Proceedings of the Jangjeon Mathematical Society
25 (2022), No. 2, pp. 245 - 252
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2022.25.2.245

SOME IDENTITIES INVOLVING DEGENERATE r-STIRLING NUMBERS

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ABSTRACT. Recently, Kim-Kim studied the unsigned degenerate r-Stirling number of the first kind and the degenerate r-Stirling number of the second kind, respectively of which are the degenerate versions of the unsigned r-Stirling numbers of the first kind and those of the r-Stirling numbers of the second kind. The aim of this paper is to derive some identities involving such special numbers from the inverse relations for them.

1. INTRODUCTION

Carlitz initiated a study of degenerate versions of some special polynomials and numbers, namely the degenerate Bernoulli and Euler polynomials and numbers (see [3]). In recent years, we have witnessed that some mathematicians have explored various degenerate versions of many special polynomials and numbers by using various tools like combinatorial methods, generating functions, differential equations, umbral calculus techniques, p-adic analysis, special functions, operator theory, probability theory, and analytic number theory. These degenerate versions include the degenerate Stirling numbers of the first and second kinds, which appear very frequently when we study degenerate versions of some special numbers and polynomials (see [5-11]).

The unsigned r-Stirling number of the first kind ${n \brack k}_r$ is the number of permutations of the set $[n] = \{1,2,3,\dots,n\}$ with exactly k disjoint cycles in such a way that the numbers $1,2,\dots,r$ are in distinct cycles, while the r-Stirling number of the second kind ${n \brace k}_r$ counts the number of partitions of the set $[n]$ into k non-empty disjoint subsets in such a way that the numbers $1,2,\dots,r$ are in distinct subsets. We remark that Border [2] studied the combinatorial and algebraic properties of the r-Stirling numbers.

The unsigned degenerate r-Stirling numbers of the first kind ${n \brack k}_{r,\lambda}$ are degenerate versions of the unsigned r-Stirling numbers of the first kind ${n \brack k}_r$ and the degenerate r-Stirling numbers of the second kind ${n \brace k}_{r,\lambda}$ are degenerate versions of the r-Stirling numbers of the second kind ${n \brace k}_r$. They can be viewed also as natural extensions of the unsigned degenerate Stirling numbers of the first kind ${n \brack k}_\lambda$ and the degenerate Stirling numbers of the second kind ${n \brace k}_\lambda$ which were introduced earlier (see [6,10,11]). The aim of this paper is to derive from the inverse relations for the degenerate r-Stirling numbers some identities involving such numbers and some special numbers which are given by the evaluations at r of the fully degenerate Bernoulli polynomials, the degenerate two variable Fubini polynomials, the degenerate Euler polynomials and the degenerate poly-Bernoulli polynomials.

The outline of this paper is as follows. In Section 1, we recall the degenerate exponential functions, the degenerate logarithms, the degenerate Bernoulli polynomials, the fully degenerate Bernoulli polynomials, the degenerate harmonic numbers, the unsigned degenerate Stirling numbers of the first kind, the degenerate Stirling numbers of the second kind, the unsigned degenerate r-Stirling numbers of the first kind, and the degenerate r-Stirling numbers of the second kind. In

2010 *Mathematics Subject Classification.* 11B73; 11B83.

Key words and phrases. degenerate r-Stirling numbers of the second kind; unsigned degenerate r-Stirling numbers of the first kind; inverse relation.

Section 2, we derive the orthogonality relations between the unsigned degenerate r-Stirling numbers of the first kind and the degenerate r-Stirling numbers of the second kind, and thereby get their inverse relations. By invoking these inverse relations we show some identities involving such degenerate r-Stirling numbers and some special numbers which are given by the evaluations at r of aforementioned special polynomials.

For any $\lambda \in \mathbb{R}$, the degenerate falling factorial sequence is defined by

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), \quad (n \geq 1). \tag{1}$$

Note that $\lim_{\lambda\to 0}(x)_{n,\lambda} = x^n$, (see [7-9]). The degenerate exponential functions are defined by

$$e_\lambda^x(t) = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \quad (\text{see } [6,10,11]). \tag{2}$$

For $x = 1$, we let $e_\lambda(t) = e_\lambda^1(t)$.

