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SOME IDENTITIES OF DEGENERATE POLY-q-BERNOULLI AND POLY-q-EULER POLYNOMIALS ARISING FROM λ-q-SHEFFER SEQUENCES

TAEKYUN KIM[1], DAE SAN KIM[2], AND HYE KYUNG KIM[3,*]

ABSTRACT. Recently, we introduced the λ-q-umbral calculus which centers around the λ-q-Sheffer sequences and the degenerate q-Sheffer sequences by introducing λ-q-linear functionals and λ-q-differential operators, respectively, instead of λ-linear functionals and λ-differential operators. In this paper, we introduce the degenerate poly-q-Bernoulli polynomials and degenerate poly-q-Euler polynomials and numbers by using the q-polylogarithm function. These new sequences are q-analogues of the degenerate poly-Bernoulli polynomials and degenerate poly-Euler polynomials, respectively. We give interesting combinatorial identities and properties of these new polynomials by using λ-q-umbral calculus.

1. INTRODUCTION

Many mathematicians have studied degenerate versions of some special polynomials and numbers. They have discovered some interesting results for the degenerate Stirling numbers of the first and second kinds, the degenerate Bernstein polynomials, the degenerate Bell numbers and polynomials, the degenerate gamma function, the degenerate gamma random variables, and so on (see [5, 14-17, 20, 22, 26, 28-33, 37]). One of the important tools in the study of degenerate polynomials and numbers is the umbral calculus based on the modern idea of linear functions, linear operators and adjoints, which was laid a rigorous foundation by Rota in the 1970s (see [12, 14, 17-19, 28, 30-33, 40, 41]). Carlitz [6] introduced q-extensions of the classical Bernoulli numbers and polynomials. Since that time, many scholars have studied these and other related subjects (see [1, 2, 4, 7, 8, 12, 13, 18-20, 23-25, 27, 34-36, 42]). Recently, Kim-Kim introduced the λ-umbral calculus [17]. Motivated by this work, Kim-Kim-Kim [28] introduced the λ-q-umbral calculus which centers around the λ-q-Sheffer sequences and the degenerate q-Sheffer sequences by introducing λ-q-linear functionals and λ-q-differential operators, respectively, instead of λ-linear functionals and λ-differential operators. As one of the generalizations of special numbers and polynomials, researches on special numbers and polynomials such as poly-Bernoulli, poly-Euler, poly-Genocchi, and poly-Bell polynomials, etc is being actively conducted (see [3, 7, 9-11, 14, 20, 21, 26, 32, 34, 35, 37, 42]). In this paper, we introduce degenerate poly-q-Bernoulli polynomials and degenerate poly-q-Euler by using polylogarithm functions, which were first introduced by Hardy [11]. The poly-q-Bernoulli polynomials and poly-q-Euler polynomials introduced in this paper are different from there is (22), (23) and (24). Also, the degenerate poly-q-Bernoulli polynomials are different from (25). We give several combinatorial identities and properties of these new polynomials by using λ-q-umbral calculus.

Let q be a fixed real number with $0<|q|<1$. The q-analogue of $n(\in\mathbb{N})$ is given by

$$[n]_q=\frac{1-q^n}{1-q},\quad \text{(see [1, 2, 4, 6-8]).} \tag{1}$$

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* is corresponding author.

The q-binomial coefficients are given by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad \text{(see [12, 13, 18, 19, 23-25])}. \tag{2}$$

for $n,k \in \mathbb{N} \cup \{0\}$ with $k \le n$, where $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$ and $[0]_q! = 1$.

The q-exponential functions are given by

$$\begin{aligned} e_q(t) &= 1 + \sum_{n=1}^{\infty} \frac{(1-q)^n x^n}{(1-q)(1-q^2)\cdots(1-q^n)} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!}, \quad \text{(see [12, 13, 18, 19, 23-25])}. \end{aligned} \tag{3}$$

Note that $e_q(t) \to e^t$ as $q \to 1$.

The q-Bernoulli polynomials, which were studied by Kupershmidt [36], are given by the generating function

$$\frac{t}{e_q(t)-1} e_q(xt) = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{[n]_q!}, \quad \text{(see [1, 4, 18])}. \tag{4}$$

When $x = 0$, $B_{n,q}(0) = B_{n,q}$ is called the n-th q-Bernoulli number.

In [24], Kim considered the q-Euler polynomials given by the generating function

$$\frac{2}{e_q(t)+1} e_q(xt) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{[n]_q!}, \quad \text{(see [19, 24])}. \tag{5}$$

For $x = 0$, $E_{n,q} = E_{n,q}(0)$, are called the q-Euler numbers.

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t) = (1+\lambda t)^{\frac{1}{\lambda}}, \quad \text{(see [14-17, 26, 29-33])}. \tag{6}$$

By Taylor expansion, we get

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad \text{(see [14-17, 26, 29-33])}, \tag{7}$$

where $(x)_{0,\lambda} = 1,\ (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda),\ (n \ge 1)$.

Recently, we considered the degenerate q-exponential functions which are given by

$$e_{q,\lambda}^x(t) = \sum_{n=0}^{\infty} \frac{(x)_{n,\lambda}}{[n]_q!} t^n, \quad \text{(see [28])}. \tag{8}$$

When $x = 1$, we have

$$e_{q,\lambda}(t) = e_{q,\lambda}^1(t) = \sum_{n=0}^{\infty} \frac{(1)_{n,\lambda}}{[n]_q!} t^n. \tag{9}$$

When $q = 1$, we note that

$$e_{1,\lambda}^x(t) = e_\lambda^x(t) \ \text{ and } \ \lim_{\lambda \to 0} e_{q,\lambda}^x(t) = e_q(xt).$$

For $r \in \mathbb{N}$, we consider the degenerate q-Bernoulli polynomials of order r given as follows:

$$\left(\frac{t}{e_{q,\lambda}(t)-1} \right)^r e_{q,\lambda}^x(t) = \sum_{n=0}^{\infty} B_{n,q}^{(r)}(x|\lambda) \frac{t^n}{[n]_q!}, \quad \text{(see [28])}. \tag{10}$$

When $x = 0$, $B_{n,q}^{(r)}(\lambda) = B_{n,q}^{(r)}(0|\lambda)$ are called the degenerate q-Bernoulli numbers of order r.

When $r = 1$, $B_{n,q}(x|\lambda) = B_{n,q}^{(1)}(x|\lambda)$ are called the degenerate q-Bernoulli polynomials.

When $\lambda \to 0$, $\lim_{\lambda\to 0} B_{n,q}^{(r)}(x|\lambda) = B_{n,q}^{(r)}(x)$ are called q-Bernoulli polynomials of order r.
From (8) and (10), we get

$$B_{n,q}^{(r)}(x|\lambda) = \sum_{l=0}^{n} \binom{n}{l}_q B_{n-l,q}^{(r)}(\lambda)(x)_{l,\lambda}. \tag{11}$$

For $r \in \mathbb{N}$, we also consider the degenerate q-Euler polynomials of order r given by the generating function

$$\left(\frac{2}{e_{q,\lambda}(t)+1}\right)^r e_{q,\lambda}^x(t) = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x|\lambda)\frac{t^n}{[n]_q!}, \quad \text{(see [28])}. \tag{12}$$

When $x = 0$, $E_{n,q}^{(r)}(\lambda) = E_{n,q}^{(r)}(0|\lambda)$ are called the degenerate q-Euler numbers of order r.
When $r = 1$, $E_{n,q}(x|\lambda) = E_{n,q}^{(1)}(x|\lambda)$ are called the degenerate q-Euler polynomials.
When $\lambda \to 0$, $\lim_{\lambda\to 0} E_{n,q}^{(r)}(x|\lambda) = E_{n,q}^{(r)}(x)$ are called q-Euler polynomials of order r.
From (8) and (12), we get

$$E_{n,q}^{(r)}(x|\lambda) = \sum_{l=0}^{n} \binom{n}{l}_q E_{n-l,q}^{(r)}(\lambda)(x)_{l,\lambda}. \tag{13}$$

For $n \geq 0$, it is well known that the Stirling numbers of the first and second kind, respectively, are given by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l \text{ and } \frac{1}{l!}(\log(1+t))^l = \sum_{n=l}^{\infty} S_1(n,l)\frac{t^n}{n!}, \quad \text{(see [5, 16, 22])}, \tag{14}$$

and

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l \text{ and } \frac{1}{l!}(e^t-1)^l = \sum_{n=l}^{\infty} S_2(n,l)\frac{t^n}{n!}, \quad \text{(see [5, 16, 22])}, \tag{15}$$

where $(x)_n = x(x-1)\dots(x-n+1)$, $(n \geq 1)$ and $(x)_0 = 1$.

