sum_{j=0}^{n}\binom{n}{j} d_{j,\lambda} W^L_{m,r}(n-j,k)(-1)^{k+n-j}. \end{aligned} \tag{80}$$

Combining with (79) and (80), we have the desired identity. □

3. Further Remark

We consider another degenerate r-Dowling-Lah polynomials as follows:

$$\sum_{n=0}^{\infty} \widetilde{D}^L_{m,r,\lambda}(n|x)\frac{t^n}{n!} = (1-mt)^{-\frac{2r}{m}} e_\lambda\left(x\frac{t}{1-mt}\right). \tag{81}$$

When $x = 1$, from (22) and (81) we note that

$$\sum_{n=0}^{\infty} \widetilde{D}^L_{m,r,\lambda}(n)\frac{t^n}{n!} = \sum_{n=0}^{\infty} D^L_{m,r,\lambda}(n)\frac{t^n}{n!}.$$

and

$$\widetilde{D}^L_{m,r,\lambda}(n) = D^L_{m,r,\lambda}(n) = \sum_{k=0}^{n} (1)_{k,\lambda} W^L_{m,r}(n,k).$$

Theorem 16. *For $n, r \in \mathbb{N}\cup\{0\}, m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, we have*

$$\begin{aligned} \widetilde{D}^L_{m,r,\lambda}(n|x) = \sum_{l=0}^{n}\sum_{j=0}^{l}\sum_{k=0}^{n-l}\binom{n}{l}(-1)^l m^{n+l} \\ \times (1)_{k,\lambda}(-\frac{2r}{m})_{j,\lambda} S_{1,\lambda}(l,j) L(n-l,k) x^k. \end{aligned}$$

Proof. From (5), (7),(10) and (14) we observe that

$$
\begin{aligned}
\sum_{n=0}^{\infty}\widetilde{D}^{L}_{m,r,\lambda}(n|x)\frac{t^n}{n!} &= (1-mt)^{-\frac{2r}{m}}e_{\lambda}(x\frac{t}{1-mt}) \\
&= e_{\lambda}^{-\frac{2r}{m}}(\log_{\lambda}(1-mt))e_{\lambda}\left(x\frac{t}{1-mt}\right) \\
&= \sum_{j=0}^{\infty}(-\frac{2r}{m})_{j,\lambda}\sum_{l=j}^{\infty}S_{1,\lambda}(l,j)(-m)^{l}\frac{t^l}{l!}e_{\lambda}\left(x\frac{t}{1-mt}\right) \\
&= \sum_{l=0}^{\infty}\left(\sum_{j=0}^{l}(-1)^{l}(-\frac{2r}{m})_{j,\lambda}m^{l}S_{1,\lambda}(l,j)\right)\frac{t^l}{l!}\sum_{k=0}^{\infty}(1)_{k,\lambda}x^{k}\frac{1}{k!}\left(\frac{t}{1-mt}\right)^{k} \\
&= \sum_{l=0}^{\infty}\left(\sum_{j=0}^{l}(-1)^{l}(-\frac{2r}{m})_{j,\lambda}m^{l}S_{1,\lambda}(l,j)\right)\frac{t^l}{l!}\sum_{h=0}^{\infty}\left(\sum_{k=0}^{h}(1)_{k,\lambda}m^{k+h}L(h,k)x^{k}\right)\frac{t^h}{h!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{j=0}^{l}\sum_{k=0}^{n-l}\binom{n}{l}(-1)^{l}m^{n+l}(1)_{k,\lambda}(-\frac{2r}{m})_{j,\lambda}S_{1,\lambda}(l,j)L(n-l,k)x^{k}\right)\frac{t^n}{n!}.
\end{aligned}
\tag{82}
$$

By comparing with the coefficients of both side of (82), we have the desired result. □

Theorem 17. *For $n,r\in\mathbb{N}\cup\{0\},m\in\mathbb{N}$ and $\lambda\in\mathbb{R}$, we have*

$$
\widetilde{D}^{L}_{m,r,\lambda}(n|x)=\sum_{l=0}^{n}\sum_{j=0}^{l}\binom{n}{l}(1)_{j,\lambda}m^{n-j}\left\langle\frac{2r}{m}\right\rangle_{n-j}L(l,j)x^{j}.
$$

