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FEW MORE RELATIONS CONNECTING RAMANUJAN-TYPE EISENSTEIN SERIES AND CUBIC THETA FUNCTIONS OF BOREWEIN

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Abstract. This paper aims to formulate identities involving Ramanujan-type Eisenstein series and the cubic theta functions of Borwein, utilizing the (p,k)-parametrization introduced by Alaca. In addition, as an application, by using the derived identities, an appealing representation for the discrete convolution sum $\sum_{i+2j=\alpha} \delta(i)\delta(j)$ have been evaluated.

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1. Introduction

Convolution is an operation in Science and Engineering that involves the computation of data in the form of multiplying two vectors and then accumulating the results. Convolution is frequently used in the field of numerical linear algebra, probability theory, numerical analysis, deep learning and in the design and implementation of finite impulse response filters in signal processing. Convolution may also be applicable in communication systems and forms the theoretical basis of Digital Signal processing. For the determination of convolution sums, mathematicians most frequently use Ramanujan's discriminant function, Gaussian hyper-geometric series, quasi-modular forms, Ramanujan-type Eisenstein series and many more.
In this article, we formulate few fascinating identities involving Eisenstein series and Borwein's cubic theta functions by adopting the parameters p and k, introduced by Alaca. Interestingly, without the help of computer, Ramanujan-type Eisenstein series have been expressed in terms of the product of cubic theta functions. Further, the derived formulas have been used to evaluate a new representation for the discrete convolution sum $\sum_{i+2j=\alpha} \delta(i)\delta(j)$, for all positive integers l.
Section 2 is intended to provide few preliminary results that are helpful in achieving the main outcomes. In Section 3, some interesting identities have been stated and proved, that are analogues of Earnest Xia's identities, but found to be new. These relations include the Ramanujan-type Eisenstein series and cubic theta functions of Borwein. In Section 4, a new representation of evaluating a discrete convolution sum have been discussed.

2. Preliminaries

Definition 2.1. *For any complex r and s, Ramanujan*[4, p.35] *documented a general theta function,*

$$f(r,s) := \sum_{\alpha=-\infty}^{\infty} r^{\alpha(\alpha+1)/2} s^{\alpha(\alpha-1)/2} := (-r;rs)_\infty(-s;rs)_\infty(rs;rs)_\infty,$$

where

$$(r;q)_\infty := \prod_{\alpha=0}^{\infty}(1-rq^\alpha), \qquad |q|<1.$$

The special case of theta function defined by Ramanujan[4, p.35]*,*

$$\varphi(q) := f(q,q) = \sum_{\alpha=-\infty}^{\infty} q^{\alpha^2} = (-q;q^2)^2_\infty(q^2;q^2)_\infty.$$

J. M. Borewein an P. B. Borewein [6] documented the following three two-dimensional theta functions in their work on a cubic counterpart of jacobi and a cubic analogue of arithmetic geometric mean iteration of Legendre and Gauss,

$$a(q) := \sum_{\alpha=-\infty}^{\infty}\sum_{\beta=-\infty}^{\infty} q^{\alpha^2+\alpha\beta+\beta^2},$$

$$b(q) := \sum_{\alpha=-\infty}^{\infty}\sum_{\beta=-\infty}^{\infty} y^{\alpha-\beta} q^{\alpha^2+\alpha\beta+\beta^2},$$

$$c(q) := \sum_{\alpha=-\infty}^{\infty}\sum_{\beta=-\infty}^{\infty} q^{(\alpha+\frac{1}{3})^2+(\alpha+\frac{1}{3})(\beta+\frac{1}{3})+(\beta+\frac{1}{3})^2},$$

where $y = exp(2\pi i/3)$ and $q \in \mathbb{C}$, the set of complex numbers.
Also, note that

$$a(0)=1, b(0)=1, c(0)=1.$$

Further, the representation for $a(q)$ in terms of q-series is recorded by Borewein brothers [6] and Berndt [5]:

$$a(q) = S(q) + 4T(q),$$

where

(1)

$$S(q) = (-q;q^2)_\infty(-q^3;q^6)^2_\infty(q^2;q^2)_\infty(q^6;q^6)_\infty \text{ and } T(q) = q\frac{(q^4;q^4)_\infty(q^{12};q^{12})_\infty}{(q^2;q^4)_\infty(q^6;q^{12})_\infty}.$$

