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A NEW CONSTRUCTION ON THE DEGENERATE HURWITZ-ZETA FUNCTION ASSOCIATED WITH CERTAIN APPLICATIONS

M. SELÇUK AYDIN, MEHMET ACIKGOZ, AND SERKAN ARACI*

Abstract. In [5], Kim and Kim defined degenerate version of gamma functions, and then introduced its new properties by making use of analytical methods in the complex analysis. With this in mind, we consider the degenerate Hurwitz-zeta, modified degenerate Hurwitz-zeta, degenerate digamma functions. We obtain several new properties and identities for these functions.

1. Introduction

Special functions play a vital role in various branches of mathematics, statistics, mathematical physics and engineering. The problems encountered in mathematical physics, particularly in engineering, stem from the use of ordinary or partial differential equations. Most of these equations can only be treated by using various families of special functions which provide new means of classical analysis. They are widely used in numerical analysis and scientific computing. With this in mind, the researchers working on special functions (or special polynomials) are moivated to consider possible extensions of new families of special functions. Another reason for working on new classes of special functions is to demonstrate the existence of natural solutions to a given set of (partial) differential equations under certain conditions that frequently occur in the treatment of electromagnetic wave propagation, quantum beam lifetime in storage rings, and so on.

Recent observations including degenerate gamma functions [6], modified version of degenerate gamma and Laplace transformation [7], certain fundamental properties of the modified degenerate gamma function [10], relations between degenerate gamma function and degenerate Stirling number of the second kind in [12] have been investigated extensively.

Throughout this paper, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ be the sets of complex numbers, real numbers, integers, and positive integers, respectively. Also let

$$\mathbb{N} = \{1, 2, 3, \cdots\} \text{ and } \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Also, as usual, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Z}^-$ denotes the set of negative integers, $\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}$ and $\mathbb{C}$ denotes the set of complex numbers.

The Bernoulli polynomials $B_n(z)$ are defined by the generating function:

$$F(t, z) := \frac{t}{e^t - 1} e^{zt} = \sum_{n=0}^{\infty} B_n(z) \frac{t^n}{n!} \qquad (|t| < 2\pi). \tag{1.1}$$

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[0]*Corresponding author: mtsrkn@hotmail.com

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Here we have $B_n := B_n(0)$ in (1.1) are called Bernoulli numbers (see, e.g., [2, 12]).
The Hurwitz-zeta function is defined by ([16, 17, 18, 19, 20])

$$\zeta(s,z)=\sum_{m=0}^{\infty}\frac{1}{(m+z)^s}\qquad\left(z\in\mathbb{C}\backslash\mathbb{Z}_0^-;\operatorname{Re}(s)>1\right). \tag{1.2}$$

The particular case $z=1$ of (1.2) is Riemann-zeta function $\zeta(s)$:

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s}\quad(\operatorname{Re}(s)>1). \tag{1.3}$$

Obviously, $\zeta(s,1)=\zeta(s)$.
The familiar Gamma function $\Gamma(s)$ is defined by

$$\Gamma(s)=\int_0^{\infty}e^{-t}t^{s-1}\mathrm{d}t\quad(\operatorname{Re}(s)>0). \tag{1.4}$$

Let

$$(x)_{0,\lambda}:=1,\ (x)_{\ell,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(\ell-1)\lambda),\quad(n\geq 1).$$

Then the exponential generating function of $(x)_{\ell,\lambda}$ is the degenerate exponential function $e_\lambda^x(t)$ given by

$$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{\ell=0}^{\infty}(x)_{\ell,\lambda}\frac{t^\ell}{\ell!}$$

which satisfies the following assumptions:

$$e_\lambda^1(t):=e_\lambda(t).$$

The pioneering of the degenerate exponential function idea without the limit case was Carlitz who considered for the classical Bernoulli polynomials as follows:

$$F_\lambda(t,z):=\frac{t}{(1+\lambda t)^{1/\lambda}-1}(1+\lambda t)^{z/\lambda}=\sum_{n=0}^{\infty}\beta_{n,\lambda}(z)\frac{t^n}{n!}, \tag{1.5}$$

where $\beta_{n,\lambda}(z)$ are the degenerate Bernoulli polynomials. In the case when $z=0$, we have $\beta_{n,\lambda}(0)=:\beta_{n,\lambda}$ that stands for the degenerate Bernoulli numbers. It follows from (1.5) that

$$\lim_{\lambda\to 0}\beta_{n,\lambda}(z)=B_n(z)\qquad(n\geq 0).$$

In [6], Kim and Kim introduced the degenerate gamma function as a degenerate version of the usual gamma function, and derived some new interesting identities. They also obtained an analytic continuation as a meromorphic function on the whole complex plane, the difference formula, the values at positive integers, some expressions following from the Weierstrass and Euler formulas for the ordinary gamma function, and an integral representation as an integral along a Hankel contour. Kim [5] introduced the degenerate Euler zeta function which is holomorphic on the complex plane, and showed that the degenerate Euler zeta function is closely related to the degenerate Euler polynomials at negative integers.