Moreover, the degenerate Stirling numbers of the first and second kind, respectively, are given by

$$(x)_n = \sum_{l=0}^{n} S_{1,\lambda}(n,l)(x)_{l,\lambda} \text{ and } \frac{1}{l!}\big(\log_\lambda(1+t)\big)^l = \sum_{n=l}^{\infty} S_{1,\lambda}(n,l)\frac{t^n}{n!}, \quad \text{(see [16, 22])}, \tag{16}$$

and

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_l \text{ and } \frac{1}{l!}\big(e_\lambda(t)-1\big)^l = \sum_{n=l}^{\infty} S_{2,\lambda}(n,l)\frac{t^n}{n!}, \quad \text{(see [16, 22])}. \tag{17}$$

Here we recall $\log_\lambda(1+t) = \frac{1}{\lambda}((1+t)^\lambda - 1)$.
In [27], the q-Stirling numbers of the second kind are given by

$$\frac{1}{[l]_q!}(e_q(t)-1)^l = \sum_{n=l}^{\infty} S_{2,q}(n,l)\frac{t^n}{[n]_q!}. \tag{18}$$

When $q = 1$, $S_{2,1}(n,l) = S_2(n,l)$.
We naturally consider the degenerate q-Stirling numbers of the second kind given by

$$\frac{1}{[l]_q!}(e_{q,\lambda}(t)-1)^l = \sum_{n=l}^{\infty} S_{2,q}(n,l|\lambda)\frac{t^n}{[n]_q!}. \tag{19}$$

When $\lim_{\lambda\to 0} S_{2,q}(n,l|\lambda) = S_{2,q}(n,l)$. When $q = 1$, $S_{2,1}(n,l|\lambda) = S_{2,\lambda}(n,l)$.

The polylogarithm functions are given by

$$\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^k}{n^k}, \quad (k \in \mathbb{Z}), \quad (\text{see [11]}). \tag{20}$$

Note that $\mathrm{Li}_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = -\log(1-x)$.
The q-polylogarithm functions are given by

$$\mathrm{Li}_{k,q}(x) = \sum_{n=1}^{\infty} \frac{x^n}{[n]_q^k}, \quad (k \in \mathbb{Z}), \quad (\text{see [34, 35]}). \tag{21}$$

We note that $\lim_{q\to 1} \mathrm{Li}_{k,q}(x) = \mathrm{Li}_k(x)$.
The q-poly-Bernoulli numbers are defined by

$$\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} b_{n,q}^{(k)} \frac{t^n}{n!}, \qquad (\text{see [35]}). \tag{22}$$

In [34], the generating function of the q-analogue of poly-Euler polynomials $E_{n,q}^{(k)}(x)$ and poly-Bernoulli polynomials $B_{n,q}^{(k)}(x)$ $(n = 0,1,2,\cdots)$, respectively, are given as follows:

$$\frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{t(e^t+1)} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}^{(k)}(x) \frac{t^n}{n!} \quad \text{and} \quad \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(k)}(x) \frac{t^n}{n!}, \quad (\text{see [34]}). \tag{23}$$

For $\lambda, q \in \mathbb{C}$ with $\lambda, q \neq 0$ and $k \in \mathbb{Z}$. The poly-Bernoulli polynomials with a q parameter $B_{n,q}^{(k)}(x)$ are defined by Cenkci [7]

$$\frac{q\mathrm{Li}_k\left(\frac{1-e^{-qt}}{q}\right)}{1-e^{-qt}} e^{xt} = \sum_{n\geq 0} b_{n,q}^{(k)}(x) \frac{t^n}{n!}, \quad (\text{see [7]}). \tag{24}$$

Kim et al considered the fully degenerate poly-Bernoulli polynomials with a q parameter $b_{n,q}^{(k)}(\lambda,x)$ given by the generating function

$$\frac{q\mathrm{Li}_k\left(\frac{1-(1+\lambda t)^{-\frac{q}{\lambda}}}{q}\right)}{1-(1+\lambda t)^{-\frac{q}{\lambda}}} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n\geq 0} b_{n,q}^{(k)}(\lambda,x) \frac{t^n}{n!}, \quad (\text{see [20]}). \tag{25}$$

2. The fundamental properties of λ-q-umbral calculus

In this section, we review briefly the fundamental properties of λ-q-umbral calculus ([28]). we introduce a family of λ-q-linear functionals on the space of polynomials and prove some basic theorems and propositions on these functionals.

Throughout this section, λ is arbitrary but is a fixed nonzero real number.

Let $f(t) = \sum_{l=0}^{\infty} \frac{a_l}{[l]_q!} t^l \in \mathfrak{F}$. Then each $\lambda \in \mathbb{R}$ gives rise to the linear functional $\langle f(t)|\cdot\rangle_{\lambda,q}$ on $\mathbb{P}$, called λ-q-linear functional given by $f(t)$, which is defined by

$$\langle f(t)|(x)_{n,\lambda}\rangle_{\lambda,q} = a_n, \quad (n \geq 0), \quad (\text{see [28]}). \tag{26}$$

From (26), we note that

$$\langle t^l|(x)_{n,\lambda}\rangle_{\lambda,q} = [n]_q!\delta_{n,l}. \tag{27}$$

By (26), for $f(t) \in \mathfrak{F}$ and $p(x)$ in $\mathbb{P}$, we have

$$f(t) = \sum_{l=0}^{\infty} \frac{\langle f(t)|(x)_{l,\lambda}\rangle_{\lambda,q}}{[l]_q!} t^l \tag{28}$$

and

$$p(x) = \sum_{k=0}^{\infty} \frac{\langle t^l|p(x)\rangle_{\lambda,q}}{[l]_q!}(x)_{l,\lambda}. \tag{29}$$

From (15) and (16), we note that

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l = \sum_{l=0}^{n}\sum_{m=0}^{l} S_2(n,l)S_{1,\lambda}(l,m)(x)_{m,\lambda}. \tag{30}$$

Thus, by (26), (30) and (31), we have

$$\langle f(t)|x^n\rangle_{\lambda,q} = \sum_{l=0}^{n}\sum_{m=0}^{l} S_2(n,l)S_{1,\lambda}(l,m)\langle f(t)|(x)_{m,\lambda}\rangle_{\lambda,q}. \tag{31}$$

In particular, we get

$$\langle t^k|x^n\rangle_{\lambda,q} = \sum_{l=0}^{n}\sum_{m=0}^{l} S_2(n,l)S_{1,\lambda}(l,m)\langle t^k|(x)_{m,\lambda}\rangle_{\lambda,q}.$$

The order $o(f(t))$ of the power series $f(t)(\neq 0)$ is the smallest integer for which a_k does not vanish. If $o(f(t)) = 0$, then $f(t)$ is called an invertible series. If $o(f(t)) = 1$, then $f(t)$ is called a delta series (see [17, 28, 41]).