In their wonderful article, Alaca et. al [3] defined the (p,k) parametrization of theta functions that has lots of importance, especially in designing duplication and triplication principle and further obtaining certain sum to product identities. These parameters p and k are defined as follows:

$$p = p(q) = \frac{\varphi^2(q)-\varphi^2(q^3)}{2\varphi^2(q^3)}, \; k = k(q) = \frac{\varphi^2(q^3)}{\varphi(q)}.$$

Lemma 2.1. [2] *The demonstrated parametric representations of* $a(q^m), b(q^m), c(q^m) (m \in 1, 2, 3, 4, 6, 8, 12)$*, as well as* $c(-q)$ *in terms of the parameters* p *and* k *are given by*

$$\begin{aligned}
a(q) &= (p^2+4p+1)k,\\
a(q^2) &= (p^2+p+1)k,\\
a(q^3) &= \frac{(p^2+4p+1+2^{2/3}(1-p)((1-p)(2+p)(1+2p))^{1/3})k}{3},\\
a(q^4) &= \frac{(-p^2+2p+2)k}{2},\\
a(q^6) &= \frac{(p^2+p+1+2^{1/3}((1-p)(2+p)(1+2p))^{2/3})k}{3},\\
b(q) &= 2^{-1/3}(1-p)((1-p)(2+p)(1+2p))^{1/3}k,\\
b(q^2) &= 2^{-2/3}((1-p)(2+p)(1+2p))^{2/3}k,\\
c(-q) &= -2^{1/3}3(p(1+p))^{1/3}k,\\
c(q) &= 2^{-1/3}3(p(1+p)^4)^{1/3},\\
c(q^2) &= 2^{-2/3}3(p(1+p))^{2/3}k,\\
c(q^4) &= 2^{-4/3}3p(p(1+p))^{1/3}k,\\
c(q^6) &= \frac{\left(p^2+p+1-2^{-2/3}((1-p)(2+p)(1+2p))^{2/3}\right)k}{3}.
\end{aligned}$$

Definition 2.2. *Ramanujan, in his second notebook* [8]*, documented certain infinite series named as Ramanujan-type Eisenstein series:*

$$\begin{aligned}
L(q) &:= 1-24\sum_{\alpha=1}^{\infty}\frac{\alpha q^\alpha}{1-q^\alpha} = 1-24\sum_{\alpha=1}^{\infty}\delta_1(\alpha)q^\alpha,\\
M(q) &:= 1+240\sum_{\alpha=1}^{\infty}\frac{\alpha^3 q^\alpha}{1-q^\alpha} = 1+240\sum_{\alpha=1}^{\infty}\delta_3(\alpha)q^\alpha,\\
N(q) &:= 1-504\sum_{\alpha=1}^{\infty}\frac{\alpha^5 q^\alpha}{1-q^\alpha} = 1-504\sum_{\alpha=1}^{\infty}\delta_5(\alpha)q^\alpha.
\end{aligned}$$

Lemma 2.2. [2] *For the above specified Eisenstein series, the representations in terms of the parameters* p *and* k *are given by*

$$\begin{aligned}
&M(q) = (1+124p(1+p^6)+964p^2(1+p^4)+2788p^3(1+p^2)+3910p^4+p^8)k^4,\\
&M(q^2) = (1+4p(1+p^6)+64p^2(1+p^4)+178p^3(1+p^2)+235p^4+p^8)k^4,\\
&M(q^3) = (1+4p(1+p^6)+4p^2(1+p^4)+28p^3(1+p^2)+70p^4+p^8)k^4,\\
&M(q^6) = (1+4p(1+p^6)+4p^2(1+p^4)-2p^3(1+p^2)-5p^4+p^8)k^4,\\
&M(q^{12}) = (1+4p(1+p)-2p^3(1+p^2)-5p^4+1/4p^6(1+p)+1/16p^8)k^4,\\
&L(-q)-L(q) = 3(8p+12p^2+6p^3+p^4)k^2,
\end{aligned}$$

