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THE PIVOT NUMBER OF A GRAPH

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Abstract. For a graph G having at least one edge, the minimum number of edges which we can remove from G such that the resulting graph has hub number larger than the hub number of G is called the pivot number $\rho(G)$ of G. The values of pivot number for several classes of graphs are computed, and we determine the pivot number of join and corona products. Also some bounds for this parameter are obtained.

Keywords: Pivot number, network, hub number, graph

MSC: 05C40, 05C69

1. Introduction

Consider $G = (V, E)$ to be a finite simple and undirected graph where $|V| = p$ and $|E| = q$ denote the order and size of G, respectively. $deg(v)$ denotes the cardinality of the set of all neighbors of $v \in V(G)$ and it is known as the degree of the vertex v in G. $\delta(G) = \min\{deg(u) : u \in V(G)\}$ and $\Delta(G) = \max\{deg(u) : u \in V(G)\}$ denote the smallest and largest vertex degrees in G.

The open neighbourhood of $v \in V(G)$ is $N(v) = \{u \in V(G) : uv \in E(G)\}$ where $N(v) \cup \{v\} = N[v]$ is the closed neighbourhood of v. For any set $U \subseteq V$, $N(U) = \bigcup_{u \in U} N(u)$. The leaf v_l in a tree T is a vertex with $deg(v_l) = 1$, sometimes called as pendant vertex. Naturally, l denotes the cardinality of leaves. For notation and definitions in graph theory not defined here, we follow [4]. For a graph G, a subset H of $V(G)$ is a hub set of G if for any $u, v \in V(G) \setminus H$, there is a $u-v$ path with all intermediate vertices are in H. The hub number $h(G)$ is the minimum cardinality among all hub sets, [16]. The theory of hubs have been developed by many researchers, [6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

For graphs G and F having V_1 and V_2 as disjoint vertex sets and E_1 and E_2 as edge sets, respectively. The join of G and F is denoted by $G+F$ and composed of $G \cup F$ together with all the edges joining V_1 with V_2, [4]. The corona product of G and F denoted by $G \circ F$ is the graph obtained from one copy of G and $|V(G)|$ copies of F, such that the j^{th} vertex of G is joined to each vertex in the j^{th} copy of F. The copy of F whose vertices are attached to the vertex v is denoted by F^v, [3].

A dominating set D of a graph G is a subset of $V(G)$ satisfying the following property: For each vertex $v \notin D$, v must be adjacent to at least one vertex in D. The smallest cardinality of a dominating set in G is called the domination number $\gamma(G)$ of G, [5]. A set X of edges, with property $\gamma(G-X) > \gamma(G)$ is called a bondage set of G. The bondage number $b(G)$ of G is the minimum cardinality of a bondage set in G, [2].

There are many applications of the domination theory. Perhaps, the most often discussed one is concerned with communications networks. A network is an arrangement establishing communication links connected to a fixed set of locations. The theory of hubs is very close to the theory of domination, and by the definition of a hub set, we can see that it may be helpful in setting up the server stations at several selected locations in a communication network, and also in selecting a minimum set of locations at which such servers are to be placed so that every two locations in the network which do not have a direct communication link are joined by a link consisting of one or more servers. In the following, we recall some helpful results:

Proposition 1.1. [16] $h(C_p) = p-3$.

Proposition 1.2. [16] *For* $G \cong T$, $h(G) = p-l$.

Proposition 1.3. [16] *Let* $G = K_{r_1,r_2,\ldots,r_w}$ *with* $w \geqslant 2$ *and* $r_i > 1$ *for some* i. *Then* $h(G)=1$ *iff some* $r_i \leqslant 2$; *otherwise* $h(G)=2$.

Proposition 1.4. [16] *Let* G *be a connected graph. Then*

$$h(G) \leqslant p - \Delta(G).$$

Proposition 1.5. [16] *For the set $L \subset V(G)$, G/L is complete if and only if L is a hub-set of G.*

Proposition 1.6. [1] *If G is a complete graph and F is a connected non-complete graph, then $h(G+F) = 1$.*

Proposition 1.7. [1] *Let G and F be two connected graphs where $|V(G)| \geqslant 2$. Then, $h(G \circ F) = |V(G)|$.*

2. Definition and some exact values

Definition 2.1. For a graph G with at least one edge, the set P of edges of G with the property that $h(G-P) > h(G)$ is called a pivot set of G. The pivot number $\rho(G)$ of G is the minimum cardinality among all pivot sets of G.