Let $\log_\lambda t$ be the degenerate logarithm which is the compositional inverse of $e_\lambda(t)$. Then we have

$$\log_\lambda(1+t) = \sum_{n=1}^{\infty}\frac{\lambda^{n-1}(1)_{n,1/\lambda}}{n!}t^n, \quad (\text{see } [6]). \tag{3}$$

Note that $e_\lambda\left(\log_\lambda(1+t)\right) = \log_\lambda\left(e_\lambda(1+t)\right) = 1+t$.

Carlitz considered the degenerate Bernoulli polynomials given by

$$\frac{t}{e_\lambda(t)-1}e_\lambda^x(t) = \sum_{n=0}^{\infty}B_{n,\lambda}(x)\frac{t^n}{n!}, \quad (\text{see } [3]). \tag{4}$$

Note that $\lim_{\lambda\to 0}B_{n,\lambda}(x) = B_n(x)$, where $B_n(x)$ are the ordinary Bernoulli polynomials given by

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \quad (\text{see } [1-13]).$$

Kim-Kim considered the fully degenerate Bernoulli polynomials $\beta_{n,\lambda}(x)$ which are given by

$$\frac{\log(1+\lambda t)}{\lambda(e_\lambda(t)-1)}e_\lambda^x(t) = \sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}, \quad (\text{see } [9]). \tag{5}$$

Note that $\lim_{\lambda\to 0}\beta_{n,\lambda}(x) = B_n(x),\ (n \geq 0)$.

The degenerate harmonic numbers are defined by Kim-Kim as

$$H_{0,\lambda} = 1, \quad H_{n,\lambda} = \sum_{k=1}^{n}\frac{1}{\lambda}\binom{\lambda}{k}(-1)^{k-1}, \quad (n \geq 1), \quad (\text{see } [8]). \tag{6}$$

Note that $\lim_{\lambda\to 0}H_{n,\lambda} = H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

From (6), we have

$$-\frac{1}{1-t}\log_\lambda(1-t) = \sum_{n=1}^{\infty}H_{n,\lambda}t^n, \quad (\text{see } [8]). \tag{7}$$

The degenerate Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{k=0}^{n}S_{1,\lambda}(n,k)(x)_{k,\lambda}, \quad (n \geq 0), \quad (\text{see } [6]). \tag{8}$$

Note that $\lim_{\lambda\to 0} S_{1,\lambda}(n,k) = S_1(n,k)$ are the Stirling numbers of the first kind. The unsigned degenerate Stirling numbers of the first kind are defined by Kim-Kim as

$$(9)\qquad \langle x\rangle_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\lambda} \langle x\rangle_{k,\lambda}, \quad (n \geq 0), \quad (\text{see } [6,10]),$$

where

$$\langle x\rangle_0 = 1, \quad \langle x\rangle_n = x(x+1)\cdots(x+n-1),$$
$$\langle x\rangle_{0,\lambda} = 1, \quad \langle x\rangle_{n,\lambda} = x(x+\lambda)\cdots(x+(n-1)\lambda), \quad (n \geq 1).$$

Note that $\begin{bmatrix} n \\ k \end{bmatrix}_{\lambda} = (-1)^{n-k} S_{1,\lambda}(n,k),\ (n,k) \geq 0.$

As the inversion formula of (8), the degenerate Stirling numbers of the second kind are defined by Kim-Kim as

$$(10)\qquad (x)_{n,\lambda} = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_{\lambda} (x)_k, \quad (n \geq 0), \quad (\text{see } [6]).$$

Note that $\lim_{\lambda\to 0} \begin{Bmatrix} n \\ k \end{Bmatrix}_{\lambda} = \begin{Bmatrix} n \\ k \end{Bmatrix}$ are the Stirling numbers of the second kind which are defined by

$$(11)\qquad x^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (x)_k, \quad (n \geq 0), \quad (\text{see } [12]).$$