Proof. From (5),(7) and (10), we observe that

$$
\begin{aligned}
\sum_{n=0}^{\infty}D^{L}_{m,r,\lambda}(n|x)\frac{t^n}{n!} &= (1-mt)^{-\frac{2r}{m}}e_{\lambda}\left(x\frac{t}{1-mt}\right) \\
&= \sum_{h=0}^{\infty}\left\langle\frac{2r}{m}\right\rangle_{h}m^{h}\frac{t^h}{h!}\sum_{j=0}^{\infty}(1)_{j,\lambda}x^{j}\frac{1}{j!}\left(\frac{t}{1-mt}\right)^{j} \\
&= \sum_{h=0}^{\infty}\left\langle\frac{2r}{m}\right\rangle_{h}m^{h}\frac{t^h}{h!}\sum_{j=0}^{\infty}(1)_{j,\lambda}x^{j}m^{-j}\sum_{l=j}^{\infty}L(l,j)\frac{m^{l}t^{l}}{l!} \\
&= \sum_{h=0}^{\infty}\left\langle\frac{2r}{m}\right\rangle_{h}m^{h}\frac{t^h}{h!}\sum_{l=0}^{\infty}\left(\sum_{j=0}^{l}(1)_{j,\lambda}m^{l-j}L(l,j)x^{j}\right)\frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{j=0}^{l}\binom{n}{l}(1)_{j,\lambda}m^{n-j}\left\langle\frac{2r}{m}\right\rangle_{n-j}L(l,j)x^{j}\right)\frac{t^n}{n!}.
\end{aligned}
\tag{83}
$$

By comparing with coefficients of both sides of (83), we have he desired result. □

Theorem 18. *For $n,r\in\mathbb{N}\cup\{0\},m\in\mathbb{N}$ and $\lambda\in\mathbb{R}$, we have*

$$
\begin{aligned}
&\widetilde{D}^{L}_{m,r,\lambda}(1|x)=(2r+x)\widetilde{D}^{L}_{m,r,\lambda}(0|x),\ \text{ and} \\
&\widetilde{D}^{L}_{m,r,\lambda}(n+1|x)=(mn+2r)\widetilde{D}^{L}_{m,r,\lambda}(n|x) \\
&\qquad\qquad +x\sum_{l=0}^{n}\binom{n}{l}(m-\lambda x)^{n-l}(n-l)!\widetilde{D}^{L}_{m,r,\lambda}(l|x),\quad \text{if}\quad n\geq 1.
\end{aligned}
$$

Proof. Differentiating with respect to t in (81), the right side of (81) is

$$\begin{aligned}&\frac{d}{dt}(1-mt)^{-\frac{2r}{m}}e_\lambda\left(x\frac{t}{1-mt}\right)\\&=\frac{2r}{1-mt}(1-mt)^{-\frac{2r}{m}}e_\lambda\left(x\frac{t}{1-mt}\right)+(1-mt)^{-\frac{2r}{m}}e_\lambda\left(x\frac{t}{1-mt}\right)\frac{x}{(1-mt+\lambda xt)(1-mt)}.\end{aligned}\tag{84}$$

On the other hand, the left side of (81) is

$$\frac{d}{dt}\sum_{n=0}^{\infty}D_{m,r,\lambda}^{L}(n|x)\frac{t^n}{n!}=\sum_{n=1}^{\infty}\widetilde{D}_{m,r,\lambda}^{L}(n|x)\frac{t^{n-1}}{(n-1)!}.\tag{85}$$

By (84) and (85), we have

$$(1-mt)\sum_{n=1}^{\infty}\widetilde{D}_{m,r,\lambda}^{L}(n|x)\frac{t^{n-1}}{(n-1)!}=2r\sum_{n=0}^{\infty}\widetilde{D}_{m,r,\lambda}^{L}(n|x)\frac{t^n}{n!}+x\frac{1}{1-(m-\lambda x)t}\sum_{n=0}^{\infty}\widetilde{D}_{m,r,\lambda}^{L}(n|x)\frac{t^n}{n!}.\tag{86}$$

From (86), we get

$$\begin{aligned}&\sum_{n=1}^{\infty}\widetilde{D}_{m,r,\lambda}^{L}(n+1|x)\frac{t^n}{n!}-m\sum_{n=1}^{\infty}\widetilde{D}_{m,r,\lambda}^{L}(n|x)n\frac{t^n}{n!}\\&=2r\sum_{n=0}^{\infty}\widetilde{D}_{m,r,\lambda}^{L}(n|x)\frac{t^n}{n!}+x\sum_{j=0}^{\infty}(m-\lambda x)^j j!\frac{t^j}{j!}\sum_{l=0}^{\infty}\widetilde{D}_{m,r,\lambda}^{L}(l|x)\frac{t^n}{l!}.\end{aligned}\tag{87}$$

From (87), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\widetilde{D}_{m,r,\lambda}^{L}(n+1|x)\frac{t^n}{n!}\\&=m\sum_{n=1}^{\infty}n\widetilde{D}_{m,r,\lambda}^{L}(n|x)\frac{t^n}{n!}+2r\sum_{n=0}^{\infty}\widetilde{D}_{m,r,\lambda}^{L}(n|x)\frac{t^n}{n!}\\&\quad+x\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}(m-\lambda x)^{n-l}(n-l)!\widetilde{D}_{m,r,\lambda}^{L}(l|x)\frac{t^n}{n!}.\end{aligned}\tag{88}$$

By comparing with the coefficients of both sides of (88), we have the desired relation. □

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