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k^{th}-ECCENTRICITY INDEX OF GRAPHS

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Abstract. The molecular topological descriptors are the numerical invariants of a molecular graph and are very useful for predicting their physical properties, chemical reactivity and bioactivity. A variety of such indices are studied and used in theoretical chemistry and pharmaceutical research related to drugs and also in different fields. The main classes of topological graph indices are those based on vertex degrees, distances, and graph parameters like eccentricity. In this paper, we introduce k^{th}-eccentricity index of graphs. Also we compute k^{th}-eccentricity index of some standard graphs including some windmill graphs and molecular graphs of cycloalkenes. Further, we obtain lower and upper bounds for the k^{th}-eccentricity index in terms of other topological indices.

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Topological indices are numbers associated with chemical structures derived from their hydrogen-depleted graphs as a tool for compact and effective description of structural formulas which are used to study and predict the structure-property correlations of organic compounds. Molecular descriptors are playing significant role in chemistry, pharmacology, etc. Among them topological indices have a prominent place [5].

In this paper, all graphs are simple, finite and connected. A graph $G = (V, E)$ is a simple graph if it has no loops, no multiple and directed edges. A graph G is said to be connected if there is a path between every pair of its vertices. As usual, we denote the number of vertices and edges of a graph G by $n = |V|$ and $m = |E|$, respectively.

The distance $d(u, v)$ or $d_G(u, v)$ is defined as the length of the shortest path between u and v in G. The eccentricity of a vertex $v \in V(G)$ is $e_v = max\{d(u, v) : u \in V(G)\}$. The radius $r(G)$ of a graph G is the minimum eccentricity of the vertices while the diameter $d(G)$ is the maximum eccentricity of all vertices in the graph [4]. A graph is equi-eccentric if all of its vertices have the same eccentricity.

Some of the graph topological indices are defined by means of vertex eccentricity. Some of these indices are as follows: The first Zagreb eccentricity index E_1 and the second Zagreb eccentricity index E_2 [6] of G are defined as $E_1(G) = \sum_{v\in V(G)} e_v^2$ and $E_2(G) = \sum_{uv\in E(G)} e_u e_v$. The total eccentricity index of G [6] is denoted by $\theta(G)$ and defined as the sum of eccentricities of all vertices of graph, i.e., $\theta(G) = \sum_{v\in V(G)} e_v$. The eccentric connectivity index [6] of G is denoted by ξ^c and defined by $\xi^c(G) = \sum_{uv\in E(G)}[e_u + e_v]$. The fourth geometric-arithmetic index [3] of G is defined by $GA_4(G) =$

$\sum_{uv\in E(G)} \frac{2\sqrt{e_u e_v}}{e_u+e_v}$. The fifth atom-bond connectivity index [2] of G is defined by $ABC_5(G) = \sum_{uv\in E(G)} \sqrt{\frac{e_u+e_v-2}{e_u e_v}}$. The Wiener index $W(G)$ of G [10] is defined by $W(G) = \sum_{u,v\subseteq V(G)} d_G(u,v)$. Finally, the eccentric harmonic index [8] $H_e(G)$ of G is defined by $H_e(G) = \sum_{uv\in E(G)} \frac{2}{e_u+e_v}$.

In this paper, we defined and computed k^{th}-eccentricity index for some graphs including windmill graphs and molecular graphs of cycloalkenes. Further we obtain lower and upper bounds for the k^{th}-eccentricity index of graphs.

1. k^{th}-ECCENTRICITY INDICES OF SOME STANDARD GRAPHS

Definition 1.1. *For a connected graph G, the k^{th}-eccentricity index of G is defined by*

$$k^{th}EI(G) = \sum_{uv\in E(G)} [e_u e_v]^k,$$

where k is any positive integer.

Proposition 1.2. *For a complete graph K_n, $(n \geq 2)$,*

$$k^{th}EI(K_n) = \frac{n(n-1)}{2}.$$

Proof. In K_n, we have $e_v = 1$ for all $v \in V(K_n)$. Hence

$$\begin{aligned} k^{th}EI(K_n) &= \textstyle\sum_{uv\in E(K_n)} (e_u e_v)^k \\ &= \underbrace{1^k 1^k + \cdots + 1^k 1^k}_{\frac{n(n-1)}{2}-times} \\ &= \tfrac{n(n-1)}{2}. \end{aligned}$$

□

Proposition 1.3. *For any cycle graph C_n,*

$$k^{th}EI(C_n) = \begin{cases} \frac{n^{2k+1}}{2^{2k}}, & \text{if } n \text{ even} \\ n\left(\frac{n-1}{2}\right)^{2k}, & \text{if } n \text{ odd.} \end{cases}$$

Proof. In C_n, we consider the following cases:

Case (i) n is even. Then $e_{v_i} = \frac{n}{2}$, for $i = 1, 2, 3, \cdots, n$. So

$$\begin{aligned} k^{th}EI(C_n) &= \textstyle\sum_{uv\in E(C_n)} (e_u e_v)^k \\ &= \underbrace{\left[\left(\frac{n}{2}\right)^k \left(\frac{n}{2}\right)^k + \cdots + \left(\frac{n}{2}\right)^k \left(\frac{n}{2}\right)^k\right]}_{n-times} \\ &= \tfrac{n^{2k+1}}{2^{2k}}. \end{aligned}$$

Case (ii) n is odd. Then $e_{v_i} = \frac{n-1}{2}$, for $i = 1, 2, 3, \cdots, n$. So

$$\begin{aligned} k^{th}EI(C_n) &= \textstyle\sum_{uv\in E(C_n)} (e_u e_v)^k \\ &= \underbrace{\left[\left(\frac{n-1}{2}\right)^k \left(\frac{n-1}{2}\right)^k + \cdots + \left(\frac{n-1}{2}\right)^k \left(\frac{n-1}{2}\right)^k\right]}_{n-times} \\ &= n\left(\tfrac{n-1}{2}\right)^{2k}. \end{aligned}$$