$$L_{1,2}(q) = \frac{1}{48}(L(-q) - L(q)) = (1/2p + 3/4p^2 + 3/8p^3 + 1/16p^4)k^2,$$
$$L(-q) - 2L(q^2) = -(1 - 10p - 12p^2 - 4p^3 - 2p^4)k^2,$$
$$L(q) - 2L(q^2) = -(1 + 14p(1+p^2) + 24p^2 + p^4)k^2,$$
$$L(q) - 3L(q^3) = -(1 + 8p(1+p^2) + 18p^2 + p^4)k^2,$$
$$L(q) - 6L(q^6) = -(5 + 22p(1+p^2) + 36p^2 + 5p^4)k^2,$$
$$L(q^2) - 3L(q^6) = -2(1 + 2p(1+p^2) + 3p^2 + p^4)k^2,$$
$$L(q^3) - 2L(q^6) = -(1 + 2p(1+p^2) + p^4)k^2.$$

In his notebook [4], Ramanujan devoted much attention to Eisenstein series, most notably to L, M, and N and supplied few interesting identities consisting of infinite series and theta functions. Using Computer, E. X. W. Xia and O. X. M. Yao [12] demonstrated some beautiful relationships among Eisenstein series and cubic theta functions of Borewein and exhibited the application of these identities in the explicit evaluation of convolution sum. Recently, similar types of identities have been established by Sruthi and B. R. Srivatsa Kumar [10]. Inspired by their work, few new sum to product identities involving $L(q^n), n = 1,2,3,6, M(q^n), n = 1,2,3,6,12$ and $L(-q^m)$ for $m = 1,3$ has been generated in this article. Moreover, employing some of these relations a discrete convolution sum is evaluated.

3. Eisenstein Series Identities

Theorem 3.1. *The relation among an infinite series and theta functions hold:*

$$\sum_{\alpha=1}^{\infty}\left[\frac{\alpha(-q)^\alpha}{1-(-q)^\alpha} + \frac{\alpha q^\alpha}{1-q^\alpha} - \frac{5\alpha q^{2\alpha}}{1-q^{2\alpha}} + \frac{3\alpha q^{6\alpha}}{1-q^{6\alpha}}\right] = \frac{a(q^4)c^2(q^4)}{3c(q^2)}.$$

Proof. Assume that

$$C_1(2L(q^2) - L(-q)) + C_2(3L(q^3) - L(q)) + C_3(6L(q^6) - L(q)) + C_4(3L(q^6) - L(q^2))$$
$$+ C_5(2L(q^6) - L(q^3)) = \frac{a(q^4)c^2(q^4)}{c(q^2)}. \tag{2}$$

We formulate a system by incorporating Lemma 2.2 and expressing the above relation in terms of (p,k) parametrization and then equalizing the coefficients of $k^2, pk^2, p^2k^2, p^3k^2$ and p^4k^2 on either side,

$$\begin{pmatrix} 1 & 2 & 5 & 2 & 1 \\ -10 & 16 & 22 & 4 & 2 \\ -12 & 36 & 6 & 6 & 0 \\ -4 & 16 & 22 & 4 & 2 \\ -2 & 2 & 5 & 2 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3/4 \\ 3/4 \\ -3/8 \end{pmatrix}.$$

Note that, the above linear system possess infinitely many solutions,

$$C_1=\frac{1}{8}, C_2=\frac{v}{3}, C_3=\frac{3-8v}{24}, C_4=-\frac{3}{8}, C_5=v,$$

where $u, v \in \mathbb{R}$.
Result follows immediately by substituting the above statistic in (2) and simplifying using Definition 2.2. □

Theorem 3.2. *One has*

$$1+4\sum_{\alpha=1}^{\infty}\left[\frac{2\alpha(-q)^{\alpha}}{1-(-q)^{\alpha}}+\frac{\alpha q^{2\alpha}}{1-q^{2\alpha}}-\frac{6\alpha q^{3\alpha}}{1-q^{3\alpha}}-\frac{3\alpha q^{6\alpha}}{1-q^{6\alpha}}\right]=\frac{1}{2^{4/3}}\frac{b^2(q)c^2(-q)}{3b(q^2)c(q^2)}.$$