From (9) and (10), we note that

$$(12)\qquad \frac{1}{k!}\big(-\log_{\lambda}(1-t)\big)^k = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{\lambda} \frac{t^n}{n!}, \quad (k \geq 0),$$

and

$$(13)\qquad \frac{1}{k!}\big(e_{\lambda}(t)-1\big)^k = \sum_{n=k}^{\infty} \begin{Bmatrix} n \\ k \end{Bmatrix}_{\lambda} \frac{t^n}{n!}, \quad (\text{see } [6]).$$

Let r be the nonnegative integer. Then the unsigned degenerate r-Stirling numbers of the first kind are defined by Kim-Kim as

$$(14)\qquad \langle x+r\rangle_n = \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} \langle x\rangle_{k,\lambda}, \quad (n \geq 0), \quad (\text{see } [6,10,11]).$$

From (14), we have

$$(15)\qquad \frac{1}{(1-t)^r}\frac{1}{k!}\big(-\log_{\lambda}(1-t)\big)^k = \sum_{n=k}^{\infty} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} \frac{t^n}{n!}.$$

Note that $\lim_{\lambda\to 0} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} = \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r}$ are the unsigned r-Stirling numbers of the first kind which are defined by

$$\frac{1}{(1-t)^r}\frac{1}{k!}\big(-\log(1-t)\big)^k = \sum_{n=k}^{\infty} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r} \frac{t^n}{n!}, \quad (\text{see } [10]).$$

In view of (10), the degenerate r-Stirling numbers of the second kind are defined by

$$(16)\qquad (x+r)_{n,\lambda} = \sum_{k=0}^{n} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_{r,\lambda} (x)_k, \quad (\text{see } [6,10,11]).$$

By (16), we easily get

$$e_\lambda^r(t)\frac{1}{k!}\big(e_\lambda(t)-1\big)^k=\sum_{n=k}^{\infty}\left\{\begin{matrix}n+r\\k+r\end{matrix}\right\}_{r,\lambda}\frac{t^n}{n!},\quad \text{(see [6,10,11])}. \tag{17}$$

2. Some identities involving degenerate r-Stirling numbers

In this section, we show orthogonality and inverse relations for the unsigned degenerate r-Stirling numbers of the first kind and the degenerate r-Stirling numbers of the second kind. Then, by using such inverse relations, we derive some identites involving the degenerate r-Stirling numbers and some special numbers which are given by the evaluations at r of several special polynomials.

From (14), we note that

$$\begin{aligned}\langle x+r\rangle_n&=\sum_{k=0}^{n}\begin{bmatrix}n+r\\k+r\end{bmatrix}_{r,\lambda}\langle x\rangle_{k,\lambda}=\sum_{k=0}^{n}\begin{bmatrix}n+r\\k+r\end{bmatrix}_{r,\lambda}(-1)^k(-x)_{k,\lambda}\\&=\sum_{k=0}^{n}\begin{bmatrix}n+r\\k+r\end{bmatrix}_{r,\lambda}(-1)^k(-x-r+r)_{k,\lambda}\\&=\sum_{k=0}^{n}\begin{bmatrix}n+r\\k+r\end{bmatrix}_{r,\lambda}(-1)^k\sum_{j=0}^{k}\left\{\begin{matrix}k+r\\j+r\end{matrix}\right\}_{r,\lambda}(-x-r)_j\\&=\sum_{k=0}^{n}\sum_{j=0}^{k}\begin{bmatrix}n+r\\k+r\end{bmatrix}_{r,\lambda}\left\{\begin{matrix}k+r\\j+r\end{matrix}\right\}_{r,\lambda}(-1)^{k-j}\langle x+r\rangle_j\\&=\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\begin{bmatrix}n+r\\k+r\end{bmatrix}_{r,\lambda}\left\{\begin{matrix}k+r\\j+r\end{matrix}\right\}_{r,\lambda}(-1)^{k-j}\right)\langle x+r\rangle_j.\end{aligned}\tag{18}$$

By comparing the coefficients on both sides of (18), we obtain the first orthogonality relation of the following theorem. The proof of the second one is similar.