□

Proposition 1.4. *For any path graph P_n,*

$$k^{th}EI(P_n) = \begin{cases} 2\left[\sum_{i=1}^{\frac{n-2}{2}} (n-i)^k (n-i-1)^k\right] + \left(\frac{n}{2}\right)^{2k}, & \text{if } n \text{ even} \\ 2\left[\sum_{i=1}^{\frac{n-1}{2}} (n-i)^k (n-i-1)^k\right], & \text{if } n \text{ odd.} \end{cases}$$

Proof. In a path graph P_n with $V(P_n) = \{v_1, v_2, v_3, \cdots, v_n\}$, we consider the following cases.
Case (i) If n is even, then we have $e_{v_1} = e_{v_n} = n-1$, $e_{v_2} = e_{v_{n-1}} = n-2, \cdots, e_{v_{\frac{n}{2}}} = e_{v_{\frac{n}{2}+1}} = \frac{n}{2}$. Hence

$$\begin{aligned} k^{th}EI(P_n) &= (n-1)^k (n-2)^k + (n-2)^k (n-3)^k + \cdots + \left(\tfrac{n}{2}+1\right)^k \left(\tfrac{n}{2}\right)^k + \left(\tfrac{n}{2}\right)^k \left(\tfrac{n}{2}\right)^k \\ &+ \left(\tfrac{n}{2}\right)^k \left(\tfrac{n}{2}+1\right)^k + \cdots + (n-3)^k (n-2)^k + (n-2)^k (n-1)^k \\ &= 2\left[(n-1)^k (n-2)^k + (n-2)^k (n-3)^k + \cdots + \left(\tfrac{n}{2}+1\right)^k \left(\tfrac{n}{2}\right)^k\right] + \left(\tfrac{n}{2}\right)^{2k} \\ &= 2\left[\sum_{i=1}^{\frac{n-2}{2}} (n-i)^k (n-i-1)^k\right] + \left(\tfrac{n}{2}\right)^{2k}. \end{aligned}$$

Case (ii) Let n be odd. We have $e_{v_1} = e_{v_n} = n-1$, $e_{v_2} = e_{v_{n-1}} = n-2, \cdots, e_{v_{\frac{n-1}{2}}} = e_{v_{\frac{n+3}{2}}} = \frac{n+1}{2}$ and $e_{v_{\frac{n+1}{2}}} = \frac{n-1}{2}$. Then

$$\begin{aligned} k^{th}EI(P_n) &= (n-1)^k (n-2)^k + (n-2)^k (n-3)^k + \cdots + \left(\tfrac{n-1}{2}+1\right)^k \left(\tfrac{n-1}{2}\right)^k \\ &+ \left(\tfrac{n-1}{2}\right)^k \left(\tfrac{n-1}{2}+1\right)^k + \cdots + (n-3)^k (n-2)^k + (n-2)^k (n-1)^k \\ &= 2\left[(n-1)^k (n-2)^k + (n-2)^k (n-3)^k + \cdots + \left(\tfrac{n-1}{2}+1\right)^k \left(\tfrac{n-1}{2}\right)^k\right] \\ &= 2\left[\sum_{i=1}^{\frac{n-1}{2}} (n-i)^k (n-i-1)^k\right]. \end{aligned}$$

□

Proposition 1.5. *For a wheel graph W_n, $(n \geq 5)$, we have*

$$k^{th}EI(W_n) = 2^k(n-1)(1+2^k).$$

Proof. Let v_1 be the central vertex of W_n. Then $e_{v_1} = 1$ and $e_{v_i} = 2$ for $i = 2, 3, \cdots, n$. Hence we have

$$\begin{aligned} k^{th}EI(W_n) &= \underbrace{1^k2^k + \cdots + 1^k2^k}_{(n-1)-times} \\ &+ \underbrace{2^k2^k + \cdots + 2^k2^k}_{(n-1)-times} \\ &= 1^k2^k(n-1) + 2^k2^k(n-1) \\ &= 2^k(n-1)(1+2^k). \end{aligned}$$

□

Proposition 1.6. *For a complete bipartite graph $K_{m,n}$, $(m, n \geq 2)$,*

$$k^{th}EI(K_{m,n}) = 2^{2k}mn.$$

Proof. In $K_{m,n}$ with $V(K_{m,n}) = U \cup V$ where $U = \{u_1, u_2, \cdots, u_m\}$ and $V = \{v_1, v_2, \cdots, v_n\}$, $e_{u_i} = 2$ for $i = 1, 2, 3, \cdots, m$ and $e_{v_j} = 2$ for $j = 1, 2, 3, \cdots, n$. Then

$$\begin{aligned} k^{th}EI(K_{m,n}) &= \underbrace{2^k 2^k + \cdots + 2^k 2^k}_{mn-times} \\ &= 2^{2k} mn. \end{aligned}$$

□

Proposition 1.7. *For a star graph $K_{1,n}$, $n \geq 2$, we have*

$$k^{th}EI(K_{1,n}) = n2^k.$$

Proof. Let v_0 be the central vertex of $K_{1,n}$. Then $e_{v_0} = 1$ and $e_{v_i} = 2$, for $i = 1, 2, 3, \cdots, n$.

$$\begin{aligned} k^{th}EI(K_{1,n}) &= \underbrace{1^k 2^k + \cdots + 1^k 2^k}_{n-times} \\ &= n2^k. \end{aligned}$$

□

There are several variants of star graphs due to their applications. Although they appear in literature, we recall their definitions not to cause any confusion:

Definition 1.8. [7] *The double star graph which is denoted by $S_{n,m}$, $m, n \geq 2$ is a graph obtained from two star graphs $K_{1,n-1}$ and $K_{1,m-1}$ by joining their centers v_0 and u_0. The vertex set $V(S_{n,m})$ is $V(K_{1,n-1}) \cup V(K_{1,m-1}) = \{v_0, v_1, \cdots, v_{n-1}, u_0, u_1, \cdots, u_{m-1}\}$ and edge set is*

$$E(S_{n,m}) = \{v_0u_0, v_0v_i, u_0u_i : 1 \leq i \leq n-1; 1 \leq j \leq m-1\}$$

as shown in Fig. 1.