Proof. Consider the inequality

$$C_1(L(q)-L(-q))+C_2(2L(q^2)-L(-q))+C_3(2L(q^6)-L(q^3))+C_4(3L(q^3)-L(q))$$
$$+C_5(3L(q^6)-L(q^2))=\frac{1}{2^{4/3}}\frac{b^2(q)c^2(-q)}{b(q^2)c(q^2)}. \tag{3}$$

Now, the following system is generated by comparing the coefficients of $k^2, pk^2, p^2k^2, p^3k^2$ and p^4k^2 on either sides,

$$\begin{pmatrix} 0 & 1 & 1 & 2 & 2\\ 24 & -10 & 2 & 16 & 4\\ 36 & -12 & 0 & 36 & 6\\ 18 & -4 & 2 & 16 & 4\\ 3 & -2 & 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} C_1\\ C_2\\ C_3\\ C_4\\ C_5\end{pmatrix}=\begin{pmatrix} 3\\ -6\\ 3\\ 0\\ 0\end{pmatrix}.$$

The infinitely many solutions obtained after solving the system are given by,

$$C_1=\frac{2v-5}{4}, C_2=\frac{2v-1}{4}, C_3=\frac{3-6v}{4}, C_4=\frac{5-2v}{4}, C_5=v,$$

where $u, v \in \mathbb{R}$.
Proof is completed by replacing the above values in (3) and further using Definition 2.2. □

Theorem 3.3. *The following relations among the series and theta functions hold:*

$$(i)\ 1+3\sum_{\alpha=1}^{\infty}\left[\frac{\alpha(-q)^{\alpha}}{1-(-q)^{\alpha}}+\frac{3\alpha(-q)^{3\alpha}}{1-(-q)^{3\alpha}}+\frac{\alpha q^{\alpha}}{1-q^{\alpha}}-\frac{4\alpha q^{2\alpha}}{1-q^{2\alpha}}+\frac{3\alpha q^{3\alpha}}{1-q^{3\alpha}}-\frac{12\alpha q^{6\alpha}}{1-q^{6\alpha}}\right]=a(q^2)a(q^4).$$

$$(ii)1+4\sum_{\alpha=1}^{\infty}\left[\frac{2\alpha(-q)^{3\alpha}}{1-(-q)^{3\alpha}}+\frac{\alpha q^{2\alpha}}{1-q^{2\alpha}}-\frac{2\alpha q^{3\alpha}}{1-q^{3\alpha}}-\frac{11\alpha q^{6\alpha}}{1-q^{6\alpha}}\right]=\frac{c^2(-q)c^2(q)}{3^2 2^{4/3}c^2(q^2)}.$$

$$(iii)\sum_{\alpha=1}^{\infty}\left[\frac{\alpha q^{\alpha}}{1-q^{\alpha}}-\frac{2\alpha q^{2\alpha}}{1-q^{2\alpha}}-\frac{\alpha q^{3\alpha}}{1-q^{3\alpha}}+\frac{2\alpha q^{6\alpha}}{1-q^{6\alpha}}\right]=-\frac{c(-q)c^2(q)c(q^4)}{3^2 2^{2/3}c^2(q^2)}.$$

Proof. Consider the relation

$$C_1 L_{1,2}(q) + C_2 L_{1,2}(q^3) + C_3(L(q) - 2L(q^2)) + C_4(L(q) - 3L(q^3)) + C_5(L(q^2) - 3L(q^6)) = a(q^2)a(q^4). \tag{4}$$

Extracting the coefficients of $k^2, pk^2, p^2k^2, p^3k^2$ and p^4k^2 on either sides, we design the system

$$\begin{pmatrix} 0 & 0 & -1 & -2 & -2 \\ 1/2 & 0 & -14 & -16 & -4 \\ 3/4 & 0 & -24 & -36 & -6 \\ 3/8 & 1/8 & -14 & -16 & -4 \\ 1/16 & 1/16 & -1 & -2 & -2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3/2 \\ 1/2 \\ -1/2 \end{pmatrix}.$$