Let $o(f_l(t)) = l$, for all $l \geq 0$ and let

$$\langle f_l(t)|p(x)\rangle_{\lambda,q} = \langle f_l(t)|q(x)\rangle_{\lambda,q}. \quad \text{Then } \; p(x) = q(x). \tag{32}$$

For $f(t),\ g(t) \in \mathfrak{F}$, we have

$$\langle f(t)g(t)|(x)_{n,\lambda}\rangle_{\lambda,q} = \sum_{l=0}^{n}\binom{n}{l}_q \langle f(t)|(x)_{l,\lambda}\rangle_{\lambda,q}\langle g(t)|(x)_{n-l,\lambda}\rangle_{\lambda,q}, \quad \text{for } n \geq 0. \tag{33}$$

For all $j \geq 0$, let $\deg p_j(x) = j$ and

$$\langle f(t)|p_j(x)\rangle_{\lambda,q} = \langle g(t)|p_j(x)\rangle_{\lambda,q}. \quad \text{Then } \; f(t) = g(t). \tag{34}$$

For each $\lambda \in \mathbb{R}$ and each nonnegative integer k, we define the λ-q-differential operator on $\mathbb{P}$ on

$$(t^l)_{\lambda,q}(x)_{n,\lambda} = \begin{cases} ([n]_q)_l(x)_{n-l,\lambda} & \text{if } l \leq n, \\ 0 & \text{if } l > n, \end{cases} \quad \text{(see [28])}, \tag{35}$$

where $([n]_q)_l = [n]_q[n-1]_q\cdots[n-(l-1)]_q$.

For any power series $f(t) = \sum_{l=0}^{\infty}\frac{a_l}{[l]_q!}t^l \in \mathfrak{F}$, the λ-q-differential operator given by $f(t)$ is

$$(f(t))_{\lambda,q}(x)_{n,\lambda} = \sum_{l=0}^{n}\binom{n}{l}_q a_l(x)_{n-l,\lambda}, \quad (n \geq 0). \tag{36}$$

and by linear extension.

Let $f_1(t) = \sum_{j=0}^{\infty}\frac{a_j}{[j]_q!}t^j$ and $f_2(t) = \sum_{l=0}^{\infty}\frac{b_l}{[l]_q!}t^l \in \mathfrak{F}$.

Then we have

$$(f_1(t)f_2(t))_{\lambda,q}(x)_{n,\lambda} = (f_1(t))_{\lambda,q}((f_2(t))_{\lambda,q}(x)_{n,\lambda}). \tag{37}$$

That is $(f_1(t)f_2(t))_{\lambda,q} = (f_1(t))_{\lambda,q}(f_2(t))_{\lambda,q}$.

For $f(t), g(t) \in \mathfrak{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|p(x)\rangle_{\lambda,q} = \langle g(t)|(f(t))_{\lambda,q}p(x)\rangle_{\lambda,q} = \langle f(t)|(g(t))_{\lambda,q}p(x)\rangle_{\lambda,q}. \tag{38}$$

For $f(t), g(t) \in \mathfrak{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 0$, there exists a unique sequence $s_{n,q}(x|\lambda)$ $(\deg s_{n,q}(x|\lambda) = n)$ of polynomials satisfying the orthogonality conditions

$$\langle g(t)f(t)^k|s_{n,q}(x|\lambda)\rangle_{\lambda,q} = [n]_q!\delta_{n,k}, \quad (n,k \geq 0). \tag{39}$$

The sequence $s_{n,q}(x|\lambda)$ is called the λ-q-Sheffer sequence for $(g(t), f(t))$, which is denoted by $s_{n,q}(x|\lambda) \sim (g(t), f(t))_{\lambda,q}$

The sequence $s_{n,q}(x|\lambda)$ is λ-q-Sheffer for $(g(t), f(t))$ if and only if

$$\frac{1}{g(\bar{f}(t))}e^y_{q,\lambda}(\bar{f}(t)) = \sum_{k=0}^{\infty} s_{k,q}(y|\lambda)\frac{t^k}{[k]_q!}, \quad \text{for all } y \in \mathbb{C}, \tag{40}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

Let $s_{n,q}(x|\lambda) \sim (g(t), f(t))_{\lambda,q}$. Then we have

$$(f(t))_{q,\lambda}s_{n,q}(x|\lambda) = [n]_q s_{n-1,q}(x|\lambda), \quad (n \geq 0). \tag{41}$$

Let $s_{n,q}(x|\lambda) \sim (g(t), f(t))_{\lambda,q}$ and $P_{n,q}(x|\lambda) \sim (1, f(t))_{\lambda,q}$. For $h(t) \in \mathfrak{F}$, we have

$$(h(t))_{\lambda,q}s_{n,q}(x|\lambda) = \sum_{k=0}^{n}\binom{n}{k}_q \langle h(t)|s_{n,q}(x|\lambda)\rangle_{\lambda,q}P_{n-k,q}(x|\lambda). \tag{42}$$

From (8) and (40), we note that

$$(x)_{n,\lambda} \sim (1,t)_{\lambda,q} \tag{43}$$

In particular, when $f(t) = t$, combining (42) with (43), we observe that

$$P_{n,q}(x|\lambda) = (x)_{n,\lambda}, \tag{44}$$

and

$$\begin{aligned} [n]_q!\delta_{n,k} &= \langle g(t)t^k|s_{n,q}(x|\lambda)\rangle_{\lambda,q} \\ &= \left\langle t^k|(g(t))_{\lambda,q}s_{n,q}(x|\lambda)\right\rangle_{\lambda,q} = \left\langle t^k|(x)_{n,\lambda}\right\rangle_{\lambda,q}. \end{aligned} \tag{45}$$

From the last equality of (45) and by using (32), we get

$$(g(t))_{\lambda,q}s_{n,q}(x|\lambda) = (x)_{n,\lambda}. \tag{46}$$

Since $e^y_{q,\lambda}(t) = \sum_{k=0}^{\infty}\frac{(y)_{k,\lambda}}{[k]_q!}t^k$, we have

$$\langle e^y_{q,\lambda}(t)|(x)_{n,\lambda}\rangle_{\lambda,q} = \sum_{k=0}^{\infty}\frac{(y)_{k,\lambda}}{[k]_q!}\langle t^k|(x)_{n,\lambda}\rangle_{\lambda,q} = (y)_{n,\lambda}, \quad (n \geq 0). \tag{47}$$

By (47), we have

$$\langle e^y_{q,\lambda}(t)|p(x)\rangle_{\lambda,q} = p(y). \tag{48}$$

Let $s_{n,q}(x|\lambda) \sim (g(t), f(t))_{\lambda,q}$ and $r_{n,q}(x|\lambda) \sim (h(t), l(t))_{\lambda,q}$. Then we have

$$s_{n,q}(x|\lambda) = \sum_{k=0}^{n} A_{n,k} r_{k,q}(x|\lambda), \quad \text{where } A_{n,k} = \frac{1}{[k]_q!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} (l(\bar{f}(t)))^k | (x)_{n,\lambda} \right\rangle_{\lambda,q}. \tag{49}$$

From (10), (12) and (40), we have λ-q-Sheffer sequences as follows.

$$B_{n,q}^{(r)}(x|\lambda) \sim \left(\left(\frac{e_{q,\lambda}(t) - 1}{t} \right)^r, t \right)_{\lambda,q} \quad \text{and} \quad E_{n,q}^{(r)}(x|\lambda) \sim \left(\left(\frac{e_{q,\lambda}(t) + 1}{2} \right)^r, t \right)_{\lambda,q}. \tag{50}$$

By (41), we note that

$$(t)_{q,\lambda} B_{n,q}^{(r)}(x|\lambda) = [n]_q B_{n-1,q}^{(r)}(x) \quad \text{and} \quad (t)_{q,\lambda} E_{n,q}^{(r)}(x|\lambda) = [n]_q E_{n-1,q}^{(r)}(x), \quad (n \geq 0). \tag{51}$$

3. Some identities of degenerate poly-q-Bernoulli and poly-q-Euler polynomials arising from λ-q-Sheffer sequences

In this section, we introduce new types of degenerate poly q-Bernoulli polynomials and degenerate poly-q-Euler polynomials, respectively. We show that each of these is represented by a finite linear combination of degenerate q-Bernoulli polynomials, degenerate q-Bernoulli polynomials of order r, degenerate q-Euler polynomials, degenerate q-Euler polynomials of order r or degenerate falling factorials, etc.