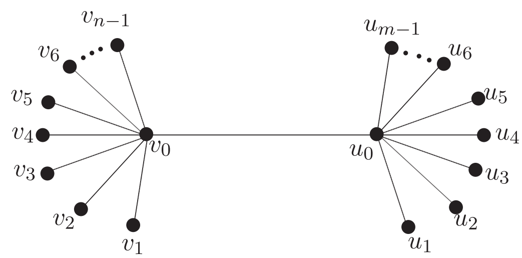


Figure 1. Double star graph $S_{n,m}$

Proposition 1.9. *The k^{th}-eccentricity index of $S_{n,m}$, $m, n \geq 2$, is*

$$(m + n - 2)2^k 3^k + 2^{2k}.$$

Proof. Clearly $e_{v_0} = e_{u_0} = 2$ and $e_{v_i} = e_{u_j} = 3$, for $i = 1, 2, \cdots, n-1$ and $j = 1, 2, \cdots, m-1$. Then

$$\begin{aligned} k^{th}EI(S_{n,m}) &= \underbrace{2^k 3^k + \cdots + 2^k 3^k}_{(n-1)-times} + \underbrace{2^k 3^k + \cdots + 2^k 3^k}_{(m-1)-times} + 2^k 2^k \\ &= 2^k 3^k (n-1) + 2^k 3^k (m-1) + 2^{2k} \\ &= (m + n - 2)2^k 3^k + 2^{2k}. \end{aligned}$$

□

Definition 1.10. [7] *The multi-star graph $K_{1,n,n,\cdots,n}$ is constructed as follows: Consider the star graph $K_{1,n}$ with vertex set $\{v_{11}, v_{12}, \cdots, v_{1n}\}$. Add an edge to each of the pendant vertices $v_{11}, v_{12}, \cdots, v_{1n}$ to get the resulting graph $K_{1,n,n}$ with vertices $\{v_0, v_{11}, \cdots, v_{1n}, v_{21}, \cdots, v_{2n}\}$. Again add an edge to each of the pendant vertices $v_{21}, \cdots, v_{2n}$ to get the graph $K_{1,n,n,n}$. Repeating this $(m-1)$ times, we get a graph $K_{1,\underbrace{n, n, \ldots, n}_{m-times}}$ called the multi-star graph with $mn+1$ vertices $v_0, v_{11}, v_{12}, \cdots, v_{1n}, v_{21}, \cdots, v_{2n}, v_{31}, \cdots, v_{3n}, \ldots, v_{m1}, \cdots, v_{mn}$ and mn edges as shown in Fig. 2.*

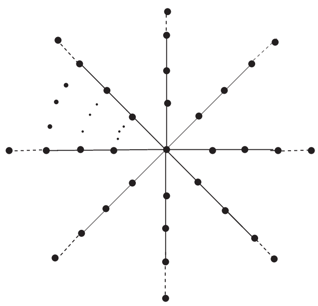


Figure 2: Multi-star graph $K_{1,\underbrace{n, n, \ldots, n}_{m-times}}$

Proposition 1.11. *The k^{th}-eccentricity index of a multi-star graph is*

$$k^{th}EI(K_{1,\underbrace{n, n, \ldots, n}_{m-times}}) = n\left[\sum_{i=1}^{m}(m+i-1)^k(m+i)^k\right].$$

Proof. We have $e_{v_0} = m$ and $e_{v_{ij}} = m+i$ for $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$

$$\begin{aligned} k^{th}EI(K_{1,\underbrace{n, n, \ldots, n}_{m-times}}) &= n\left[m^k(m+1)^k + (m+1)^k(m+2)^k + \cdots + (m+m-1)^k(m+m)^k\right] \\ &= n\left[\textstyle\sum_{i=1}^{m}(m+i-1)^k(m+i)^k\right]. \end{aligned}$$

□

Definition 1.12. [7] *The graph Pl_n, $(n \geq 3)$, is obtained as the join of P_{n-2} and P_2 as shown in Fig. 3.*

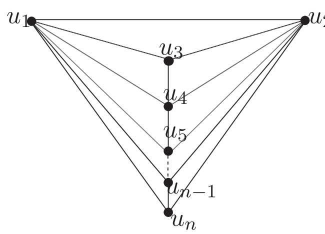


Figure 3: Pl_n graph

Proposition 1.13. *For any Pl_n, $n \geq 6$ graph,*

$$k^{th}EI(Pl_n) = 1 + 2^{k+1}(n-2) + 2^{2k}(n-3).$$

Proof. Here $e_{u_1} = e_{u_2} = 1$ and $e_{u_i} = 2$ where $i = 3, 4, \cdots, n$. We have

$$\begin{aligned} k^{th}EI(Pl_n) &= 1^k 1^k + \underbrace{1^k 2^k + \cdots + 1^k 2^k}_{2(n-2)-times} + \underbrace{2^k 2^k + \cdots + 2^k 2^k}_{(n-3)-times} \\ &= 1 + 2^{k+1}(n-2) + 2^{2k}(n-3). \end{aligned}$$

□

Definition 1.14. [7] *The lollipop graph denoted by $L_{n,d}$ is obtained from a complete graph K_{n-d} and a path P_d by joining one of the end vertices of P_d to all the vertices of K_{n-d} as shown in Fig. 4.*