In view of the above definition, we consider graphs with at least one edge throughout this article. This concept is close to the bondage number of a graph which got high attention from researchers. Also there are various useful applications of the bondage number. Our concern in introducing the concept of pivot number is that pivot number of a graph is associated to the hub sets just as the bondage number is associated to the dominating sets. We start our study of the pivot number by determining its value for some standard graphs. First we start by computing its value for the path graph P_p.

Lemma 2.1. *The pivot number of the path P_p is given by*

$$\rho(P_p) = p - 1.$$

Proof. Since $h(P_p) = p-2$, the hub number of any graph will not exceed $p-1$. If there remains only one edge after removing some edges from P_p, then the hub number will be $p-2$, which is same as the hub number of the path, hence we must remove all edges from the path to increase the hub number of the resulting graph. Therefore, $\rho(P_p) = p-1$. □

In the next lemma, we determine the pivot number of the cycle C_p.

Lemma 2.2. *For any cycle C_p,*

$$\rho(C_p) = 1.$$

Proof. By the definition of a cycle, if we remove one edge from C_p, the resulting graph is a path P_p, since $h(C_p) = p-3$ and $h(P_p) = p-2$, we get the result. □

In the complete graph K_p $(p \geqslant 2)$, any two vertices are adjacent, so removal of one edge will make two non-adjacent vertices. Hence we can establish the pivot number of the complete graph in the next result:

Lemma 2.3. *If $G \cong K_p$ with $p \geqslant 2$, then*

$$\rho(K_p) = 1.$$

Now we will establish the pivot number of a complete $m-$partite graph $K_{k_1,k_2,\cdots,k_m}$.

Lemma 2.4. *Let $G = K_{k_1,k_2,\cdots,k_m}$ be a complete $m-$partite graph, where $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_m$, and $k_i > 1$ for some i. Then*

$$\rho(G) = \begin{cases} \lceil \frac{m}{2} \rceil, & if\ k_1 = k_2 = \cdots = k_m = 2; \\ \lceil t \rceil + \lceil \frac{\frac{t}{2}+s}{2} \rceil, & if\ k_t = 1, k_{t+1} = \cdots = k_{t+s} = 2; \\ & \quad and\ k_{t+s+1} \geqslant 3\ for\ 1 \leqslant s+t \leqslant m-1; \\ \sum\limits_{i=1}^{m-1} k_i, & otherwise. \end{cases}$$

Proof. Let $G = K_{k_1,k_2,\cdots,k_m}$ be a complete $m-$partite graph where $p = \sum\limits_{i=1}^{m} k_i$, and $V(G) = V_{k_1} \cup V_{k_2} \cup \cdots \cup V_{k_m}$. We now consider three possibilities depending on the order of partite sets.

Case 1: Let $k_1 = k_2 = \cdots = k_m = 2$. By Theorem 1.3, $h(G) = 1$, let F be a spanning subgraph of G obtained by removing less than $\lceil \frac{m}{2} \rceil$ edges (may be, there are no such edges) from G, then F contains a partite set of two vertices, say v_1, v_2, such that v_1 and v_2 are of degree $p-2$. Note that v_1 and v_2 are adjacent to all vertices in all partite sets apart from the partite set to which they belong. So each of the sets $\{v_1\}$ and $\{v_2\}$ forms a hub set for G, and accordingly, $h(G) = 1$. Therefore, $\rho(G) \geqslant \lceil \frac{m}{2} \rceil$. Hence, if we have a vertex of degree $p-3$ in each partite set in the spanning subgraph F, then $h(F) \geqslant h(G)$.

Subcase 1.1: Let m be even. Removal of $\frac{m}{2}$ independent edges between distinct pairs of partite sets of G will reduce the degree of one vertex in each partite set to $p-3$, which produces a graph F with $h(F) = 2$.