Solving for C_1, C_2, C_3, C_4, C_5, we note that, the generated system has a unique solution, given by

$$C_1 = -6, C_2 = -18, C_3 = -\frac{1}{2}, C_4 = \frac{1}{4}, C_5 = -\frac{1}{2}.$$

Now the required result follows by substituting these entries in (4) and applying Definition 2.2.
Likewise, utilizing the same technique, the following identities are deduced.

$$-288L_{1,2}(q^3) - 4(L(q) - L(q^2)) + 4(L(q) - 3L(q^3)) - 11(L(q^2) - 3L(q^6)) = \frac{c^2(-q)c^2(q)}{2^{1/3}c^2(q^2)}. \tag{5}$$

$$\frac{1}{4}(L(q) - L(q^2)) + \frac{1}{8}(L(q) - 3L(q^3)) - \frac{1}{4}(L(q^2) - 3L(q^6)) = \frac{c(-q)c^2(q)c(q^4)}{2^{2/3}c^2(q^2)}. \tag{6}$$

The identities (ii) and (iii) follows from (5) and (6), by employing Definition 2.2.

□

Theorem 3.4. *One has*

(i) $\displaystyle \frac{1}{3} + 4\sum_{\alpha=1}^{\infty}\left[\frac{\alpha q^{\alpha}}{1-q^{\alpha}} - \frac{3\alpha q^{3\alpha}}{1-q^{3\alpha}}\right] = \left(a(q^3) - \frac{2}{3}b(q)\right)a(q).$

(ii) $\displaystyle \frac{1}{3} + 4\sum_{\alpha=1}^{\infty}\left[\frac{\alpha q^{2\alpha}}{1-q^{2\alpha}} - \frac{3\alpha q^{6\alpha}}{1-q^{6\alpha}}\right] = \left(a(q^6) - \frac{2}{3}b(q^2)\right)a(q^2).$

(iii) $\displaystyle \frac{1}{3} + 2\sum_{\alpha=1}^{\infty}\left[\frac{\alpha q^{\alpha}}{1-q^{\alpha}} - \frac{2\alpha q^{2\alpha}}{1-q^{2\alpha}} + \frac{3\alpha q^{3\alpha}}{1-q^{3\alpha}} - \frac{6\alpha q^{6\alpha}}{1-q^{6\alpha}}\right] = \left(c(q^6) + \frac{1}{3}b(q^2)\right)a(q).$

Proof. We assume that

$$C_1(2L(q^2) - L(-q)) + C_2(3L(q^3) - L(q)) + C_3(6L(q^6) - L(q)) + C_4(3L(q^6) - L(q^2)) + C_5(2L(q^6) - L(q^3)) = \left(c(q^2) + \frac{1}{3}b(q)\right)a(q). \tag{7}$$

With the help of Lemma 2.2, expressing (7) in terms of (p,k) parametrization and then comparing the coefficient of $k^2, pk^2, p^2k^2, p^3k^2$ and p^4k^2on either sides, we generate the system,

$$\begin{pmatrix} 1 & 2 & 5 & 2 & 1 \\ -10 & 16 & 22 & 4 & 2 \\ -12 & 36 & 36 & 6 & 0 \\ -4 & 16 & 22 & 4 & 2 \\ -2 & 2 & 5 & 2 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 8/3 \\ 6 \\ 8/3 \\ 1/3 \end{pmatrix}.$$

Solving for C_1, C_2, C_3, C_4, C_5, we find that, the system posses infinitely many solutions, given by

$$C_1 = 0, C_2 = \frac{2v+1}{6}, C_3 = \frac{-v}{3}, C_4 = 0, C_5 = v.$$

where $u, v \in \mathbb{R}$.
(i) follows by replacing these values in (7) and further simplifying using the definition of Eisenstein series.
Likewise, using the same method, the following identities can be derived.