First, we consider a new type of degenerate poly q-Bernoulli polynomials defined by

$$\frac{\mathrm{Li}_{k,q}(1 - e^{-t})}{e_{q,\lambda}(t) - 1} e_{q,\lambda}^x(t) = \sum_{n=0}^{\infty} \mathscr{B}_{n,q}^{(k)}(x|\lambda) \frac{t^n}{[n]_q!}, \quad (k \in \mathbb{Z}). \tag{52}$$

When $x = 0$, $\mathscr{B}_{n,q}^{(k)}(\lambda) := \mathscr{B}_{n,q}^{(k)}(0|\lambda)$ are called degenerate poly q-Bernoulli numbers.
When $\lambda \to 0$, we have the generating function of poly-q-Bernoulli polynomials given by

$$\frac{\mathrm{Li}_{k,q}(1 - e^{-t})}{e_q(t) - 1} e_q^x(t) = \sum_{n=0}^{\infty} \mathscr{B}_{n,q}^{(k)}(x) \frac{t^n}{[n]_q!}, \quad (k \in \mathbb{Z}). \tag{53}$$

When $x = 0$, $\mathscr{B}_{n,q}^{(k)} := \mathscr{B}_{n,q}^{(k)}(0)$ are called the poly-q-Bernoulli numbers.
When $q = 1 = k$, we note that $\lim_{\lambda \to 0} \mathscr{B}_{n,1}^{(1)}(x|\lambda) = B_n(x)$ are the Bernoulli polynomials.
When $q = 1 = k$, we note that $\mathscr{B}_{n,1}^{(1)}(x|\lambda) = B_{n,\lambda}(x)$ are the degenerate Bernoulli polynomials.
When $k = 1$, we have $\mathscr{B}_{n,q}^{(1)}(x|\lambda) = B_{n,q}(x|\lambda)$ are the degenerate q-Bernoulli polynomials.

Second, we consider a new type of degenerate poly-q-Euler polynomials defined by

$$\frac{2\mathrm{Li}_{k,q}(1 - e^{-t})}{t(e_{q,\lambda}(t) + 1)} e_{q,\lambda}^x(t) = \sum_{n=0}^{\infty} \mathscr{E}_{n,q}^{(k)}(x|\lambda) \frac{t^n}{[n]_q!}. \tag{54}$$

When $x = 0$, $\mathscr{E}_{n,q}^{(k)}(\lambda) := \mathscr{E}_{n,q}^{(k)}(0|\lambda)$ are called the degenerate poly-q-Euler numbers.
When $\lambda \to 0$, we have the generating function of poly-q-Euler polynomials given by

$$\frac{2\mathrm{Li}_{k,q}(1 - e^{-t})}{t(e_q(t) + 1)} e_q^x(t) = \sum_{n=0}^{\infty} \mathscr{E}_{n,q}^{(k)}(x) \frac{t^n}{[n]_q!}, \tag{55}$$

when $x = 0$, $\mathscr{E}_{n,q}^{(k)} := \mathscr{E}_{n,q}^{(k)}(0)$ are called the poly-q-Euler numbers.
When $q = 1 = k$, we note that $\lambda \to 0 \mathscr{E}_{n,1}^{(1)}(x|\lambda) = E_n(x)$ are the Euler polynomials.

When $k=1$, we have $\mathscr{E}^{(1)}_{n,q}(x|\lambda)=E_{n,q}(x|\lambda)$ are the degenerate q-Euler polynomials. When $q=k=1$, we note that $\mathscr{E}^{(1)}_{n,1}(x|\lambda)=E_{n,\lambda}(x)$ are the degenerate Euler polynomials.

For $n\geq 0$, from (8), (52) and (54) we observe that

$$\mathscr{B}^{(k)}_{n,q}(x|\lambda)=\sum_{j=0}^{n}\binom{n}{j}_q\mathscr{B}^{(k)}_{n-j,q}(\lambda)(x)_{j,\lambda} \tag{56}$$

and

$$\mathscr{E}^{(k)}_{n,q}(x|\lambda)=\sum_{j=0}^{n}\binom{n}{j}_q\mathscr{E}^{(k)}_{n-j,q}(\lambda)(x)_{j,\lambda}. \tag{57}$$

From (40), (52) and (54), we have two λ-q-sheffer sequences as follows:

$$\mathscr{B}^{(k)}_{n,q}(x|\lambda)\sim\left(\frac{e_{q,\lambda}(t)-1}{\mathrm{Li}_{k,q}(1-e^{-t})},\, t\right)_{\lambda,q} \tag{58}$$

and

$$\mathscr{E}^{(k)}_{n,q}(x|\lambda)\sim\left(\frac{t(e_{q,\lambda}(t)+1)}{2\mathrm{Li}_{k,q}(1-e^{-t})},\, t\right)_{\lambda,q}. \tag{59}$$

By (46), (58) and (59), we have

$$\left(\frac{e_{q,\lambda}(t)-1}{\mathrm{Li}_{k,q}(1-e^{-t})}\right)_{\lambda,q}\mathscr{B}^{(k)}_{n,q}(x|\lambda)=(x)_{n,\lambda} \tag{60}$$

and

$$\left(\frac{t(e_{q,\lambda}(t)+1)}{2\mathrm{Li}_{k,q}(1-e^{-t})}\right)_{\lambda,q}\mathscr{E}^{(k)}_{n,q}(x|\lambda)=(x)_{n,\lambda}. \tag{61}$$

Theorem 1. *For $s_{n,q}(x|\lambda)\sim(g(t),f(t))_{\lambda,q}$, we have*

$$s_{n,q}(x|\lambda)=\sum_{l=0}^{n}\frac{1}{[l]_q!}\left\langle\frac{1}{g(\bar{f}(t))}(\bar{f}(t))^l\middle|(x)_{n,\lambda}\right\rangle_{\lambda,q}(x)_{l,\lambda}.$$

Proof. For $s_{n,q}(x|\lambda)\sim(g(t),f(t))_{\lambda,q}$, by (40) and (41), we observe that, for any $y\in\mathbb{C}$

$$\begin{aligned}&\left\langle\frac{1}{g(\bar{f}(t))}e^{y}_{q,\lambda}(\bar{f}(t))\middle|(x)_{n,\lambda}\right\rangle_{\lambda,q}\\&=\sum_{l=0}^{\infty}s_{l,q}(y|\lambda)\frac{1}{[l]_q!}\langle t^l|(x)_{n,\lambda}\rangle_{\lambda,q}=s_{n,q}(y|\lambda),\quad(n\geq 0).\end{aligned} \tag{62}$$

On the other hand, we note that

$$\begin{aligned}&\left\langle\frac{1}{g(\bar{f}(t))}e^{y}_{q,\lambda}(\bar{f}(t))\middle|(x)_{n,\lambda}\right\rangle_{\lambda,q}\\&=\sum_{l=0}^{n}\frac{1}{[l]_q!}\left\langle\frac{1}{g(\bar{f}(t))}(\bar{f}(t))^l\middle|(x)_{n,\lambda}\right\rangle_{\lambda}(y)_{l,\lambda}\\&=\sum_{l=0}^{n}\frac{1}{[l]_q!}\left\langle\frac{1}{g(\bar{f}(t))}(\bar{f}(t))^l\middle|(x)_{n,\lambda}\right\rangle_{\lambda,q}(y)_{l,\lambda}.\end{aligned} \tag{63}$$

By (62) and (63), we get the desired result.

