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LE-DEGREE SUM ENERGY OF GRAPH

S. VIJAY, K.N PRAKASHA, P.S HEMAVATHI, AND SMITHA G KINI

ABSTRACT. In the midst of 20^{th} century, Ivan Gutman invented the concept of energy of graph which is depends on the numbers 0 (if there is no edge between two vertices) and 1 (If two vertices are adjacent) in adjacency matrix. In the current paper we are introducing a degree based new matrix and energy. We define the matrix as follows (i, j)-entry is equal to $\ln(d_i+d_j)+e^{(d_i+d_j)}$ if $i \neq j$ and 0 otherwise.(where d_i, d_j be the degree of its i^{th} and j^{th} vertices respectively) In this paper we compute the LE-Degree sum energy for graphs and some graph operations. Few important properties and bounds for $DS_{LE}E(G)$ are also discussed.

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1. INTRODUCTION

If any two vertices v_i and v_j in a graph G are adjacent, then we use the notation $v_i \sim v_j$. The degree of a vertex will be denoted by d_i. The energy of graph was introduced by Ivan Gutman in 1978 as the sum of the absolute values of eigenvalues of adjacency matrix with respect that graph.[2]. Details on energy of graph can be found in [2] and [3]. Motivated by the energy of graph, in this paper we are introducing a new matrix, as

$$DS_{LE} = \begin{cases} \ln(d_i + d_j) + e^{d_i+d_j} & \text{if } i \neq j, \\ 0 & otherwise \end{cases}$$

We call this matrix as Logarithm and Exponential Degree sum matrix.

Lemma 1.1. [5] *If a,b,c and d are real numbers, then the determinant of the form*

$$\begin{vmatrix} (\lambda + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1\times n_2} \\ -dJ_{n_2\times n_1} & (\lambda + b)I_{n_2} - bJ_{n_2} \end{vmatrix}.$$

of order $n_1 + n_2$ can be expressed in the simplified form as

$$(\lambda + a)^{n_1-1}(\lambda + b)^{n_2-1}\left((\lambda - (n_1 - 1)a)(\lambda - (n_2 - 1)b) - n_1n_2cd\right).$$

2. THE LE-DEGREE SUM ENERGY OF GRAPH

Let G be a simple graph of order n with vertex set $V = \{v_1, v_2, v_3, ..., v_n\}$ and edge set E. Consider $DS_{LE}(G)$ as the LE-Degree sum matrix, it's characteristic polynomial would be given by $P_{DS_{LE}}(G, \lambda) = det(\lambda I - DS_{LE}(G))$. Here the LE-Degree sum matrix is real and symmetric, hence its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 > \lambda_2 > \cdots \lambda_n$.

The LE-Degree sum energy is given by

$$DS_{LE}E(G) = \sum_{i=1}^{n} |\lambda_i| \tag{1}$$

3. LE-Degree sum energy of few graph structures

Theorem 3.1. *Let W_n be the wheel graph, then it's characteristic polynomial would be*

$$P_{DS_{LE}}(W_n)(\lambda) = (\lambda+(\ln 6+e^6))^{n-2}\left(\lambda[\lambda-(n-2)(\ln 6+e^6)]-(n-1)[\ln(n+2)+e^{n+2}]^2\right)$$

Proof. The wheel graph W_n has two types of vertices, $n-1$ rim vertices are of degree 3 and remaining central vertex has degree $n-1$. Thus,

$$DS_{LE}(W_n) = \begin{pmatrix} (\ln 6+e^6)(J_{n-1}-I_{n-1}) & (\ln(n+2)+e^{(n+2)})J_{n-1\times 1} \\ (ln(n+2)+e^{(n+2)})J_{1\times n-1} & [\ln 2(n-1)+e^{2(n-1)}](J_1-I_1) \end{pmatrix}.$$

Thus by lemma 1.1 one can get

$$P_{DS_{LE}}(W_n)(\lambda) = (\lambda+(\ln 6+e^6))\left(\lambda[\lambda-(n-2)(\ln 6+e^6)]-(n-1)[\ln(n+2)+e^{n+2}]^2\right)$$

□

The following two results can be proven in the similar way.

Theorem 3.2. *For a prism graph of order $2n$, the polynomial would be*

$$P_{DS_{LE}}(\lambda) = (\lambda+\ln 6+e^6)^{2n-1}(\lambda-(2n-1)(\ln 6+e^6))$$

Theorem 3.3. *For a T_n triangular snake, then it's characteristic polynomial would be*

$$P_{DS_{LE}}(T_n)(\lambda) = (\lambda+(\ln 4+e^4))^n(\lambda+\ln 8+e^8)^{n-3}$$
$$\left(\lambda-n(\ln 4+e^4)[\lambda-(n-3)(\ln 8+e^8)-2(\ln 8+e^8)(n+1)(n-2)]\right)$$