$$\frac{v}{3}(3L(q^3) - L(q)) - \frac{v}{3}(6L(q^6) - L(q)) + \frac{1}{6}(3L(q^6) - L(q^2)) + v(2L(q^6) - L(q^3)) = \left(a(q^6) - \frac{2}{3}b(q^2)\right) a(q^2), \tag{8}$$

$$\left(\frac{4v-1}{12}\right)(3L(q^3) - L(q)) + \left(\frac{1-2v}{6}\right)(6L(q^6) - L(q)) - \frac{1}{6}(3L(q^6) - L(q^2)) + v(2L(q^6) - L(q^3)) = \left(c(q^6) + \frac{1}{3}b(q^2)\right) a(q), \tag{9}$$

where $u, v \in \mathbb{R}$.
Relations (ii) and (iii) follows by the simplification of (8) and (9) and further employing Definition 2.2. □

Theorem 3.5. *One has*

$$(i)\ (1-u-4v) + 24\sum_{\alpha=1}^{\infty}\left[\frac{(1-2u-4v)\alpha^3 q^\alpha}{1-q^\alpha} - \frac{8u\alpha^3 q^{2\alpha}}{1-q^{2\alpha}} + \frac{9(1-4v)\alpha^3 q^{3\alpha}}{1-q^{3\alpha}}\right]$$
$$+ u\left[-1 - 24\sum_{\alpha=1}^{\infty}\left(\frac{\alpha q^\alpha}{1-q^\alpha} - \frac{2\alpha q^{2\alpha}}{1-q^{2\alpha}}\right)\right]^2 + v\left[-2 - 24\sum_{\alpha=1}^{\infty}\left(\frac{\alpha q^\alpha}{1-q^\alpha} - \frac{3\alpha q^{3\alpha}}{1-q^{3\alpha}}\right)\right]^2$$
$$= \left(3a(q^3) - 2b(q)\right)^4.$$

$$(ii)\ (1-u-4v) - 24\sum_{\alpha=1}^{\infty}\left[\frac{2(u+2v)\alpha^3q^\alpha}{1-q^\alpha} - \frac{(1-8u)\alpha^3q^{2\alpha}}{1-q^{2\alpha}} + \frac{36v\alpha^3q^{3\alpha}}{1-q^{3\alpha}} - \frac{9\alpha^3q^{6\alpha}}{1-q^{6\alpha}}\right]$$
$$+u\left[-1-24\sum_{\alpha=1}^{\infty}\left(\frac{\alpha q^\alpha}{1-q^\alpha} - \frac{2\alpha q^{2\alpha}}{1-q^{2\alpha}}\right)\right]^2 + v\left[-2-24\sum_{\alpha=1}^{\infty}\left(\frac{\alpha q^\alpha}{1-q^\alpha} - \frac{3\alpha q^{3\alpha}}{1-q^{3\alpha}}\right)\right]^2$$
$$= \left(3a(q^6) - 2b(q^2)\right)^4.$$

where $u, v, w \in \mathbb{R}$.

Proof. Consider the relation

$$C_1M(q) + C_2M(q^2) + C_3M(q^3) + C_4M(q^6) + C_5M(q^{12}) + C_6\left(L(-q) - L(q)\right)^2$$
(10)
$$+ C_7\left(L(q) - 2L(q^2)\right)^2 + C_8\left(L(q) - 3L(q^3)\right)^2 = \left(3a(q^3) - 2b(q)\right)^4.$$

The system generated by equating the coefficient of $k^4, pk^4, p^2k^4, p^3k^4, p^4k^4, p^5k^4, p^6k^4, p^7k^4$ and p^8k^4 on either sides,

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 1 & 4 \\ 124 & 4 & 4 & 4 & 4 & 0 & 28 & 64 \\ 964 & 64 & 4 & 4 & 4 & 576 & 244 & 400 \\ 2788 & 178 & 28 & -2 & -2 & 1728 & 700 & 1216 \\ 3910 & 235 & 70 & -5 & -5 & 2160 & 970 & 1816 \\ 2788 & 178 & 28 & -2 & -2 & 1440 & 700 & 1216 \\ 964 & 64 & 4 & 4 & 1/4 & 540 & 244 & 400 \\ 124 & 4 & 4 & 4 & 1/4 & 108 & 28 & 64 \\ 1 & 1 & 1 & 1 & 1/16 & 9 & 1 & 4 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{pmatrix} = \begin{pmatrix} 1 \\ 16 \\ 100 \\ 304 \\ 454 \\ 304 \\ 100 \\ 16 \\ 1 \end{pmatrix}.$$