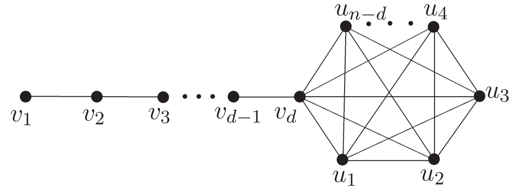


Figure 4: Lollipop graph $L_{n,d}$

Proposition 1.15. *For any $L_{n,d}$ graph,*

$$k^{th}EI(L_{n,d}) = \begin{cases} (n-d+1)[d^k(d-1)^k] + 2\left[\sum_{i=1}^{\frac{d}{2}-1} (d-i)^k (d-i-1)^k\right] & \\ + \left[\frac{(n-d-1)(n-d)}{2}\right] d^{2k}, & \text{if } n \text{ even} \\ (n-d+1)[d^k(d-1)^k] + 2\left[\sum_{i=1}^{\frac{d-1}{2}-1} (d-i)^k (d-i-1)^k\right] & \\ + \left[\frac{(n-d-1)(n-d)}{2}\right] d^{2k} + \left(\frac{d+1}{2}\right)^{2k}, & \text{if } n \text{ odd.} \end{cases}$$

Proof. Let $V(L_{n,d}) = \{v_1, v_2, \cdots, v_d, u_1, u_2, \cdots, u_{n-d}\}$. We have the following cases.

Case (i) d is even. Then $e_{v_1} = d$, $e_{v_2} = e_{v_d} = d-1$, $e_{v_3} = e_{v_{d-1}} = d-2, \cdots, e_{v_{\frac{d}{2}}} = e_{v_{\frac{d}{2}+2}} = \frac{d}{2}+1$, $e_{v_{\frac{d}{2}+1}} = \frac{d}{2}$, and $e_{u_1} = e_{u_2} = \cdots = e_{u_{n-d}} = d$.

$$\begin{aligned} k^{th}EI(L_{n,d}) &= d^k (d-1)^k + (d-1)^k (d-2)^k + \cdots + \left(\tfrac{d}{2}+1\right)^k \left(\tfrac{d}{2}\right)^k \\ &+ \left(\tfrac{d}{2}\right)^k \left(\tfrac{d}{2}+1\right)^k + \cdots + (d-2)^k (d-1)^k \\ &+ \underbrace{d^k (d-1)^k + \cdots + d^k (d-1)^k}_{(n-d)-times} + \underbrace{d^k d^k + \cdots + d^k d^k}_{\frac{(n-d)(n-d-1)}{2}-times} \\ &= d^k (d-1)^k + 2\left[(d-1)^k (d-2)^k + \cdots + \left(\tfrac{d}{2}+1\right)^k \left(\tfrac{d}{2}\right)^k\right] \\ &+ (n-d)\, d^k (d-1)^k + \frac{(n-d)(n-d-1)}{2} d^k d^k \\ &= (n-d+1)\left[d^k (d-1)^k\right] + 2\left[\sum_{i=1}^{\frac{d}{2}-1} (d-i)^k (d-i-1)^k\right] \\ &+ \left[\frac{(n-d)(n-d-1)}{2}\right] d^{2k}. \end{aligned}$$

Case (ii) d is odd. Then $e_{v_1} = d$, $e_{v_2} = e_{v_d} = d-1, e_{v_3} = e_{v_{d-1}} = d-2, \ldots, e_{v_{\frac{d+1}{2}-1}} = e_{v_{\frac{d+1}{2}+2}} = \frac{d-1}{2}+2$, $e_{v_{\frac{d+1}{2}}} = e_{v_{\frac{d+1}{2}+1}} = \frac{d-1}{2}+1$ and $e_{u_1} = e_{u_2} = \cdots = e_{u_{n-d}} = d$. Then

$$\begin{aligned}
k^{th}EI(L_{n,d}) &= d^k(d-1)^k + (d-1)^k(d-2)^k + \cdots + \left(\tfrac{d+1}{2}+1\right)^k\left(\tfrac{d+1}{2}\right)^k \\
&+ \left(\tfrac{d+1}{2}\right)^k\left(\tfrac{d+1}{2}\right)^k + \left(\tfrac{d+1}{2}\right)^k\left(\tfrac{d+1}{2}+1\right)^k + \cdots + (d-2)^k(d-1)^k \\
&+ \underbrace{d^k(d-1)^k + \cdots + d^k(d-1)^k}_{(n-d)-times} + \underbrace{d^kd^k + \cdots + d^kd^k}_{\frac{(n-d)(n-d-1)}{2}-times} \\
&= d^k(d-1)^k + 2\left[(d-1)^k(d-2)^k + \cdots + \left(\tfrac{d+1}{2}+1\right)^k\left(\tfrac{d+1}{2}\right)^k\right] + \left(\tfrac{d+1}{2}\right)^{2k} \\
&+ (n-d)\,d^k(d-1)^k + \tfrac{(n-d)(n-d-1)}{2}d^kd^k \\
= \quad & (n-d+1)\left[d^k(d-1)^k\right] + 2\left[\textstyle\sum_{i=1}^{\frac{d-1}{2}-1}(d-i)^k(d-i-1)^k\right] \\
&+ \left[\tfrac{(n-d)(n-d-1)}{2}\right]d^{2k} + \left(\tfrac{d+1}{2}\right)^{2k}.
\end{aligned}$$

□

Definition 1.16. [7] *The broom graph denoted by $B_{n,d}$ is obtained from a path P_d together with $n-d$ end vertices and all adjacent to the same end vertex of P_d as shown in Fig.* 5.