2. Degenerate Hurwitz-Zeta Function

Let $\lambda \in \mathbb{R}\backslash \{0\}$ and $|1+\lambda t|<1$ with $|\lambda t|<1$. We begin with the following computations:

$$\begin{aligned}\sum_{n=0}^{\infty}\beta_{n,\lambda}(z)\frac{t^n}{n!} &= \frac{-t}{1-(1+\lambda t)^{\frac{1}{\lambda}}}(1+\lambda t)^{\frac{z}{\lambda}}\\ &= -t\sum_{m=0}^{\infty}(1+\lambda t)^{\frac{m+z}{\lambda}}\\ &= \sum_{n=1}^{\infty}\left(-n\sum_{m=0}^{\infty}(m+z)_{n-1,\lambda}\right)\frac{t^n}{n!}.\end{aligned}$$

Thus, we have

$$-\frac{\beta_{n,\lambda}(z)}{n}=\sum_{m=0}^{\infty}(m+z)_{n-1,\lambda},\ (n\geq 1). \tag{2.1}$$

As λ tends to 0 in (2.1), we obtain the well-known identity

$$\zeta(-n,z)=-\frac{B_{n+1}(z)}{n+1} \qquad (n\in\mathbb{N}), \tag{2.2}$$

which interpolates the Bernoulli polynomials by the Riemann zeta function at negative integers.

For $\lambda\in(0,1)$ and $s\in\mathbb{C}$ with $0<\mathrm{Re}(s)<\frac{1}{\lambda}$, Kim defined the degenerate gamma function by the following expression ([6, 9]):

$$\Gamma_{\lambda}(s)=\int_{0}^{\infty}(1+\lambda t)^{-\frac{1}{\lambda}}t^{s-1}\mathrm{d}t, \tag{2.3}$$

which satisfies the following property with the assumptions $n\in\mathbb{N}$ and $\lambda\in\left(0,\frac{1}{n}\right)$:

$$\Gamma_{\lambda}(n)=\frac{(n-1)!}{(1)_{n+1,\lambda}}. \tag{2.4}$$

By (2.3), we consider a new generalization of Hurwitz-Zeta function by the following Definition:

Definition 1. The degenerate Hurwitz-zeta function is defined by

$$\zeta_{\lambda}(s,z)=\frac{1}{\Gamma_{\lambda}(s)}\sum_{m=0}^{\infty}\frac{\Gamma_{\frac{\lambda}{m+z}}(s)}{(m+z)^s}\,text \qquad \left(z\notin\mathbb{Z}_0^-;s\in\mathbb{C};\mathrm{Re}(s)>1\right).$$

Theorem 2.1. *Let $s\in\mathbb{C}$. The degenerate Hurwitz-zeta function holds true:*

$$\zeta_{\lambda}(s,z)=\frac{1}{\Gamma_{\lambda}(s)}\int_{0}^{\infty}F_{-\lambda}(-t,z)\,t^{s-2}dt.$$

Proof. It is proved by using Definition 1 and change of variables as $u = (m+z)\,t$ for the following third equation that

$$\begin{aligned}
\zeta_\lambda(s,z) &= \sum_{m=0}^{\infty} \frac{\frac{\Gamma_{\frac{\lambda}{m+z}}(s)}{\Gamma_\lambda(s)}}{(m+z)^s} \\
&= \frac{1}{\Gamma_\lambda(s)} \sum_{m=0}^{\infty} \frac{1}{(m+z)^s} \int_0^\infty \left(1+\frac{\lambda}{m+z}u\right)^{-\frac{m+z}{\lambda}} u^{s-1}\mathrm{d}u \\
&= \frac{1}{\Gamma_\lambda(s)} \int_0^\infty \sum_{m=0}^{\infty} \left(1+\frac{m+z}{m+z}\lambda t\right)^{-\frac{m+z}{\lambda}} t^{s-1}\mathrm{d}t \\
&= \frac{1}{\Gamma_\lambda(s)} \int_0^\infty \sum_{m=0}^{\infty} (1+\lambda t)^{-\frac{m+z}{\lambda}} t^{s-1}\mathrm{d}t \\
&= \frac{1}{\Gamma_\lambda(s)} \int_0^\infty \frac{-t}{(1+\lambda t)^{\frac{-1}{\lambda}}-1} (1+\lambda t)^{\frac{-z}{\lambda}} t^{s-2}\mathrm{d}t \\
&= \frac{1}{\Gamma_\lambda(s)} \int_0^\infty F_{-\lambda}(-t,z)\, t^{s-2}\mathrm{d}t.
\end{aligned}$$