On solving,

$$C_1 = \frac{1}{10} - \frac{1}{5}u - \frac{2}{5}v, C_2 = -\frac{4}{5}u, C_3 = \frac{9}{10} - \frac{18}{5}v, C_4 = 0, C_5 = 0, C_6 = 0, C_7 = u, C_8 = v,$$

where $u, v, w \in \mathbb{R}$.
The necessary result follows, by substituting these entries in (10).
Likewise, utilizing the same technique, the following identity is deduced.

$$-\left(\frac{1}{5}u + \frac{2}{5}v\right)M(q) + \left(\frac{1}{10} - \frac{4}{5}u\right)M(q^2) - \frac{18}{5}vM(q^3) + \frac{9}{10}M(q^6) + u\left(L(q) - 2L(q^2)\right)^2$$
$$+ v\left(L(q) - 3L(q^3)\right)^2 = \left(3a(q^3) - 2b(q)\right)^4,$$

where $u, v \in \mathbb{R}$.
The essential result (ii) follows immediately by employing Definition 2.2. □

4. Application to convolution

We start with the Convolution Sum description. For $k, n \in \mathbb{N}$, we set

$$\delta_k(n) = \sum_{d/n} d^k,$$

where d runs through the positive integers divisors of n. For $i, j, \alpha \in \mathbb{N}$ with $i \leq j$, the convolution sum is defined as

$$W_{i,j}(\alpha) := \sum_{il+jm=\alpha} \delta(l)\delta(m).$$

For all α, the convolution $\sum_{il+jm=\alpha} \delta(l)\delta(m)$ has been evaluated explicitly for various values of i and j, by Alaca et. al. [1, 11], H. C. Vidya and B. R. Srivtasa Kumar [9], and E. X. W. Xia and O. X. M. Yao [12]. Imperative to our proof are the claims of J. W. L. Glaisher [7],

$$L^2(q) = 1 + \sum_{\alpha=1}^{\infty}(240\delta_3(\alpha) - 288\alpha\delta_1(\alpha))q^\alpha. \tag{11}$$

Theorem 4.1. *For any $\alpha \in \mathbb{N}$, $u, v \in \mathbb{R} - \{0\}$ and $S(q), T(q)$ as defined in(1), one has*

$$\begin{aligned}\sum_{i+2j=\alpha} \delta(i)\delta(j) = &\frac{1}{24}\delta_1(\alpha) - \frac{1}{8}l\delta_1(\alpha) - \frac{1}{24}\delta_1(\alpha/2) - \frac{1}{4}\alpha\delta_1(\alpha/2)\\ &+ \frac{1}{12}(1 - \frac{v}{2u})\delta_3(\alpha) + \frac{1}{3}\left(1 + \frac{1}{32u}\right)\delta_3(\alpha/2) - \frac{3}{8}\frac{v}{u}\delta_3(\alpha/3)\\ &+ \frac{3}{32u}\delta_3(\alpha/6) - \left(\frac{A(\alpha) - vB(\alpha)}{2304u}\right),\end{aligned}$$

where

$$1 + \sum_{\alpha=1}^{\infty} A(\alpha)q^\alpha = \left[S(q^2) + 4T(q^2)\right]^4 \text{ and } 1 + \sum_{\alpha=1}^{\infty} B(\alpha)q^\alpha = \left[S(q) + 4T(q)\right]^2.$$