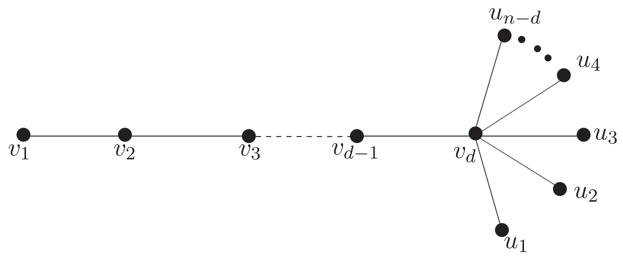


Figure 5: Broom graph $B_{n,d}$

Corollary 1.17. *For any $B_{n,d}$ graph,*

$$k^{th}EI(B_{n,d}) = \begin{cases} (n-d+1)d^k(d-1)^k + 2\left[\sum_{i=1}^{\frac{d}{2}-1}(d-i)^k(d-(i+1))^k\right], & \textit{if } n \textit{ is even} \\ (n-d+1)d^k(d-1)^k + 2\left[\sum_{i=1}^{\frac{d-1}{2}-1}(d-i)^k(d-(i+1))^k\right] + \left(\frac{d+1}{2}\right)^{2k}, & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. The proof follows from Proposition 2.15 by deleting K_{n-d} and joining $n-d$ pendant edges to an end vertex of P_d. □

Definition 1.18. [1] *The Dutch windmill graph D_m^n ($m \geq 3$, $n \geq 2$) is defined as the graph having n-copies of C_m with a common vertex v_0.*

Note that $|V(D_m^n)| = n(m-1)+1$ and $|E(D_m^n)| = mn$ as shown in Fig. 6.

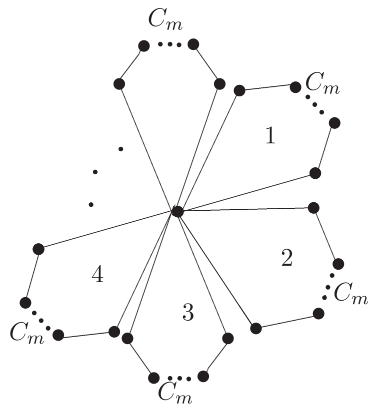


Figure 6. Dutch windmill graph D_m^n

Proposition 1.19. *For a D_m^n $(m \geq 3,\ n \geq 2)$,*

$$k^{th}EI(D_m^n) = \begin{cases} 2n\left[\sum_{i=1}^{\frac{m-1}{2}}\left(\frac{m-1}{2}+i-1\right)^k\left(\frac{m-1}{2}+i\right)^k\right] + n\left[\frac{m-1}{2}+\frac{m-1}{2}\right]^{2k}, & \textit{if } n \textit{ is odd,} \\ 2n\left[\sum_{i=1}^{\frac{m}{2}}\left(\frac{m}{2}+i-1\right)^k\left(\frac{m}{2}+i\right)^k\right]^k, & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. In D_m^n, $|V(G)| = n(m-1)+1$. Let v_0 be the central vertex of D_m^n. We consider the following cases:
Case (i) Let m be even. Then v_0 has eccentricity $\frac{m}{2}$, the n vertices at distance $\frac{m}{2}$ from v_0 have eccentricity m and the remaining $(m-2)n$ vertices have eccentricity $\frac{m}{2}+i$ where $1 \leq i \leq \frac{m}{2}-1$. Hence

$$\begin{aligned} k^{th}EI(D_m^n) &= 2n\left[\left(\frac{m}{2}\right)^k\left(\frac{m}{2}+1\right)^k + \cdots + \left(\frac{m}{2}+\frac{m}{2}-1\right)^k\left(\frac{m}{2}+\frac{m}{2}\right)^k\right] \\ &= 2n\left[\sum_{i=1}^{\frac{m}{2}}\left(\frac{m}{2}+i-1\right)^k\left(\frac{m}{2}+i\right)^k\right] \end{aligned}$$

Case(ii) Let m be odd. Then v_0 has eccentricity $\frac{m-1}{2}$, the n vertices at distance $\frac{m-1}{2}$ from v_0 have eccentricity $m-1$ and the remaining $(m-1)n$ vertices have eccentricity $\frac{m-1}{2}+i$ where $1 \leq i \leq \frac{m-1}{2}-1$. Hence

$$\begin{aligned} k^{th}EI(D_m^n) &= 2n\left[\left(\frac{m-1}{2}\right)^k\left(\frac{m-1}{2}+1\right)^k + \cdots + \left(\frac{m-1}{2}+\frac{m-1}{2}-1\right)^k\left(\frac{m-1}{2}+\frac{m-1}{2}\right)^k\right] \\ &\quad + n\left[\left(\frac{m-1}{2}\right)+\left(\frac{m-1}{2}\right)\right]^{2k} \\ &= 2n\left[\sum_{i=1}^{\frac{m-1}{2}}\left(\frac{m-1}{2}+i-1\right)^k\left(\frac{m-1}{2}+i\right)^k\right] \\ &\quad + n\left[\frac{m-1}{2}+\frac{m-1}{2}\right]^{2k}. \end{aligned}$$

□

Definition 1.20. [1] *A Kulli path windmill graph P_{m+1}^n $(m, n \geq 2)$ is the graph constructed by joining n copies of the graph $K_1 + P_m$ $(m \geq 2)$ with a vertex K_1 in common.*

Here $|V(P_{m+1}^n)| = mn+1$ and $|E(P_{m+1}^n)| = 2mn - n$ as seen in Fig. 7.

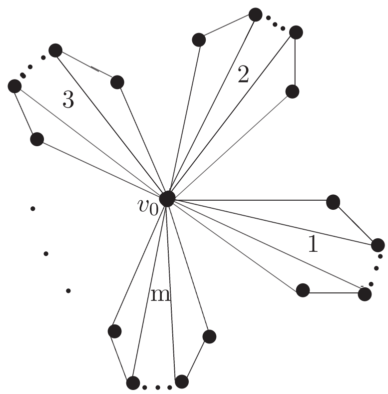


Figure 7: Kulli path windmill graph P^n_{m+1}

Proposition 1.21. *For any P^n_{m+1},*

$$k^{th}EI(P^n_{m+1}) = n[m2^k + (m-1)2^{2k}].$$

Proof. Let v_0 be the central vertex. Then $e_{v_0} = 1$ and $e_{v_i} = 2$ for $i = 1, 2, 3, \cdots, mn$. Hence

$$\begin{aligned} k^{th}EI(P^n_{m+1}) &= n[\underbrace{1^k2^k + \cdots + 1^k2^k}_{m-times} + \underbrace{2^k2^k + \cdots + 2^k2^k}_{(m-1)-times}] \\ &= n[m2^k + (m-1)2^{2k}]. \end{aligned}$$