Thus we arrive at the desired result. □

By (2.4), we have the following corollary.

Corollary 2.2. Let $n \in \mathbb{N}$ and $\lambda \in \left(0, \frac{1}{n}\right)$. Then we have

$$\zeta_\lambda(n,z) = \sum_{m=0}^{\infty} (m+z)^{-n} \frac{(1)_{n+1,\lambda}}{(1)_{n+1,\frac{m+z}{\lambda}}}.$$

Remark 2.3. Putting $z=1$ in Definition 1 yields

$$\zeta_\lambda(s,1) := \zeta_\lambda(s) = \frac{1}{\Gamma_\lambda(s)} \sum_{m=1}^{\infty} \frac{\Gamma_{\frac{\lambda}{m}}(s)}{m^s}, \tag{2.5}$$

where $\zeta_\lambda(s)$ is degenerate version of the Riemann-zeta function.

Remark 2.4. Kim and Kim [8] first introduced the degenerate Hurwitz-zeta function as follows:

$$\zeta_\lambda(s,\delta) = \sum_{n=0}^{\infty} \frac{(1)_{n,\lambda}}{(n+\delta)^s}.$$

We would like to note that our degenerate Hurwitz-zeta function is different from Kim and Kim's degenerate Hurwitz-zeta function.

We are now in a position to state that the degenerate Hurwitz-zeta function interpolates the degenerate Bernoulli polynomials at nonpositive integers by the following theorem.

Theorem 2.5. *Let n be a nonpositive integer. Then we have*

$$\zeta_\lambda(1-n,z) = -\frac{\beta_{n,-\lambda}(z)}{n\,(1)_{n-1,-\lambda}}.$$

Proof. By the same method given in [5], we prove this theorem. Then, by the Definition 1, we have

$$\begin{aligned}
\zeta_\lambda(s,z) &= \frac{1}{\Gamma_\lambda(s)}\int_0^\infty \frac{t}{1-(1+\lambda t)^{-\frac{1}{\lambda}}}(1+\lambda t)^{-\frac{z}{\lambda}}t^{s-2}\mathrm{d}t \\
&= \frac{1}{\Gamma_\lambda(s)}\int_0^\infty \sum_{m=0}^\infty \beta_{m,-\lambda}(z)\frac{(-1)^m}{m!}t^{m+s-1}\mathrm{d}t \\
&= \frac{1}{\Gamma_\lambda(s)}\sum_{m=0}^\infty \beta_{m,-\lambda}(z)\frac{(-1)^m}{m!}\int_0^\infty t^{m+s-2}\mathrm{d}t.
\end{aligned}$$

From here we see that

$$\zeta_\lambda(1-n,z) = 2\pi i\frac{1}{\Gamma_\lambda(1-n)}\beta_{n,-\lambda}(z)\frac{(-1)^n}{n!}. \tag{2.6}$$

For $\Gamma_\lambda(1-n)$, we compute

$$\begin{aligned}
\Gamma_\lambda(1-n) &= \int_0^\infty (1+\lambda t)^{-\frac{1}{\lambda}}t^{-n}\mathrm{d}t \\
&= \sum_{p=0}^\infty\left(-\frac{1}{\lambda}\right)\left(-\frac{1}{\lambda}-1\right)\left(-\frac{1}{\lambda}-2\right)\cdots\left(-\frac{1}{\lambda}-(p-1)\lambda\right)\frac{\lambda^p}{p!}\int_0^\infty t^{p-n}\mathrm{d}t \\
&= 2\pi i\frac{(-1)^{n-1}}{(n-1)!}(1)_{n-1,-\lambda}. \qquad (2.7)
\end{aligned}$$

By (2.6) and (2.7), we complete the proof of theorem. □

3. Further modification for Degenerate Hurwitz-Zeta Function

In this section, motivated and inspired by [7], we consider the modified degenerate Hurwitz-zeta function which is similar to that of He *et al.*'s ([2]) modified version of degenerate gamma function and modified degenerate Bernoulli polynomials. Thus we give some new properties for them.