Theorem 1. *For $n\ge 0$, we have the following orthogonality relations:*

$$\sum_{k=j}^{n}(-1)^{n-k}\begin{bmatrix}n+r\\k+r\end{bmatrix}_{r,\lambda}\left\{\begin{matrix}k+r\\j+r\end{matrix}\right\}_{r,\lambda}=\delta_{n,j},$$
$$\sum_{k=j}^{n}(-1)^{k-j}\left\{\begin{matrix}n+r\\k+r\end{matrix}\right\}_{r,\lambda}\begin{bmatrix}k+r\\j+r\end{bmatrix}_{r,\lambda}=\delta_{n,j},$$

where $\delta_{n,j}$ is the Kronecker's delta.

Theorem 2. *For $n\ge 0$, we have the following inverse relations:*

$$a_{n,\lambda}=\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n+r\\k+r\end{bmatrix}_{r,\lambda}b_{k,\lambda}\iff b_{k,\lambda}=\sum_{k=0}^{n}\left\{\begin{matrix}n+r\\k+r\end{matrix}\right\}_{r,\lambda}a_{k,\lambda}.$$

Proof. ($\Longrightarrow$) Assume that

$$a_{n,\lambda}=\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n+r\\k+r\end{bmatrix}_{r,\lambda}b_{k,\lambda}.$$

Then we have

$$\begin{aligned}\sum_{k=0}^{n}\left\{\begin{matrix}n+r\\k+r\end{matrix}\right\}_{r,\lambda}a_{k,\lambda}&=\sum_{k=0}^{n}\left\{\begin{matrix}n+r\\k+r\end{matrix}\right\}_{r,\lambda}(-1)^k\sum_{j=0}^{k}(-1)^j\begin{bmatrix}k+r\\j+r\end{bmatrix}_{r,\lambda}b_{j,\lambda}\\&=\sum_{j=0}^{n}b_{j,\lambda}\sum_{k=j}^{n}(-1)^{k-j}\left\{\begin{matrix}n+r\\k+r\end{matrix}\right\}_{r,\lambda}\begin{bmatrix}k+r\\j+r\end{bmatrix}_{r,\lambda}\\&=b_{n,\lambda}.\end{aligned}$$

($\Longleftarrow$) Let $b_{n,\lambda}=\sum_{k=0}^{n}\left\{\begin{matrix}n+r\\k+r\end{matrix}\right\}_{r,\lambda}a_{k,\lambda}$. Then we note that

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n+r\\k+r\end{bmatrix}_{r,\lambda}b_{k,\lambda}&=\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n+r\\k+r\end{bmatrix}_{r,\lambda}\sum_{j=0}^{k}\left\{\begin{matrix}k+r\\j+r\end{matrix}\right\}_{r,\lambda}a_{j,\lambda}\\&=\sum_{j=0}^{n}a_{j,\lambda}\sum_{k=j}^{n}(-1)^{n-k}\begin{bmatrix}n+r\\k+r\end{bmatrix}_{r,\lambda}\left\{\begin{matrix}k+r\\j+r\end{matrix}\right\}_{r,\lambda}\\&=a_{n,\lambda}.\end{aligned}$$

□

In [9], it is shown that

$$\beta_{n,\lambda}(r)=\sum_{k=0}^{n}(-1)^k\left\{\begin{matrix}n+r\\k+r\end{matrix}\right\}_{r,\lambda}\frac{k!}{k+1}. \tag{19}$$

Therefore, by Theorem 2 and (19), we obtain the following corollary.