□

Definition 1.22. [7] *The friendship graph C^n_3, $n \geq 2$, is the graph obtained by identifying one vertex of each of n copies of C_3 as shown in Fig. 8.*

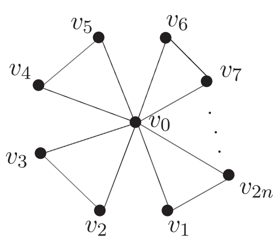


Figure 8: Friendship graph C^n_3

Corollary 1.23. *For any friendship graph C^n_3, $k^{th}EI(C^n_3) = n(2^{k+1} + 2^{2k})$.*

Proof. The proof follows from Case (ii) of Proposition 2.19 by substituting $m = 3$ or from Proposition 2.21 by substituting $m = 2$. □

Definition 1.24. [1] *The C^m_{n+1} is a Kulli cycle windmill graph which is constructed by taking m copies of the graph $K_1 + C_n (n \geq 3)$ with K_1 in common. Here $|V(C^m_{n+1})| = mn + 1$ and $|E(C^m_{n+1})| = 2mn$, as shown in Fig. 9.*

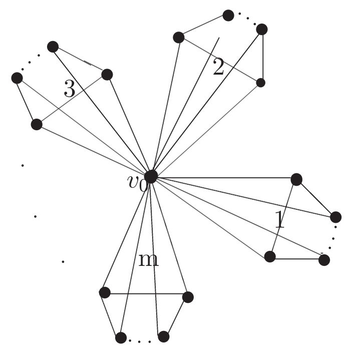


Figure 9: Kulli cycle windmill graph C_{n+1}^m

Proposition 1.25. *For any C_{n+1}^m,*

$$k^{th}EI(C_{n+1}^m) = mn(2^k + 2^{2k}).$$

Proof. Let v_0 be the central vertex. Then $e_{v_0} = 1$ and $e_{v_i} = 2$, for $i = 1, 2, \cdots, mn$.

$$\begin{aligned} k^{th}EI(C_{n+1}^m) &= m(\underbrace{1^k2^k + \cdots + 1^k2^k}_{n-times}) + m(\underbrace{2^k2^k + \cdots + 2^k2^k}_{n-tmes}) \\ &= mn(2^k + 2^{2k}). \end{aligned}$$

□

Definition 1.26. [1] *The French windmill graph F_n^m is constructed by joining $m \geq 2$ copies of the complete graph K_n, $(n \geq 2)$ with a vertex in common.*

Here $|V(F_n^m)| = m(n-1) + 1$ and $|E(F_n^m)| = \frac{mn(n-1)}{2}$ as shown in Fig. 10.

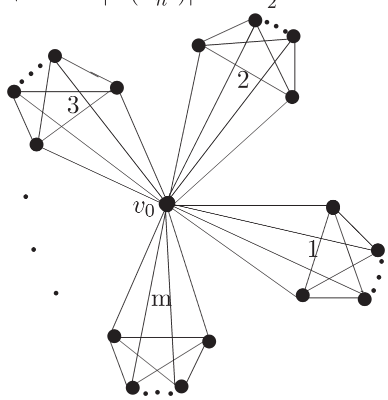


Figure 10: French wndmill graph F_n^m

Proposition 1.27. *For any F_n^m, we have*

$$k^{th}EI(F_n^m) = m(n-1)\left[2^k + (n-2)2^{2k-1}\right].$$

Proof. Let v_0 be the central vertex. Then $e_{v_0} = 1$ and $e_{v_i} = 2$ for $i = 1, 2, 3, \cdots, n$. Hence

$$\begin{aligned} k^{th}EI(F_n^m) &= m(\underbrace{1^k 2^k + \cdots + 1^k 2^k}_{(n-1)-times}) + m(\underbrace{2^k 2^k + \cdots + 2^k 2^k}_{\frac{(n-1)(n-2)}{2}-times}) \\ &= m(n-1)\left(2^k + (n-2)2^{2k-1}\right). \end{aligned}$$

□

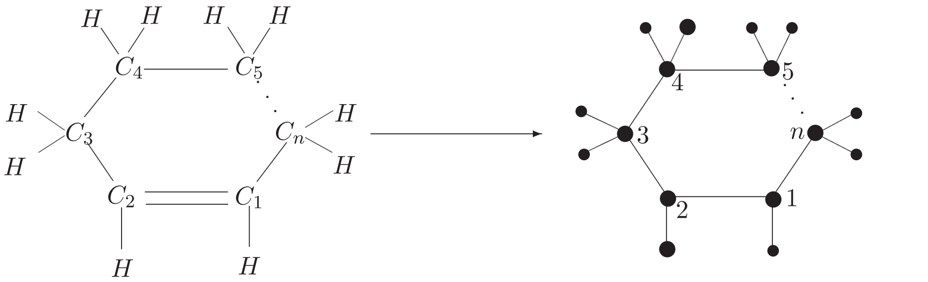
2. Bounds for k^{th}-eccentric index

We observe the following bounds by the definitions of topological indices:

(1) Let G be a unicyclic, equi-eccentric graph. Then
  (a) $k^{th}EI(G) = E_2(G) = E_1(G)$ when $k = 1$.
  (b) $k^{th}EI(G) > E_2(G) = E_1(G)$ when $k \geq 2$ and $e_u \neq 1$ for all $u \in V(G)$.
(2) If $G = K_n$, $n \geq 2$, and $k = 1$, then $k^{th}EI(G) < \xi^c(G)$.
(3) If $G = K_n$, $n \geq 2$ and $k = 1$, then $ABC_5(G) < k^{th}EI(G) < \xi^c(G)$.
(4) For any connected graph G, $E_2(G) \leq k^{th}EI(G)$.
(5) $k^{th}EI(G) = E_2(G) = GA_4(G) = H_e(G) = W(G)$ if and only if $G = K_n$.
(6) $k^{th}EI(G) = E_2(G) = E_1(G) = GA_4(G) = H_e(G) = W(G)$ if and only if $G = K_3$.
(7) For any graph G with $|V(G)| = 2$ *or* 3, , we have $k^{th}EI(G) = W(G)$.
(8) $k^{th}EI(G) = \theta(G)$ if and only if $G = K_3$.
(9) Let G be a (n, m) equi-eccentric graph. If $m > n$, then
  (a) $k^{th}EI(G) = E_2(G) > E_1(G)$ for $k = 1$.
  (b) $k^{th}EI(G) > E_2(G) > E_1(G)$ for $k \geq 3$ and $e_u \neq 1$ for all $u \in V(G)$.
(10) Let G be an (n, m) graph, $n \geq 2$. Then $k^{th}EI(G) = \xi^c(G) = E_2(G)$ if and only if $k = 1$ and $e_u = 2$ for all $u \in V(G)$.

3. k^{th}-eccentricity index of some molecular graphs

Consider the cycloalkene C_n^{2n-2} having n carbon atoms and $2n-2$ hydrogen atoms. The molecular graph of cycloalkene is obtained by attaching $2n-2$ pendant vertices corresponding to hydrogen atoms to vertices on a cycle corresponding to carbon atoms. Fig. $11(a)$ displays the molecular structure and Fig. $11(b)$ shows the molecular graph of cycloalkene, [9].



(a) Molecular structure of cycloalkene (b) Molecular graph representing the chemical compound of cycloalkene

Figure 11: Molecular structure and Molecular graph of C_n^{2n-2}

Proposition 3.1. *Let $n \geq 3$ be a positive integer. Then*

$$k^{th}EI(C_n^{2n-2}) = \begin{cases} (2n-2)(\frac{n}{2}+1)^k(\frac{n}{2}+2)^k + n(\frac{n}{2}+1)^{2k}, & \text{if } n \text{ is even} \\ (2n-2)(\frac{n-1}{2}+1)^k(\frac{n-1}{2}+2)^k + n(\frac{n-1}{2}+1)^{2k}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We have the following two cases:
Case(i) Let n be even. Then the eccentricity of each vertex corresponding to a carbon atom is $\frac{n}{2}+1$ and the eccentricity of each vertex corresponding to hydrogen atoms is $\frac{n}{2}+2$. We have

$$\begin{aligned} k^{th}EI(C_n^{2n-2}) &= \sum_{uv \in E(C_n^{2n-2})} (e_u e_v)^k \\ &= \underbrace{\left(\frac{n}{2}+1\right)^k \left(\frac{n}{2}+2\right)^k + \cdots + \left(\frac{n}{2}+1\right)^k \left(\frac{n}{2}+2\right)^k}_{(2n-2)-times} \\ &+ \underbrace{\left(\frac{n}{2}+1\right)^k \left(\frac{n}{2}+1\right)^k + \cdots + \left(\frac{n}{2}+1\right)^k \left(\frac{n}{2}+1\right)^k}_{n-times} \\ &= (2n-2)\left(\frac{n}{2}+1\right)^k \left(\frac{n}{2}+2\right)^k + n\left(\frac{n}{2}+1\right)^{2k} \end{aligned}$$

Case(ii) Let n be odd. Then eccentricity of each vertex corresponding to carbon atoms is $\frac{n-1}{2}+1$ and the eccentricity of each vertex corresponding to hydrogen atoms is $\frac{n-1}{2}+2$. Hence

$$\begin{aligned} k^{th}EI(C_n^{2n-2}) &= \sum_{uv \in E(C_n^{2n-2})} (e_u e_v)^k \\ &= \underbrace{\left(\frac{n-1}{2}+1\right)^k \left(\frac{n-1}{2}+2\right)^k + \cdots + \left(\frac{n-1}{2}+1\right)^k \left(\frac{n-1}{2}+2\right)^k}_{(2n-2)-times} \\ &+ \underbrace{\left(\frac{n-1}{2}+1\right)^k \left(\frac{n-1}{2}+1\right)^k + \cdots + \left(\frac{n-1}{2}+1\right)^k \left(\frac{n-1}{2}+1\right)^k}_{n-times} \\ &= (2n-2)\left(\frac{n-1}{2}+1\right)^k \left(\frac{n-1}{2}+2\right)^k + n\left(\frac{n-1}{2}+1\right)^{2k}. \end{aligned}$$

□

In [9], a chemical compound denoted by $C_n^{R_r}$ was obtained by joining an alkyl R_r in place of each hydrogen atom in Cycloalkene as shown in Fig. 12(a) and Fig. 12(b).