The modified degenerate gamma function is considered by ([2])

$$\Gamma^*_\lambda(z) = \int_0^\infty (1+\lambda)^{-\frac{t}{\lambda}}t^{z-1}dt \qquad (\lambda\in(0,\infty);\mathrm{Re}(z)>0;z\in\mathbb{C}), \tag{3.1}$$

which is closely-related to well-known gamma function as follows:

$$\Gamma^*_\lambda(z) = \left(\frac{\log(1+\lambda)}{\lambda}\right)^{-z}\Gamma(z), \text{ see [2]}, \tag{3.2}$$

where $\log(z)$ is the principal branch of the complex logarithm $\log(s)$ with the imaginary part restricted by

$$-\pi < \mathrm{Im}(\log(z)) \leq \pi.$$

Under conditions $\lambda \in (0,1)$, $\mathrm{Re}(z) > 0$ and $z \in \mathbb{C}$, the modification version of degenerate Bernoulli polynomials is also considered by

$$F_\lambda^*(t,z) = \frac{t}{(1+\lambda)^{\frac{t}{\lambda}} - 1}(1+\lambda)^{\frac{tz}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}^*(z)\frac{t^n}{n!}. \tag{3.3}$$

Definition 2. Let $s \in \mathbb{C}, z \in \mathbb{C} - \{0,-1,-2,\cdots\}, n \in \mathbb{N}$ and $\lambda \in \left(0,\frac{1}{n}\right)$ with $\mathrm{Re}(s) > 0$. Then the modified degenerate Hurwitz-Zeta function is defined by

$$\zeta_\lambda^*(s,z) = \frac{\Gamma_\lambda^*(s)}{\Gamma(s)}\sum_{m=0}^{\infty}\frac{1}{(m+z)^s}. \tag{3.4}$$

The following identity comes from (3.4) that

$$\zeta_\lambda^*(s,z) = \frac{\Gamma_\lambda^*(s)}{\Gamma(s)}\zeta(s,z), \tag{3.5}$$

where $\zeta(s,z)$ is Hurwitz-zeta function. In the case when $z=1$ in (3.4), we have $\zeta_\lambda^*(s,1) = \zeta_\lambda^*(s)$ that can be called modified degenerate Riemann-zeta function written as

$$\zeta_\lambda^*(s,1) = \frac{\Gamma_\lambda^*(s)}{\Gamma(s)}\sum_{m=1}^{\infty}\frac{1}{m^s} = \frac{\Gamma_\lambda^*(s)}{\Gamma(s)}\zeta(s,1). \tag{3.6}$$

Theorem 3.1. *Let $n,m \in \mathbb{N}$ and $\lambda \in \left(0,\frac{1}{n}\right)$. Then we have*

$$\zeta_\lambda^*(1-m,z) = -\frac{\beta_{m,\lambda}^*(z)}{m}.$$

Proof. To prove this theorem, we firstly consider

$$\zeta_\lambda^*(s,z) = \frac{1}{\Gamma(s)}\int_0^\infty F_\lambda^*(-t,z)\,t^{s-2}\mathrm{d}t.$$

Then we have

$$\begin{aligned} \zeta_\lambda^*(s,z) &= \frac{1}{\Gamma(s)}\int_0^\infty \sum_{m=0}^{\infty}\beta_{m,\lambda}^*(z)\frac{(-1)^m}{m!}t^{m+s-2}\mathrm{d}t \\ &= \frac{1}{\Gamma(s)}\sum_{n=0}^{\infty}\beta_{n,\lambda}^*\frac{(-1)^n}{n!}\int_0^\infty t^{n+s-2}\mathrm{d}t. \end{aligned} \tag{3.7}$$

By (3.7), we see that

$$\zeta_\lambda^*(1-m,z) = 2\pi i\frac{1}{\Gamma(1-m)}\beta_{m,\lambda}^*(z)\frac{(-1)^m}{m!}. \tag{3.8}$$

The function $\Gamma(1-m)$ in (3.8) is equal to

$$\Gamma(1-m) = \int_0^\infty e^{-t}t^{-m}dt = \sum_{\ell=0}^\infty \frac{(-1)^\ell}{\ell!}\int_0^\infty t^{\ell-m}\mathrm{d}t. \tag{3.9}$$