Definition 3.4. [4] *The union $G_1 \cup G_2$ of graphs G_1 and G_2 is a graph whose vertex set is $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and an edge set is $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.*

Theorem 3.5. *If G is r_1-regular graph of order n_1 and H is r_2-regular graph of order n_2, then the characteristic polynomial of $G \cup H$ is*

$$P_{DS_{LE}}(G \cup H)(\lambda) = (\lambda+\ln 2r_1+e^{2r_1})^{n_1-1}(\lambda+\ln 2r_2+e^{2r_2})^{n_2-1}$$
$$(\lambda-(n_1-1)(\ln 2r_1+e^{2r_1}))(\lambda-(n_2-1)(\ln 2r_2+e^{2r_2}))-n_1n_2(\ln(r_1+r_2)+e^{r_1+r_2})^2.$$

Proof. *The graph $G \cup H$ has two types of vertices, the n_1 vertices of degree r_1 and the remaining n_2 vertices are of degree r_2. Hence the matrix would be*

$$\begin{pmatrix} (\ln 2r_1+e^{2r_1})(J_{n_1}-I_{n_1}) & (\ln(r_1+r_2)+e^{r_1+r_2})J_{n_1\times n_2} \\ (\ln(r_1+r_2)+e^{r_1+r_2})J_{n_2\times n_1} & (\ln 2r_2+e^{2r_2})(J_{n_2}-I_{n_2}) \end{pmatrix}.$$

Thus by lemma 1.1, one can get the result.

Definition 3.6. *The corona $G_1 \circ G_2$ of graphs G_1 and G_2 is a graph obtained from G_1 and G_2 by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and then joining by an edge each vertex of the i^{th} copy of G_2 is named (G_2, i) with the i_{th} vertex of G_1.*

Theorem 3.7. *If G is r_1-regular graph of order n_1 and H is r_2-regular graph of order n_2, then the characteristic polynomial of $G \circ H$ is*

$$\left(\lambda+\ln(2r_1)+e^{2r_1}\right)^{n_1-1}\left(\lambda+\ln(2r_2)+e^{2r_2}\right)^{n_1n_2-1}((\lambda-(n_1-1)[\ln(2r_1)+e^{2r_1}])$$
$$(\lambda-(n_1n_2-1)[\ln(2r_2)+e^{2r_2}])-n_2{n_1}^2\lambda-(n_1-1)[\ln(r_1+r_2)+e^{r_1+r_2}]$$

Proof. The graph GoH has two types of vertices, the n_1 vertices with degree $a=r_1+n_2$ and the remaining n_1n_2 vertices are of degree $b=r_2+1$. Hence, the matrix would be

$$\begin{pmatrix} (\ln 2a+e^{2a})(J_{n_1}-I_{n_1}) & (\ln(a+b)+e^{a+b})J_{n_1\times n_2} \\ (\ln(a+b)+e^{a+b})J_{n_2\times n_1} & (\ln 2b+e^{2b})(J_{n_2}-I_{n_2}) \end{pmatrix}.$$

Thus by lemma 1.1, one can get the result. □

Definition 3.8. *The splitting graph $S(G)$ of a graph G is obtained by adding a new vertex u to each vertex v such that u is adjacent to every vertex that is adjacent to v in G.*

Theorem 3.9. *For any regular graph G with regularity r,*

$$DS_{LE}E(S(G))\le 2\left[(n-1)[\ln 4r+\ln 2r+e^{4r}+e^{2r}]+n(\ln 3r+e^{3r})\right]$$

Proof. The LE-Degree sum matrix for $S(G)$ is given by

$$DS_{LE}(S(G))=\begin{bmatrix}(\ln 4r+e^{4r})(J_n-I_n) & (\ln 3r+e^3r)(J_{n\times n}) \\ (\ln 3r+e^3r)(J_{n\times n})) & (\ln 2r+e^2r)(J_n-I_n)\end{bmatrix}$$

Thus

$$DS_{LE}E(S(G))\le 2\left[(n-1)[\ln 4r+\ln 2r+e^{4r}+e^{2r}]+n(\ln 3r+e^{3r})\right]$$

□

Theorem 3.10. *The regular graph has only one positive eigenvalue i.e., $\lambda=(n-1)(\ln(2r)+e^{2r})$ and its energy would be twice of the largest eigen value.*

Proof.

$$\begin{pmatrix} (\ln 2r+e^{2r})\,(J-I) \end{pmatrix}.$$

Characteristic equation is

$$\left(\lambda+\ln(2r)+e^{2r}\right)^{n-1}\left(\lambda-[\ln(2r)+e^{2r}]\right)=0$$

and the spectrum is $Spec_{DS_{LE}}(G)=\begin{pmatrix}\ln(2r)+e^{2r} & (n-1)[\ln(2r)+e^{2r}] \\ n-1 & 1\end{pmatrix}$.