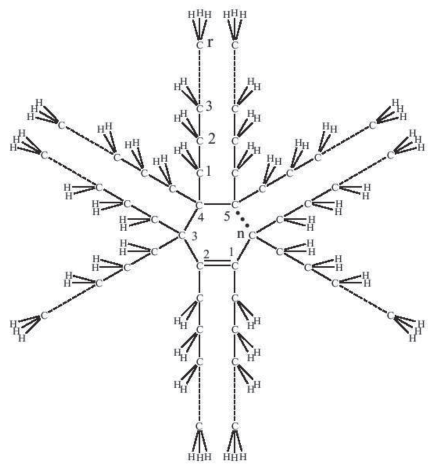


Figure 12(a): Molecular structure of $C_n^{R_r}$

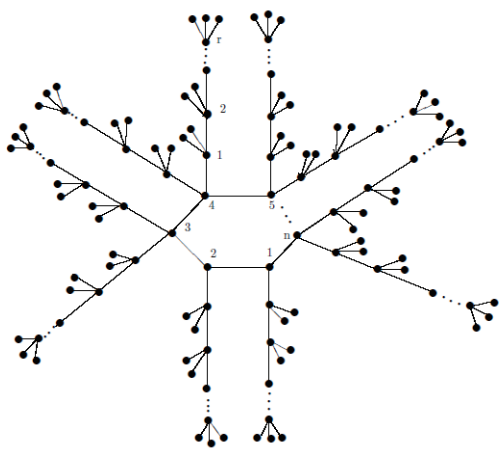


Figure 12(b): Molecular graph of $C_n^{R_r}$

Proposition 3.2. *Let n and r be positive integers with $n \geq 3$, then*

$$k^{th}EI(C_n^{R_r}) = \begin{cases} n\left(\frac{n}{2}+r+1\right)^{2k} + 2n-2\left(\frac{n}{2}+r+1\right)^k\left(\frac{n}{2}+r+2\right)^k \\ +2n-2\sum_{i=1}^{r-1}\left(\frac{n}{2}+r+1+i\right)^k\left(\frac{n}{2}+r+i+2\right)^k \\ +2(2n-2)\sum_{i=1}^{r}\left(\frac{n}{2}+r+i+2\right)^k\left(\frac{n}{2}+r+1+i\right)^k \\ +2n-2\left(\frac{n}{2}+r+r+2\right)^k\left(\frac{n}{2}+r+1+r\right)^k, & \text{if } n \text{ is even} \\ \\ n\left(\frac{n-1}{2}+r+1\right)^{2k} + 2n-2\left(\frac{n-1}{2}+r+1\right)^k\left(\frac{n-1}{2}+r+2\right)^k \\ +2n-2\sum_{i=1}^{r-1}\left(\frac{n-1}{2}+r+1+i\right)^k\left(\frac{n-1}{2}+r+i+2\right)^k \\ +2(2n-2)\sum_{i=1}^{r}\left(\frac{n-1}{2}+r+i+2\right)^k\left(\frac{n-1}{2}+r+1+i\right)^k \\ +2n-2\left(\frac{n-1}{2}+r+r+2\right)^k\left(\frac{n-1}{2}+r+1+r\right)^k, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. In $C_n^{R_r}$, $|V(C_n^{R_r})| = 6nr+3n-6r-2$ in which $n+2nr-2r$ vertices corresponding to carbon atoms and $4nr+2n-4r-2$ pendant vertices corresponding to hydrogen atoms in $C_n^{R_r}$ as shown in Fig. 12(b).

Case (i) Let n be even. Then the eccentricity of each of the n vertices corresponding to carbon atoms on the cycle is $\frac{n}{2}+r+1$, the eccentricity of each of the $2nr-2r$ vertices corresponding to carbon atoms not on the cycle is $\frac{n}{2}+r+1+i$ where $1 \leq i \leq r$ and the eccentricity of each of the $4nr+2n-4r-2$ vertices corresponding to hydrogen atoms is $\frac{n}{2}+r+i+2$ where $1 \leq i \leq r$. So,

$$\begin{aligned} k^{th}EI(C_n^{R_r}) &= \sum_{uv\in E(C_n^{R_r})}(e_u e_v)^k \\ &= n\left(\frac{n}{2}+r+1\right)^k\left(\frac{n}{2}+r+1\right)^k \\ &+(2n-2)\left(\frac{n}{2}+r+1\right)^k\left(\frac{n}{2}+r+2\right)^k \\ &+(2n-2)\sum_{i=1}^{r-1}\left(\frac{n}{2}+r+1+i\right)^k\left(\frac{n}{2}+r+i+2\right)^k \\ &+(2n-2)2\sum_{i=1}^{r}\left(\frac{n}{2}+r+i+2\right)^k\left(\frac{n}{2}+r+1+i\right)^k \\ &+(2n-2)\left(\frac{n}{2}+r+r+2\right)^k\left(\frac{n}{2}+r+1+r\right)^k \end{aligned}$$

Case (ii) Let n be odd. Then the eccentricity of each of the n vertices corresponding to carbon atoms on the cycle is $\frac{n-1}{2}+r+1$, the eccentricity of each of the $2nr-2r$ vertices corresponding to carbon atoms not on the cycle is $\frac{n-1}{2}+r+1+i$ where $1 \leq i \leq r$ and the eccentricity of each of the $4nr+2n-4r-2$ vertices corresponding to a hydrogen atom is $\frac{n-1}{2}+r+i+2$ where $1 \leq i \leq r$. So

$$\begin{aligned} k^{th}EI(C_n^{R_r}) &= \sum_{uv\in E(C_n^{R_r})}(e_u e_v)^k \\ &= n\left(\frac{n-1}{2}+r+1\right)^{2k} \\ &+(2n-2)\left(\frac{n-1}{2}+r+1\right)^k\left(\frac{n-1}{2}+r+2\right)^k \\ &+(2n-2)\sum_{i=1}^{r-1}\left(\frac{n-1}{2}+r+1+i\right)^k\left(\frac{n-1}{2}+r+i+2\right)^k \\ &+(2n-2)2\sum_{i=1}^{r}\left(\frac{n-1}{2}+r+i+2\right)^k\left(\frac{n-1}{2}+r+1+i\right)^k \\ &+(2n-2)\left(\frac{n-1}{2}+r+r+2\right)^k\left(\frac{n-1}{2}+r+1+r\right)^k. \end{aligned}$$

□

4. Conclusions

In this paper, we initiated the study of a new eccentricity based topological index called the k^{th}-eccentricity index of a graph G denoted by $k^{th}EI(G)$. Also we computed k^{th}-eccentricity index for some well-known standard graphs including windmill graphs and molecular graphs of cycloalkenes. We have also obtained bounds on k^{th}-eccentricity index of graphs in terms of some other topological indices.

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