□

Theorem 2. *For $n \geq 0$, $k \in \mathbb{N}$, we have*

$$\mathscr{B}_{n,q}^{(k)}(1|\lambda) - \mathscr{B}_{n,q}^{(k)}(\lambda) = \sum_{j=1}^{n} \frac{j![n]_q!}{n![j]_q!}(-1)^{n+j} S_2(n,j).$$

Proof. By using (8) and (52), we observe that

$$\begin{aligned} \mathrm{Li}_{k,q}(1-e^{-t}) &= (e_{q,\lambda}(t)-1)\sum_{l=0}^{\infty} \mathscr{B}_{l,q}^{(k)}(\lambda)\frac{t^l}{[l]_q!} \\ &= \left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\frac{t^m}{[m]_q!}-1\right)\left(\sum_{l=0}^{\infty}\mathscr{B}_{l,q}^{(k)}(\lambda)\frac{t^l}{[l]_q!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}_q (1)_{m,\lambda}\mathscr{B}_{n-m,q}^{(k)}(\lambda) - \mathscr{B}_{n,q}^{(k)}(\lambda)\right)\frac{t^n}{[n]_q!} \\ &= \sum_{n=1}^{\infty}\left(\mathscr{B}_{n,q}^{(k)}(1|\lambda)-\mathscr{B}_{n,q}^{(k)}(\lambda)\right)\frac{t^n}{[n]_q!}. \end{aligned} \tag{64}$$

On the other hand, from (15) and (21), we have

$$\begin{aligned} \mathrm{Li}_{k,q}(1-e^{-t}) &= \sum_{j=1}^{\infty}\frac{(1-e^{-t})^j}{[j]_q!} = \sum_{j=1}^{\infty}\frac{j!(-1)^j}{[j]_q!}\sum_{n=j}^{\infty}S_2(n,j)\frac{(-1)^n t^n}{n!} \\ &= \sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}\frac{j![n]_q!}{n![j]_q!}(-1)^{n+j}S_2(n,j)\right)\frac{t^n}{[n]_q!}. \end{aligned} \tag{65}$$

Therefore, by comparing the coefficients of (64) and (65), we get what we wanted.

□

Theorem 3. *For $n \geq 0$ and $k \geq 1$, we have*

$$\mathscr{B}_{n,q}^{(k)}(x|\lambda) = \sum_{j=0}^{n}\sum_{l=1}^{n-j+1}\frac{[n]_q!l!}{[j]_q![l]_q!(n-j+1)!}(-1)^{l+n-j+1}S_2(n-j+1)B_{j,q}(x|\lambda).$$

Where $B_{n,q}(x|\lambda)$ are the degenerate q-Bernoulli polynomials.

Proof. By (10), (15), (21), (58) and Theorem 1, we observe that

$$
\begin{aligned}
\mathscr{B}_{n,q}^{(k)}(u|\lambda) &= \left\langle \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{e_{q,\lambda}(t)-1} e_{q,\lambda}^{u}(t) \middle| (x)_{n,\lambda} \right\rangle_{\lambda,q} \\
&= \left\langle \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{t} \middle| \left(\frac{t}{e_{q,\lambda}(t)-1} e_{q,\lambda}^{u}(t) \right)_{\lambda,q} (x)_{n,\lambda} \right\rangle_{\lambda,q} \\
&= \sum_{j=0}^{n} \binom{n}{j}_q B_{j,q}(u|\lambda) \left\langle \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{t} \middle| (x)_{n-j,\lambda} \right\rangle_{\lambda,q} \\
&= \sum_{j=0}^{n} \binom{n}{j}_q \frac{1}{[n-j+1]_q} B_{j,q}(u|\lambda) \langle \mathrm{Li}_{k,q}(1-e^{-t}) | (x)_{n-j+1,\lambda} \rangle_{\lambda,q} \\
&= \sum_{j=0}^{n} \binom{n}{j}_q \frac{1}{[n-j+1]_q} B_{j,q}(u|\lambda) \sum_{i=1}^{n-j+1} \sum_{l=1}^{i} \frac{l!(-1)^{l+i}}{i![l]_q!} S_2(i,l) \langle t^i | (x)_{n-j+1,\lambda} \rangle_{\lambda,q} \\
&= \sum_{j=0}^{n} \sum_{l=1}^{n-j+1} \binom{n}{j}_q [n-j]_q! \frac{l!(-1)^{l+n-j+1}}{(n-j+1)![l]_q!} S_2(n-j+1,l) B_{j,q}(u|\lambda).
\end{aligned} \tag{66}
$$

From (66), we get the desired identity. □

Theorem 4. *For $n \geq 0$ and $r,k \geq 1$, we have*

$$
\mathscr{B}_{n,q}^{(k)}(x|\lambda) = \sum_{l=0}^{n} \left(\sum_{j=0}^{n-l} \binom{n}{l}_q \binom{n-l}{j}_q \left(\binom{j+r}{r}_q \right)^{-1} S_{2,q}(j+r,\ r|\lambda) \mathscr{B}_{n-l-j,q}^{(k)}(\lambda) \right) B_{l,q}^{(r)}(x|\lambda).
$$

where $B_{n,q}^{(r)}(x)$ are the degenerate q-Bernoulli of order polynomials.

Proof. We consider two λ-q-Sheffer sequences as follows:

$$
\mathscr{B}_{n,q}^{(k)}(x|\lambda) \sim \left(\frac{e_{q,\lambda}(t)-1}{Li_{k,q}(1-e^{-t})},\ t \right)_{\lambda,q} \quad \text{and} \quad B_{n,q}^{(r)}(x|\lambda) \sim \left(\left(\frac{e_{q,\lambda}(t)-1}{t} \right)^{r},\ t \right)_{\lambda,q}. \tag{67}
$$

From (49),and (67), we have

$$
\mathscr{B}_{n,q}^{(k)}(x|\lambda) = \sum_{l=0}^{n} z_{n,l} B_{l,q}^{(r)}(x|\lambda), \tag{68}
$$

where, by (19) and (52), we get

$$
\begin{aligned}
z_{n,l} &= \frac{1}{[l]_q!}\left\langle \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{e_{q,\lambda}(t)-1}\frac{(e_{q,\lambda}(t)-1)^r}{t^r}t^l \middle| (x)_{n,\lambda}\right\rangle_{\lambda,q} \\
&= \binom{n}{l}_q \left\langle \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{e_{q,\lambda}(t)-1}\frac{(e_{q,\lambda}(t)-1)^r}{t^r} \middle| (x)_{n-l,\lambda}\right\rangle_{\lambda,q} \\
&= \binom{n}{l}_q \left\langle \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{e_{q,\lambda}(t)-1}\frac{[r]_q!}{t^r}\sum_{j=r}^{\infty} S_{2,q}(j,\, r|\lambda)\frac{t^j}{[j]_q!} \middle| (x)_{n-l,\lambda}\right\rangle_{\lambda,q} \\
&= \binom{n}{l}_q \sum_{j=0}^{n-l}\frac{[r]_q!}{[j+r]_q!}S_{2,q}(j+r,\, r|\lambda)\left\langle \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{e_{q,\lambda}(t)-1}t^j \middle| (x)_{n-l,\lambda}\right\rangle_{\lambda,q} \qquad (69)\\
&= \binom{n}{l}_q \sum_{j=0}^{n-l}\frac{[r]_q![j]_q!}{[j+r]_q!}\binom{n-l}{j}_q S_{2,q}(j+r,\, r|\lambda)\left\langle \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{e_{q,\lambda}(t)-1} \middle| (x)_{n-l-j,\lambda}\right\rangle_{\lambda,q} \\
&= \binom{n}{l}_q \sum_{j=0}^{n-l}\frac{\binom{n-l}{j}_q}{\binom{j+r}{r}_q}S_{2,q}(j+r,\, r|\lambda)\mathscr{B}^{(k)}_{n-l-j,q}(\lambda).
\end{aligned}
$$

Combining (68) with (69), we get the desired identity.