Corollary 3. *For $n\geq 0$, we have*

$$\frac{n!}{n+1}=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n+r\\k+r\end{bmatrix}_{r,\lambda}\beta_{k,\lambda}(r).$$

In [5], the degenerate two variable Fubini polynomials are given by

$$\frac{1}{1-x(e_\lambda(t)-1)}e_\lambda^y(t)=\sum_{n=0}^{\infty}F_{n,\lambda}(x|y)\frac{t^n}{n!}. \tag{20}$$

We observe that

$$\begin{aligned}\frac{1}{1-x(e_\lambda(t)-1)}e_\lambda^r(t)&=\sum_{k=0}^{\infty}x^kk!\frac{1}{k!}\big(e_\lambda(t)-1\big)^ke_\lambda^r(t)\\&=\sum_{k=0}^{\infty}x^kk!\sum_{n=k}^{\infty}\left\{\begin{matrix}n+r\\k+r\end{matrix}\right\}_{r,\lambda}\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}x^kk!\left\{\begin{matrix}n+r\\k+r\end{matrix}\right\}_{r,\lambda}\frac{t^n}{n!}.\end{aligned} \tag{21}$$

From (20) and (21), we note that

$$F_{n,\lambda}(x|r)=\sum_{k=0}^{n}x^kk!\left\{\begin{matrix}n+r\\k+r\end{matrix}\right\}_{r,\lambda}. \tag{22}$$

Therefore, by Theorem 2 and (22), we obtain the following theorem.

Theorem 4. *For $n \geq 0$, we have*

$$x^n n! = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} F_{k,\lambda}(x|r).$$

In [3], Carlitz introduced the degenerate Euler polynomials $\mathscr{E}_{n,\lambda}(x)$ which are given by

$$\frac{2}{e_\lambda(t)+1} e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathscr{E}_{n,\lambda}(x) \frac{t^n}{n!}. \tag{23}$$

By (20), we get

$$\begin{aligned} \sum_{n=0}^{\infty} F_{n,\lambda}\big(-\frac{1}{2} \mid r\big) \frac{t^n}{n!} &= \frac{1}{1+\frac{1}{2}(e_\lambda(t)-1)} e_\lambda^r(t) \\ &= \frac{2}{e_\lambda(t)+1} e_\lambda^r(t) = \sum_{n=0}^{\infty} \mathscr{E}_{n,\lambda}(r) \frac{t^n}{n!}. \end{aligned} \tag{24}$$

Thus, we have

$$\mathscr{E}_{n,\lambda}(r) = F_{n,\lambda}\big(-\frac{1}{2} \mid r\big) = \sum_{k=0}^{n} \big(-\frac{1}{2}\big)^k k! \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_{r,\lambda}.$$

Theorem 5. *For $n \geq 0$, we have*

$$\mathscr{E}_{n,\lambda}(r) = \sum_{k=0}^{n} \big(-\frac{1}{2}\big)^k k! \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_{r,\lambda}.$$

From Theorem 2 and Theorem 5, we note that

$$\big(-\frac{1}{2}\big)^n n! = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} \mathscr{E}_{k,\lambda}(r). \tag{25}$$

Therefore, by (25), we obtain the following theorem.

Theorem 6. *For $n \geq 0$, we have*

$$\frac{n!}{2^n} = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} \mathscr{E}_{k,\lambda}(r).$$

Recently, Kim-Kim introduced the degenerate polylogarithm function of index k which is given by

$$\mathrm{Li}_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{(n-1)!n^k} x^n, \quad (k \in \mathbb{Z}), \quad (\text{see [6]}). \tag{26}$$

In [9], the degenerate poly-Bernoulli polynomials of index k are defined by

$$\frac{\mathrm{Li}_{k,\lambda}(1-e_\lambda(-t))}{1-e_\lambda(-t)} e_\lambda^{-x}(-t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}. \tag{27}$$

For $p \in \mathbb{Z}$, we have

$$\beta_{n,\lambda}^{(p)}(-r) = (-1)^n \sum_{k=0}^{n} \frac{\lambda^k (1)_{k+1,1/\lambda}}{(k+1)^p} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_{r,\lambda}. \tag{28}$$

From Theorem 2 and (2), we have

$$\frac{\lambda^n(1)_{n+1,1/\lambda}}{(n+1)^p}=\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n+r\\k+r\end{bmatrix}_{r,\lambda}(-1)^k\beta_{k,\lambda}^{(p)}(-r)=\sum_{k=0}^{n}(-1)^n\begin{bmatrix}n+r\\k+r\end{bmatrix}_{r,\lambda}\beta_{k,\lambda}^{(p)}(-r). \tag{29}$$

Therefore, by (29), we obtain the following theorem.