So, by (3.8) and (3.9), we have

$$\zeta_\lambda^*(1-m,z) = -\frac{\beta^*_{m,\lambda}(z)}{m}, \tag{3.10}$$

which is desired result. □

By (3.2) and (3.5), we have

$$\zeta_\lambda^*(1-m,z) = \left(\frac{\lambda}{\log(1+\lambda)}\right)^z \zeta(1-m,z) \text{ and } \beta^*_{m,\lambda}(z) = \left(\frac{\lambda}{\log(1+\lambda)}\right)^z B_m(z).$$

4. Degenerate Digamma Function

The digamma function denoted by $\psi(s)$ is the logarithmic derivative of the gamma function:

$$\psi(s) = \frac{d}{ds}\log\Gamma(s) = \frac{\Gamma'(s)}{\Gamma(s)}, \tag{4.1}$$

which appears in the regularization of the divergent integrals, and where $\log(z)$ is the principal branch of the complex logarithm $\log(s)$ with the imaginary part restricted by

$$-\pi < \mathrm{Im}(\log(z)) \leq \pi.$$

In the Ref. [6], one has for $\lambda \in (0,1)$, $z \in \mathbb{C}$ and $\mathrm{Re}(z) > 0$

$$\Gamma_\lambda(z+1) = \frac{z}{(1-\lambda)^{z+1}}\Gamma_{\frac{\lambda}{1-\lambda}}(z) \tag{4.2}$$

and

$$\Gamma_\lambda(z) = \frac{\lambda^{-z}\Gamma(z)\Gamma\left(\frac{1}{\lambda}-z\right)}{\Gamma\left(\frac{1}{\lambda}\right)}. \tag{4.3}$$

Since the digamma function is the logarithmic derivative of the gamma function, we consider the degenerate digamma function by the following form:

$$\psi_\lambda(z) = \frac{d}{dz}\log(\Gamma_\lambda(z)) = \frac{\Gamma'_\lambda(z)}{\Gamma_\lambda(z)}. \tag{4.4}$$

We now obtain certain fundamental properties of the degenerate digamma function.

Theorem 4.1. *Let* $\lambda \in (0,1)$, $z \in \mathbb{C}$ *and* $\mathrm{Re}(z) > 0$, *we have*

$$\psi_\lambda(z+1) = \psi_{\frac{\lambda}{1-\lambda}}(z) + \frac{1}{z} - \log(1-\lambda).$$

Proof. It immediately follows from (4.2) and (4.4). □

By the Theorem 4.1, we see that

$$\begin{aligned}
\psi_{\lambda}(z+1) &= \psi_{\frac{\lambda}{1-\lambda}}(z)+\frac{1}{z}-\log(1-\lambda)\\
\psi_{\lambda}(z+2) &= \psi_{\frac{\lambda}{1-2\lambda}}(z)+\frac{1}{z}+\frac{1}{z+1}-\log(1-2\lambda)\\
\psi_{\lambda}(z+3) &= \psi_{\frac{\lambda}{1-3\lambda}}(z)+\frac{1}{z}+\frac{1}{z+1}+\frac{1}{z+2}-\log(1-3\lambda).
\end{aligned}$$

Then we have

$$\psi_{\lambda}(z+n)=\psi_{\frac{\lambda}{1-n\lambda}}(z)-\log(1-n\lambda)+\sum_{m=0}^{n-1}\frac{1}{z+m}.$$

By (4.3) and (4.4), we have the following Theorem.

Theorem 4.2. *Let* $\lambda\in(0,1)$, $z\in\mathbb{C}$ *and* $\operatorname{Re}(z)>0$, *we have*

$$\psi_{\lambda}(z)=\psi(z)-\psi\left(\frac{1}{\lambda}-z\right)-\log\lambda.$$

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Institute of Science, Gaziantep University, TR-27310 Gaziantep, Turkey
E-mail address: mselcukaydin@hotmail.com

Department of Mathematics, Faculty of Arts and Science, University of Gaziantep, TR-27310 Gaziantep, Turkey
E-mail address: acikgoz@gantep.edu.tr

Department of Economics, Faculty of Economics, Administrative and Social Sciences, Hasan Kalyoncu University, TR-27410 Gaziantep, Turkey
E-mail address: mtsrkn@hotmail.com; serkan.araci@hku.edu.tr