□

Theorem 5. *For $n \geq 0$ and $r,k \geq 1$, we have*

$$\mathscr{B}^{(k)}_{n,q}(x|\lambda) = \sum_{l=0}^{n}\left(\sum_{j=0}^{r}\frac{1}{2^r}\binom{n}{l}_q\binom{n}{j}\mathscr{B}^{(k)}_{n-l,q}(j|\lambda)\right)E^{(r)}_{l,q}(x|\lambda),$$

where $E^{(r)}_{n,q}(x|\lambda)$ are the degenerate q-Euler polynomials of order r.

Proof. We consider two λ-q-sheffer sequences as follows:

$$\mathscr{B}^{(k)}_{n,q}(x|\lambda) \sim \left(\frac{e_{q,\lambda}(t)-1}{\mathrm{Li}_{k,q}(1-e^{-t})},\, t\right)_{\lambda,q} \quad \text{and} \quad E^{(r)}_{n,q}(x|\lambda) \sim \left(\left(\frac{e_{q,\lambda}(t)+1}{2}\right)^r,\, t\right)_{\lambda,q}. \qquad (70)$$

From (49) and (70), we have

$$\mathscr{B}^{(k)}_{n,q}(x|\lambda) = \sum_{l=0}^{n} z_{n,l}E^{(r)}_{l,q}(x|\lambda), \qquad (71)$$

where, from (52), we get

$$
\begin{aligned}
z_{n,l} &= \frac{1}{[l]_q!}\left\langle \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{e_{q,\lambda}(t)-1}\left(\frac{e_{q,\lambda}(t)+1}{2}\right)^r t^l \middle| (x)_{n,\lambda}\right\rangle_{\lambda,q} \\
&= \frac{1}{2^r}\binom{n}{l}_q \left\langle \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{e_{q,\lambda}(t)-1}(e_{q,\lambda}(t)+1)^r \middle| (x)_{n-l,\lambda}\right\rangle_{\lambda,q} \\
&= \frac{1}{2^r}\binom{n}{l}_q \sum_{j=0}^{r}\binom{n}{j}\left\langle \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{e_{q,\lambda}(t)-1}e^j_{q,\lambda}(t) \middle| (x)_{n-l,\lambda}\right\rangle_{\lambda,q} \qquad (72)\\
&= \frac{1}{2^r}\binom{n}{l}_q \sum_{j=0}^{r}\binom{n}{j}\left\langle \sum_{m=0}^{\infty}\mathscr{B}^{(k)}_{m,q}(j|\lambda)\frac{t^m}{[m]_q!} \middle| (x)_{n-l,\lambda}\right\rangle_{\lambda,q} \\
&= \frac{1}{2^r}\binom{n}{l}_q \sum_{j=0}^{r}\binom{n}{j}\mathscr{B}^{(k)}_{n-l,q}(j|\lambda).
\end{aligned}
$$

Thus, by (71) and (72), we get the desired identity. □

Theorem 6. *For $n \geq 0$, $k \in \mathbb{Z}$, we have*

$$\sum_{m=0}^{n-1} \binom{n}{m}_q (1)_{m,\lambda}[n-m]_q \mathscr{E}_{n-m-1,q}^{(k)}(\lambda) + [n]_q \mathscr{E}_{n-1,q}^{(k)}(\lambda) = 2\sum_{j=1}^{n} \frac{j![n]_q!}{n![j]_q!}(-1)^{n+j} S_2(n,j).$$

Proof. By using (8) and (54), we observe that

$$\begin{aligned}
2\mathrm{Li}_{k,q}(1-e^{-t}) &= t(e_{q,\lambda}(t)+1)\sum_{l=0}^{\infty} \mathscr{E}_{l,q}^{(k)}(\lambda)\frac{t^l}{[l]_q!} \\
&= t\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\frac{t^m}{[m]_q!}+1\right)\left(\sum_{l=0}^{\infty}\mathscr{E}_{l,q}^{(k)}(\lambda)\frac{t^l}{[l]_q!}\right) \\
&= \left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\frac{t^m}{[m]_q!}+1\right)\left(\sum_{l=1}^{\infty}\mathscr{E}_{l-1,q}^{(k)}(\lambda)\frac{t^l}{[l-1]_q!}\right) \\
&= \sum_{n=1}^{\infty}\left(\sum_{m=0}^{n-1}\binom{n}{m}_q (1)_{m,\lambda}[n-m]_q\mathscr{E}_{n-m-1,q}^{(k)}(\lambda)+[n]_q\mathscr{E}_{n-1,q}^{(k)}(\lambda)\right)\frac{t^n}{[n]_q!}.
\end{aligned} \tag{73}$$

On the other hand, from (65), we have

$$2\mathrm{Li}_{k,q}(1-e^{-t}) = 2\sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}\frac{j![n]_q!}{n![j]_q!}(-1)^{n+j}S_2(n,j)\right)\frac{t^n}{[n]_q!}. \tag{74}$$

Therefore, by comparing the coefficients of (73) and (74), we get what we wanted. □

Theorem 7. *For $n \geq 0$ and $k \geq 1$, we have*

$$\mathscr{E}_{n,q}^{(k)}(x|\lambda) = \sum_{j=0}^{n}\sum_{l=1}^{n-j+1}\frac{[n]_q!l!(-1)^{n-j+l+1}}{[j]_q![l]_q!(n-j+1)!}S_2(n-j+1,l)E_{j,q}(u|\lambda),$$

where $E_{n,q}(x|\lambda)$ are the degenerate q-Bernoulli polynomials.

Proof. By (12), (65) and Theorem 1, we observe that

$$
\begin{aligned}
\mathscr{E}^{(k)}_{n,q}(u|\lambda) &= \left\langle \frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{t(e_{q,\lambda}(t)+1)} e^{u}_{q,\lambda}(t) \middle| (x)_{n,\lambda} \right\rangle_{\lambda,q} \\
&= \left\langle \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{t} \middle| \left(\frac{2}{e_{q,\lambda}(t)+1} e^{u}_{q,\lambda}(t) \right)_{\lambda,q} (x)_{n,\lambda} \right\rangle_{\lambda,q} \\
&= \sum_{j=0}^{n} \binom{n}{j}_q E_{j,q}(u|\lambda) \left\langle \frac{1}{t} \sum_{i=1}^{\infty} \sum_{l=1}^{i} \frac{l!(-1)^{l+i}}{i![l]_q!} S_2(i,l) t^i \middle| (x)_{n-j,\lambda} \right\rangle_{\lambda,q} \\
&= \sum_{j=0}^{n} \binom{n}{j}_q E_{j,q}(u|\lambda) \left\langle \sum_{i=0}^{\infty} \sum_{l=1}^{i+1} \frac{l!(-1)^{l+i+1}}{(i+1)![l]_q!} S_2(i+1,l) t^i \middle| (x)_{n-j,\lambda} \right\rangle_{\lambda,q} \\
&= \sum_{j=0}^{n} \binom{n}{j}_q E_{j,q}(u|\lambda) \sum_{l=1}^{n-j+1} \frac{l!(-1)^{n-j+l+1}[n-j]_q!}{(n-j+1)![l]_q!} S_2(n-j+1,l) \\
&= \sum_{j=0}^{n} \sum_{l=1}^{n-j+1} \frac{[n]_q! l! (-1)^{n-j+l+1}}{[j]_q![l]_q!(n-j+1)!} S_2(n-j+1,l) E_{j,q}(u|\lambda).
\end{aligned}
\tag{75}
$$

From (75), we get the desired identity. □

Theorem 8. *For $n \geq 0$ and $r, k \geq 1$, we have*

$$
\mathscr{E}^{(k)}_{n,q}(x|\lambda) = \sum_{l=0}^{n} \left(\sum_{i=0}^{n} \frac{1}{2^r} \binom{n}{l}_q \binom{n}{i} \mathscr{E}^{(k)}_{n-l,q}(i|\lambda) \right) E^{(r)}_{l,q}(x),
$$

where $E^{(r)}_{n,q}(x)$ are the degenerate q-Euler polynomials of order r.