Proof. Reformulating Theorem 3.5 (ii) using Definition 2.2, we deduce

$$\begin{aligned}&1 - 48(u + 2v)\sum_{\alpha=1}^{\infty}\delta_3(\alpha)q^\alpha + 24(1 - 8u)\sum_{\alpha=1}^{\infty}\delta_3(\alpha)q^{2\alpha} - 864v\sum_{\alpha=1}^{\infty}\delta_3(\alpha)q^{3\alpha} + 216\sum_{\alpha=1}^{\infty}\delta_3(\alpha)q^{6\alpha}\\ &+ 576u\left(\sum_{\alpha=1}^{\infty}\delta_1(\alpha)q^\alpha\right)^2 + 2304u\left(\sum_{\alpha=1}^{\infty}\delta_1(\alpha)q^{2\alpha}\right)^2 - 2304u\sum_{\alpha=1}^{\infty}\delta_1(\alpha)q^\alpha\sum_{\alpha=1}^{\infty}\delta_1(\alpha)q^{2\alpha}\\ &+ 48u\sum_{\alpha=1}^{\infty}\delta_1(\alpha)q^\alpha - 96u\sum_{\alpha=1}^{\infty}\delta_1(\alpha)q^{2\alpha} + 576v\left(\sum_{\alpha=1}^{\infty}\delta_1(\alpha)q^\alpha\right)^2 + 5184v\left(\sum_{\alpha=1}^{\infty}\delta_1(\alpha)q^{3\alpha}\right)^2\\ &- 3456v\sum_{\alpha=1}^{\infty}\delta_1(\alpha)q^\alpha\sum_{\alpha=1}^{\infty}\delta_1(\alpha)q^{3\alpha} + 96v\sum_{\alpha=1}^{\infty}\delta_1(\alpha)q^\alpha - 288v\sum_{\alpha=1}^{\infty}\delta_1(\alpha)q^{3\alpha} = \sum_{\alpha=1}^{\infty}A(\alpha)q^\alpha.\end{aligned}$$

Extracting the coefficient of q^α on either sides, we obtain

$$48(u + 2v)\delta_3(\alpha) + 24(1 - 8u)\delta_3(\alpha/2) - 864v\delta_3(\alpha/3) + 216\delta_3(\alpha/6) + 48u\delta_1(\alpha)$$

$$- 96u\delta_1(\alpha/2) + 96v\delta_1(\alpha) - 288v\delta_1(\alpha/3) + 576(u+v)\sum_{i+j=\alpha}\delta(i)\delta(j)$$

$$+ 2304u\sum_{i+j=\alpha/2}\delta(i)\delta(j) - 2304u\sum_{i+2j=\alpha}\delta(i)\delta(j) + 5184v\sum_{i+j=\alpha/3}\delta(i)\delta(j)$$

(12)

$$- 3456v\sum_{i+3j=\alpha}\delta(i)\delta(j) = A(\alpha),$$

where

$$1 + \sum_{\alpha=1}^{\infty} A(\alpha)q^{\alpha} = \left[S(q^2) + 4T(q^2)\right]^4,$$

$S(q)$ and $T(q)$ are as defined in (1).
Expressing Theorem 3.4(i) in terms of Eisenstein series and then squaring, we get

$$L_1^2 - 6L_1L_3 + 9L_3^2 = 36\left(c(q^2) + \frac{1}{3}b(q)\right)^2 a^2(q).$$

Incorporating (11), Definition 2.2 and further equating the coefficient of q^{α} on either sides, we find that

(13)

$$\sum_{i+3j=\alpha}\delta(i)\delta(j) = \frac{5}{72}\delta_3(\alpha) - \frac{1}{12}\alpha\delta_3(\alpha) + \frac{5}{8}\delta_3(\alpha/3) - \frac{1}{4}\alpha\delta_1(\alpha/3) + \frac{1}{24}\delta_1(\alpha) - \frac{vB(\alpha)}{3456},$$

where

$$1 + \sum_{\alpha=1}^{\infty} B(\alpha)q^{\alpha} = \left[S(q) + 4T(q)\right]^2.$$

Note from ([12], Theorem 4.1), the convolution sum

$$\sum_{i+j=\alpha}\delta(i)\delta(j) = \frac{5}{12}\delta_3(\alpha) - \frac{\alpha}{2}\delta_1(\alpha) + \frac{\delta_1(\alpha)}{12}. \tag{14}$$

The desired result follows by substituting (13) and (14) in (12). □

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