Therefore, $DS_{LE}(G)=2(n-1)[\ln(2r)+e^{2r}]$. □

4. Properties of LE-Degree sum energy of a graph

As we have the definition of LE-Degree sum matrix, absolutely the diagonal entries would be zero. Suppose if you consider the dominating sets such as minimum dominating, equitable dominating, boundary dominating etc., or covering set, hub set, then the matrix could be having the non zero diagonal entries.

We use the following inequality given by Cauchy-Schwartz [1] in the next results. It is given by

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2\le\left(\sum_{i=1}^{n}{a_i}^2\right)\left(\sum_{i=1}^{n}{b_i}^2\right). \tag{2}$$

Proposition 4.1. *The initial three coefficients of $P_{DS_{LE}}(G,\lambda)$ are*

- *unity*
- *null*
- $-(\sum_{i<j}(\ln(d_i+d_j)+e^{d_i+d_j})^2$

Proof. (i) By the definition of characteristic polynomial we get $a_0 = 1$.

(ii) The trace of $DS_{LE}(G)$ is equal to the sum of determinants of all 1×1 principal submatrices.
$\Rightarrow a_1 = (-1)^1$ trace of $[DS_{LE}(G)] = 0$.

(iii)

$$\begin{aligned}(-1)^2 a_2 &= \sum_{1\le i<j\le n}\begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj}\end{vmatrix} \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - a_{ji}a_{ij} \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - \sum_{1\le i<j\le n} a_{ji}a_{ij} \\ &= -(\sum_{i<j}(\ln(d_i+d_j)+e^{d_i+d_j})^2)\end{aligned}$$

□

Proposition 4.2. *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the LE-Degree sum eigenvalues of $DS_{LE}(G)$, then*

(3) $$\sum_{i=1}^{n} {\lambda_i}^2 = 2\sum_{i<j}(\ln(d_i+d_j)+e^{d_i+d_j})^2$$

Let X be any set of diagonal entries of the matrix, then

$$\sum_{i=1}^{n} {\lambda_i}^2 = |X| + 2\sum_{i<j}(\ln(d_i+d_j)+e^{d_i+d_j})^2$$

Proof. We know that

$$\begin{aligned}\sum_{i=1}^{n}\lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\ &= 2\sum_{i<j}(a_{ij})^2 + \sum_{i=1}^{n}(a_{ii})^2 \\ &= 2\sum_{i<j}(a_{ij})^2 + |X| \\ &= 2(\sum_{i<j}(\ln(d_i+d_j)+e^{d_i+d_j})^2 + |X|.\end{aligned}$$

□

Theorem 4.3. *If G is any graph of order n and* λ_1 *is the largest eigenvalue, then* $\lambda_1 \leq \sqrt{\frac{2(n-1)}{n} \sum_{i<j} (\ln(d_i + d_j) + e^{d_i+d_j})^2}$

Proof. Using the equation 2, we get

$$\left(\sum_{i=1}^{n} \lambda_i \right)^2 \leq (n-1) \left(\sum_{i=2}^{n} \lambda_i^2 \right)$$

Using the equation 3, we get

$$\lambda_1^2 \leq (n-1) \left(2 \sum_{i<j} (\ln(d_i + d_j) + e^{d_i+d_j})^2 - \lambda_1^2 \right)$$

Thus the proof follows. □

Theorem 4.4. *Let G be a graph with n vertices and then*

$$\sqrt{2 \sum_{i<j} (\ln(d_i + d_j) + e^{d_i+d_j})^2 + n(n-1)(|DS_{LE}|)^{\frac{2}{n}}}$$
$$\leq DS_{LE}E(G) \leq \sqrt{2n \sum_{i<j} (\ln(d_i + d_j) + e^{d_i+d_j})^2}.$$

Proof. Substitute $a_i = 1$, $b_i =\mid \lambda_i \mid$ to the equation2, and using the equation 3, we get

$$DS_{LE}E = \sum_{i=1}^{n} |\lambda_i| = \sqrt{\left(\sum_{i=1}^{n} |\lambda_i| \right)^2} \leq \left(n \sum_{i=1}^{n} |\lambda_i|^2 \right)$$

hence we get

$$[DS_{LE}E] \leq \sqrt{2n \sum_{i<j} (\ln(d_i + d_j) + e^{d_i+d_j})^2}$$

as an upper bound. Lower bound can be easily proven by using arithmetic mean and geometric mean inequality. □

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[1]Department of Mathematics, Government Science College, Hassan-573 201, INDIA
Email address: vijayshivanna82@gmail.com

[2]Department of Mathematics,, Vidyavardhaka College of Engineering, Mysuru-570002, INDIA
Email address: prakashamaths@gmail.com

[3]Department of Mathematics,, Siddaganga Institute of Technology, Tumkur-572103, INDIA
Email address: hemavathisuresh@gmail.com

[4]Department of Mathematics, Sri Jayachamarajendra College of Engineering, Mysuru-570006, INDIA
Email address: smithakini3@gmail.com