Proof. We consider two λ-q-Sheffer sequences as follows:

$$
\mathscr{E}^{(k)}_{n,q}(x|\lambda) \sim \left(\frac{t(e_{q,\lambda}(t)+1)}{2\mathrm{Li}_{k,q}(1-e^{-t})},\ t \right)_{\lambda,q} \quad \text{and} \quad E^{(r)}_{n,q}(x|\lambda) \sim \left(\left(\frac{e_{q,\lambda}(t)+1}{2} \right)^r,\ t \right)_{\lambda,q}. \tag{76}
$$

From (49) and (76), we have

$$
\mathscr{E}^{(k)}_{n,q}(x|\lambda) = \sum_{l=0}^{n} z_{n,l} E^{(r)}_{l,q}(x), \tag{77}
$$

where, by (54), we have

$$
\begin{aligned}
z_{n,l} &= \frac{1}{2^r [l]_q!} \left\langle \frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{t(e_{q,\lambda}(t)+1)} (e_{q,\lambda}(t)+1)^r t^l \middle| (x)_{n,\lambda} \right\rangle_{\lambda,q} \\
&= \frac{1}{2^r} \binom{n}{l}_q \left\langle \frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{t(e_{q,\lambda}(t)+1)} (e_{q,\lambda}(t)+1)^r \middle| (x)_{n-l,\lambda} \right\rangle_{\lambda,q} \\
&= \frac{1}{2^r} \binom{n}{l}_q \sum_{i=0}^{r} \binom{n}{i} \left\langle \frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{t(e_{q,\lambda}(t)+1)} e^{i}_{q,\lambda}(t) \middle| (x)_{n-l,\lambda} \right\rangle_{\lambda,q} \\
&= \frac{1}{2^r} \binom{n}{l}_q \sum_{i=0}^{r} \binom{n}{i} \left\langle \sum_{j=0}^{\infty} \mathscr{E}^{(k)}_{j,q}(i|\lambda) \frac{t^j}{[j]_q!} \middle| (x)_{n-l,\lambda} \right\rangle_{\lambda,q} \\
&= \frac{1}{2^r} \binom{n}{l}_q \sum_{i=0}^{n} \binom{n}{i} \mathscr{E}^{(k)}_{n-l,q}(i|\lambda).
\end{aligned}
\tag{78}
$$

By (77) and (78), we get the desired identity.

□

Theorem 9. *For $n \geq 0$ and $r, k \geq 1$, we have*

$$\mathscr{E}_{n,q}^{(k)}(x|\lambda) = \sum_{l=0}^{n} \left(\sum_{j=0}^{n-l} \frac{\binom{n}{l}_q \binom{n-l}{j}_q}{\binom{j+r}{r}_q} S_{2,q}(j+r,\ r|\lambda) \mathscr{E}_{n-l-j,q}^{(k)}(\lambda) \right) B_{l,q}^{(r)}(x|\lambda).$$

Proof. We consider two λ-q-Sheffer sequences as follows:

$$\mathscr{E}_{n,q}^{(k)}(x|\lambda) \sim \left(\frac{t(e_{q,\lambda}(t)+1)}{2\mathrm{Li}_{k,q}(1-e^{-t})},\ t \right)_{\lambda,q} \quad \text{and} \quad B_{n,q}^{(r)}(x|\lambda) \sim \left(\left(\frac{e_{q,\lambda}(t)-1}{t} \right)^r,\ t \right)_{\lambda,q}. \tag{79}$$

From (76) and (79), we have

$$\mathscr{E}_{n,q}^{(k)}(x|\lambda) = \sum_{l=0}^{n} z_{n,l} B_{l,q}^{(r)}(x|\lambda), \tag{80}$$

Where, by (19) and (54), we have

$$\begin{aligned} z_{n,l} &= \frac{1}{[l]_q!} \left\langle \frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{t(e_{q,\lambda}(t)+1)} \frac{(e_{q,\lambda}(t)-1)^r}{t^r} t^l \middle| (x)_{n,\lambda} \right\rangle_{\lambda,q} \\ &= \binom{n}{l}_q \left\langle \frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{t(e_{q,\lambda}(t)+1)} \frac{[r]_q!}{t^r} \sum_{j=r}^{\infty} S_{2,q}(j,r|\lambda) \frac{t^j}{[j]_q!} \middle| (x)_{n-l,\lambda} \right\rangle_{\lambda,q} \\ &= \binom{n}{l}_q \sum_{j=0}^{n-l} \frac{[r]_q!}{[j+r]_q!} S_{2,q}(j+r,r|\lambda) \left\langle \frac{2\mathrm{Li}_{k,q}(1-e^{-t})}{t(e_{q,\lambda}(t)+1)} t^j \middle| (x)_{n-l,\lambda} \right\rangle_{\lambda,q} \\ &= \binom{n}{l}_q \sum_{j=0}^{n-l} \frac{\binom{n-l}{j}_q}{\binom{j+r}{r}_q} S_{2,q}(j+r,r|\lambda)\ \mathscr{E}_{n-l-j,q}^{(k)}(\lambda). \end{aligned} \tag{81}$$

Combining (80) with (81), we get the desired result. □

4. Further Remark

In this section, we consider new types of degenerate poly-q-Bernoulli polynomials and degenerate poly-q-Euler polynomials different from those introduced in Section 3.

For $k \in \mathbb{Z}$ and $0 \neq \lambda \in \mathbb{R}$, the degenerate polylogarithm functions are defined by

$$\mathrm{Li}_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}(1)_{n,\frac{1}{\lambda}}}{(n-1)!n^k} x^n, \quad \text{(see [26, 32])}. \tag{82}$$

The compositional inverse $\log_\lambda(t)$ of $e_\lambda(t)$ is given by

$$\log_\lambda(t) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{n!} (1)_{n,\frac{1}{\lambda}} (t-1)^n, \quad \text{(see [26, 32])}. \tag{83}$$

From (82) and (83), we note that

$$\mathrm{Li}_{1,\lambda}(x) = -\log_\lambda(1-x) \ \text{ and } \ \lim_{\lambda \to 0} \mathrm{Li}_{k,\lambda} = \mathrm{Li}_k(x),$$

where $\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^k}{n^k}$.

Now, we consider a new type of degenerate poly-q-Bernoulli polynomials defined by

$$\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{e_{q,\lambda}(t)-1}e_\lambda^x(t)=\sum_{n=0}^{\infty}\beta_{n,q}^{(k)}(x|\lambda)\frac{t^n}{[n]_q!}. \tag{84}$$

We note that $e_\lambda(-t)=e_{1,\lambda}(-t)$.

When $x=0$, $\beta_{n,q}^{(k)}(\lambda)=\beta_{n,q}^{(k)}(0|\lambda)$ are called the degenerate poly-q-Bernoulli numbers.

When $q=1$, we note that $\beta_{n,1}^{(k)}(x|\lambda)=B_{n,\lambda}^{(k)}(x)$ are the degenerate poly-Bernoulli polynomials of Kim-Kim's [26].

We also consider the degenerate poly-q-Euler polynomials defined by

$$\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-2t))}{t(e_{q,\lambda}(t)+1)}e_\lambda^x(t)=\sum_{n=0}^{\infty}\xi_{n,q}^{(k)}(x|\lambda)\frac{t^n}{[n]_q!}. \tag{85}$$

When $x=0$, $\xi_{n,q}^{(k)}(\lambda)=\xi_{n,q}^{(k)}(0|\lambda)$ are called the degenerate poly-q-Euler numbers.