Theorem 7. *For $n\geq 0$, we have*

$$\frac{(-\lambda)^n(1)_{n+1,1/\lambda}}{(n+1)^p}=\sum_{k=0}^{n}\begin{bmatrix}n+r\\k+r\end{bmatrix}_{r,\lambda}\beta_{k,\lambda}^{(p)}(-r).$$

3. Further remark

The degenerate Bernoulli polynomials of the second kind $b_{n,\lambda}(x)$ are defined by

$$\frac{t}{\log_\lambda(1+t)}(1+t)^x=\sum_{n=0}^{\infty}b_{n,\lambda}(x)\frac{t^n}{n!}. \tag{30}$$

In view of (7), we consider the degenerate hyperharmonic numbers which are defined by

$$H_{n,\lambda}^{(1)}=H_{n,\lambda},\quad H_{n,\lambda}^{(r)}=\sum_{k=1}^{n}H_{k,\lambda}^{(r-1)},\ (n\geq 1,r\geq 2),\quad H_{0,\lambda}^{(r)}=0,\ (r\geq 2). \tag{31}$$

Assume that $r\geq 2$ and that $-\frac{1}{(1-t)^{r-1}}\log_\lambda(1-t)=\sum_{n=1}^{\infty}H_{n,\lambda}^{(r-1)}t^n$. From (31), we note that

$$\sum_{n=1}^{\infty}H_{n,\lambda}^{(r)}t^n=\sum_{n=1}^{\infty}\sum_{k=1}^{n}H_{k,\lambda}^{(r-1)}t^n=\frac{1}{1-t}\sum_{k=1}^{\infty}H_{k,\lambda}^{(r-1)}t^k=\frac{1}{1-t}\left(-\frac{1}{(1-t)^{r-1}}\log_\lambda(1-t)\right)=-\frac{1}{(1-t)^r}\log_\lambda(1-t). \tag{32}$$

Therefore, by the induction step in (32), we obtain the following theorem.

Proposition 8. *Let r be a positive integer. Then we have*

$$-\frac{1}{(1-t)^r}\log_\lambda(1-t)=\sum_{n=1}^{\infty}H_{n,\lambda}^{(r)}t^n.$$

Now, by using (30) and Proposition 8, we observe that

$$t=\frac{-t}{\log_\lambda(1-t)}(1-t)^r\frac{-\log_\lambda(1-t)}{(1-t)^r}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}b_{k,\lambda}(r)t^k\sum_{l=0}^{\infty}H_{l,\lambda}^{(r)}t^l=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^k}{k!}b_{k,\lambda}(r)H_{n-k,\lambda}^{(r)}\right)t^n. \tag{33}$$

Comparing the coefficients on both sides of (33), we have

$$\sum_{k=0}^{n}\frac{(-1)^k}{k!}b_{k,\lambda}(r)H_{n-k,\lambda}^{(r)}=\delta_{n,1}. \tag{34}$$

4. Conclusion

The unsigned degenerate r-Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_{r,\lambda}$ are degenerate versions of the unsigned r-Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_{r}$ and the degenerate r-Stirling numbers of the second kind $\begin{Bmatrix} n \\ k \end{Bmatrix}_{r,\lambda}$ are degenerate versions of the r-Stirling numbers of the second kind $\begin{Bmatrix} n \\ k \end{Bmatrix}_{r}$. We derived from the inverse relations for the degenerate r-Stirling numbers some identities involving such numbers and some special numbers which are given by the evaluations at r of the fully degenerate Bernoulli polynomials, the degenerate two variable Fubini polynomials, the degenerate Euler polynomials and the degenerate poly-Bernoulli polynomials.

It is one of our future projects to continue to explore degenerate versions of some special numbers and polynomials and their applications not only in mathematics but also in other disciplines like statistics, physics, engineering and social sciences.

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