Subcase 1.2: Let m be odd. Removal of $\lfloor \frac{m}{2} \rfloor$ independent edges between distinct pairs of partite sets of G will reduce the degree of one vertex in each of $m-1$ partite sets to $p-3$, and leaves one partite set with vertices of degree $p-2$. So removing one edge incident with one of the vertices in this partite set yields a graph F with $h(F)=2$.

Hence if m is even or odd, the resulting graph F obtained by deleting $\lceil \frac{m}{2} \rceil$ edges from G has $h(F)=2$. Therefore, $\rho(G)=\lceil \frac{m}{2} \rceil$.

Case 2: Let $k_t=1$ and $k_{t+1}=\cdots=k_{t+s}=2$ and $k_{t+s+1}\geqslant 3$. Then by Theorem 1.3, $h(G)=1$ and any partite set consisting of only one vertex forms a hub set for G. Note that, removing one edge between any two distinct partite sets each having only one vertex will reduce the degree of the two vertices in these two partite sets by one and this will establish a new partite set of two vertices. The number of desired partite sets is $\lceil \frac{t}{2} \rceil$. Consequently, we have now $\lceil \frac{\frac{t}{2}+s}{2} \rceil$ partite sets each of order two. Furthermore, each vertex of these partite sets is of degree $p-2$ and each one of these vertices forms a hub set for G. Using the same argument as in the previous case, we prove the assertion.

Case 3: $k_1 \geqslant 3$. By Theorem 1.3, $h(G)=2$. Let $f=\sum_{i=1}^{m-1} k_i$. Let P be a pivot set for G. Then every vertex of each partite set $V_{k_1},\cdots,V_{k_{m-1}}$ and at least one vertex of the partite set V_{k_m} must be incident with at least one edge of P. For if $v\in V_{k_i}, 1\leqslant i\leqslant m-1$, v is not incident with any edge of P. Then v along with $u\in V_{k_m}$ will form a hub set for G. So $h(G-P)=2$ which is a contradiction to our assumption. Thus, every vertex in each partite set $V_{k_1},\cdots,V_{k_{m-1}}$ and at least one vertex of the partite set V_{k_m} must be incident with at least one edge of P.

On the other hand, let $u,u_1,\cdots,u_n$ be the vertices of the partite set V_{k_m} such that u_1 is not incident with any edge of P. Then u_1 is adjacent in $G-P$ to each vertex except $u,u_2,\cdots,u_n$. Now let $w\in (V(G)\setminus V_{k_m})$ such that w is neighbor to all vertices $u,u_1,\cdots,u_n$. Hence $\{u_1,w\}$ is a hub set for $G-P$. Since $h(G)<h(G-P)$, this is a contradiction. Therefore, every vertex which is not in V_{k_m} must be nonadjacent in $G-P$ with at least one of $u,u_1,\cdots,u_n$. So it follows that $|P|\geqslant f$. Since $k_i\leqslant k_m$ for each $i\leqslant m$, deleting all edges incident with $u\in V_{k_m}$ will make u an isolated vertex. This implies that the hub number of this graph is equal to 3. Thus, $\rho(G)=f$. □

Theorem 2.1. *If $G \cong T$ with $l \geqslant 3$, then $\rho(G) = 1$.*

Proof. Let s be the number of internal vertices of T and $l \geqslant 3$. Then there is $v \in V(T)$ with $deg(v) \geqslant 3$. Since $l \geqslant 3$, either two leaves are adjacent to the same internal vertex and the degree of this vertex will be greater than or equal to 3 or every leaf is adjacent to a different internal vertex. So there exists an internal vertex which is adjacent to at least two internal vertices and one leaf or it is neighbor to at least three internal vertices. It is clear by Theorem 1.2 that the hub number of this tree is s. If we remove one edge from T which is incident to v where u is the other vertex incident with this edge, the following cases are possible:

Case 1: Let u be a leaf in the resulting graph. As every edge in a tree is a bridge, the resulting graph is disconnected and consists of two trees T_1, T_2. Let $v \in T_1$ and $u \in T_2$. Since $deg(v) \geqslant 3$, v is still an internal vertex. As u was an internal vertex and became a leaf in the resulting graph, we have $s-1$ internal vertices which must be contained in any minimum hub set of $T_1 \cup T_2$. Since this graph is disconnected, one of the sets $V(T_1), V(T_2)$ should be contained in any minimum hub set. But T_1 and T_2 are trees, so every one has at least two leaves. Therefore, in addition to $s-1$ internal vertices, the minimum hub set contains at least two leaves. So $h(T_1 \cup T_2) \geqslant s+1 > h(T)$.