When $k=q=1$, $\xi_{n,1}^{(1)}(x|\lambda)=E_n(x|\lambda)$ are the degenerate Euler polynomials.

From (40), (84) and (85), we have two λ-q-sheffer sequences as follows:

$$\beta_{n,q}^{(k)}(x|\lambda)\sim\left(\frac{e_{q,\lambda}(t)-1}{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))},\ t\right)_{\lambda,q} \tag{86}$$

and

$$\xi_{n,q}^{(k)}(x|\lambda)\sim\left(\frac{t(e_{q,\lambda}(t)+1)}{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-2t))},\ t\right)_{\lambda,q}. \tag{87}$$

For $n\geq 0$, by using (8), (86) and (87) we note that

$$\beta_{n,q}^{(k)}(x|\lambda)=\sum_{j=0}^{n}\binom{n}{j}_q\beta_{n-j,q}^{(k)}(\lambda)(x)_{j,\lambda} \tag{88}$$

and

$$\xi_{n,q}^{(k)}(x|\lambda)=\sum_{j=0}^{n}\binom{n}{j}_q\xi_{n-j,q}^{(k)}(\lambda)(x)_{j,\lambda}. \tag{89}$$

From (46), (86) and (87), we have

$$\left(\frac{e_{q,\lambda}(t)-1}{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}\right)_{\lambda,q}\beta_{n,q}^{(k)}(x|\lambda)=(x)_{n,\lambda} \tag{90}$$

and

$$\left(\frac{t(e_{q,\lambda}(t)+1)}{\mathrm{Li}_{k}(1-e_\lambda(-2t))}\right)_{\lambda,q}\xi_{n,q}^{(k)}(x|\lambda)=(x)_{n,\lambda}. \tag{91}$$

In a similar way to Section 3, we can obtain the following theorems:

Theorem 10. *For $n\geq 0$, $k\in\mathbb{N}$, we have*

$$\beta_{n,q}^{(k)}(1|\lambda)-\beta_{n,q}^{(k)}(\lambda)=\sum_{j=1}^{n}\frac{(1)_{j,\frac{1}{\lambda}}(-1)^{n-1}}{j^{k-1}}\lambda^{j-1}S_{2,q}(n,j|\lambda).$$

Theorem 11. *For $n \geq 0$ and $k \geq 1$, we have*

$$\beta_{n,q}^{(k)}(x|\lambda) = \sum_{j=0}^{n} \sum_{l=1}^{n-j+1} \frac{[n]_q!(-1)^{n-j}\lambda^{l-1}(1)_{n,\frac{1}{\lambda}}}{[j]_q!(n-j+1)!(l-1)^{k-1}} S_{2,\lambda}(n-j+1) B_{j,q}(x|\lambda),$$

where $B_{n,q}(x|\lambda)$ are the degenerate q-Bernoulli polynomials.

Theorem 12. *For $n \geq 0$ and $r, k \geq 1$, we have*

$$\beta_{n,q}^{(k)}(x|\lambda) = \sum_{l=0}^{n} \left(\sum_{j=0}^{n-l} \binom{n}{l}_q \binom{n-l}{j}_q \left(\binom{j+r}{r}_q \right)^{-1} S_{2,q}(j+r,\ r|\lambda) \beta_{n-l-j,q}^{(k)}(\lambda) \right) B_{l,q}^{(r)}(x),$$

where $B_{n,q}^{(r)}(x)$ are the degenerate q-Bernoulli polynomials of order r.

Theorem 13. *For $n \geq 0$ and $r, k \geq 1$, we have*

$$\beta_{n,q}^{(k)}(x|\lambda) = \sum_{l=0}^{n} \left(\sum_{j=0}^{r} \frac{1}{2^r} \binom{n}{l}_q \binom{n}{j} \beta_{n-l,q}^{(k)}(j|\lambda) \right) E_{l,q}^{(r)}(x|\lambda),$$

where $E_{n,q}^{(r)}(x|\lambda)$ are the degenerate q-Euler polynomials of order r.

Theorem 14. *For $n \geq 0$, $k \in \mathbb{Z}$, we have*

$$\begin{aligned}\sum_{m=0}^{n-1} \binom{n}{m}_q (1)_{m,\lambda} [n-m]_q \xi_{n-m-1,q}^{(k)}(\lambda) + [n]_q \xi_{n-1,q}^{(k)}(\lambda) \\ = \sum_{j=1}^{n} \frac{(1)_{j,\frac{1}{\lambda}}(-1)^{n-1}}{j^{k-1}} \lambda^{j-1} 2^n S_{2,q}(n, j|\lambda).\end{aligned}$$

Theorem 15. *For $n \geq 0$ and $k \geq 1$, we have*

$$\xi_{n,q}^{(k)}(x|\lambda) = \sum_{j=0}^{n} \sum_{l=0}^{n-j+1} \frac{[n]_q!(-2)^{n-j}\lambda^{l-1}(1)_{n,\frac{1}{\lambda}}}{[j]_q!(n-j+1)!(l-1)^{k-1}} S_{2,\lambda}(n-j+1, l) E_{j,q}(u|\lambda),$$

where $E_{n,q}(x|\lambda)$ are the degenerate q-Bernoulli polynomials.

Theorem 16. *For $n \geq 0$ and $r, k \geq 1$, we have*

$$\xi_{n,q}^{(k)}(x|\lambda) = \sum_{l=0}^{n} \left(\sum_{i=0}^{n} \frac{1}{2^r} \binom{n}{l}_q \binom{n}{i} \xi_{n-l,q}^{(k)}(i|\lambda) \right) E_{l,q}^{(r)}(x),$$

where $E_{n,q}^{(r)}(x)$ are the degenerate q-Euler polynomials of order r.

Theorem 17. *For $n \geq 0$ and $r, k \geq 1$, we have*

$$\xi_{n,q}^{(k)}(x|\lambda) = \sum_{l=0}^{n} \left(\sum_{j=0}^{n-l} \frac{\binom{n}{l}_q \binom{n-l}{j}_q}{\binom{j+r}{r}_q} S_{2,q}(j+r,\ r|\lambda) \xi_{n-l-j,q}^{(k)}(\lambda) \right) B_{l,q}^{(r)}(x|\lambda).$$

5. Conclusion

In this paper, we introduced degenerate poly q-Bernoulli polynomials and degenerate poly-q-Euler polynomials. As applications of λ-q-Sheffer sequences, the degenerate poly q-Bernoulli polynomials are represented by a finite linear combination of the degenerate q-Bernoulli polynomials in Theorem 3, degenerate q-Bernoulli polynomials of order r in Theorem 4, and degenerate q-Euler polynomials of order r in Theorem 5. The degenerate poly q-Euler polynomials are also represented by a finite linear combination of the degenerate q-Euler polynomials in Theorem 7, and degenerate q-Bernoulli polynomials of order r in Theorem 9. For future projects, we would like to study many more interesting combinatorial identities and properties related to the degenerate poly q-Bernoulli polynomials and the degenerate poly-q-Euler polynomials considered in this paper.

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DEPARTMENT OF MATHEMATICS, KWANGWOON UNIVERSITY, SEOUL 139-701, REPUBLIC OF KOREA
E-mail address: kwangwoonmath@kw.ac.kr

DEPARTMENT OF MATHEMATICS, SOGANG UNIVERSITY, SEOUL 121-742, REPUBLIC OF KOREA
E-mail address: dskim@sogang.ac.kr

DEPARTMENT OF MATHEMATICS EDUCATION, DAEGU CATHOLIC UNIVERSITY, GYEONGSAN 38430, REPUBLIC OF KOREA
E-mail address: hkkim@cu.ac.kr