Case 2: Let u be an internal vertex in the resulting graph. Then no change in the number of internal vertices occurs. So by Case 1, we get $h(T_1 \cup T_2) \geqslant s+2 > h(T)$.

Case 3: Let u be an isolated vertex in the resulting graph. So u is a leaf in T. Hence there is no change in the number of internal vertices, and the hub number of the resulting graph is $s+1$.

From all the above cases, we get that the removal of such an edge will increase the hub number. This assertion obviously implies the theorem. □

The following corollaries are obvious:

Corollary 2.1. *For the star $K_{1,n-1}$ of order $p \geqslant 4$,*

$$\rho(K_{1,n-1}) = 1.$$

Corollary 2.2. *For the double star $S_{n,m}$ of order $p \geqslant 5$,*

$$\rho(S_{n,m}) = 1.$$

3. Some bounds on pivot number

In this section, we obtain some bounds on the pivot number. Our first result is obvious:

Observation 3.1. *For a (p,q) graph G, $1 \leqslant \rho(G) \leqslant q$.*

The next theorem relates the pivot number of G to both $h(G)$ and order of G, and the upper bound on pivot number established in this result is the main bound in our paper:

Theorem 3.1. *For any graph G with order p,*

$$\rho(G) \leqslant \binom{p}{h(G)}(h(G)+1).$$

Proof. Let the hub set H of G be the minimum among all hub sets. Then it is not necessary that H is a unique minimum hub set for G. So for some graphs, we can find many minimum hub sets, say $H_1, H_2, \cdots, H_s$ where $|H_i| = h(G)$ and $H_i \cap H_j$ need not be empty for every $1 \leqslant i, j \leqslant s$. Since the vertices in each hub set are chosen from $V(G)$ which has p vertices, then the maximum number of minimum hub sets of G is $\binom{p}{h(G)}$. Choose an arbitrary minimum hub set H_k of G. By Theorem 1.5, the graph G/H_k should be a complete graph. Suppose $v \in V(G) \setminus H_k$ and assume that $v \in N(H_k)$. If we delete all edges which are connecting v with the vertices of H_k, then we have $v \in V(G) \setminus N[H_k]$ in the resulting graph. Note that if H_k is still a hub set of the resulting graph, it suffices to delete one edge between v and any other vertex $w \in V(G) \setminus H_k$. We denote the set of these desired edges by E_h. Since $|H_k| = h(G)$, there is $h(G)$ neighbors of v in H_k. This implies that the removal of $h(G)+1$ desired edges leaves the resulting graph $(G - E_h)/H_k$ having at least one vertex v with $deg(v) < p - |H_k| - 1$. So $(G - E_h)/H_k$ is not a complete graph and H_k is not a hub set for the resulting graph $(G - E_h)$. Note that we have at most $\binom{p}{h(G)}$ different minimum hub sets for G. Hence if we repeat the previous procedure for each one, we conclude that the resulting graph F after removal of at most $\binom{p}{h(G)}(h(G)+1)$ edges from G has $h(F) > h(G)$. □

We observe that the previous bound is sharp for the complete graph. By Theorem 1.4 and the previous theorem, we easily compute the following bound:

Corollary 3.1. *For any connected graph G,*

$$\rho(G) \leqslant \binom{p}{h(G)}(p - \Delta(G) + 1).$$

4. Some results on pivot number of join and corona products

Graph products or graph operations are very useful tools in enumerative combinatorics as they help us to calculate some property of a large graph in terms of the same properties of some smaller graphs. They are widely used in all aspects related to graph parameters. In this section, to illustrate this with two examples for the pivot number, we calculate the pivot number of two graph products, join and corona. The similar results can be obtained for other graph products. We start by the following result:

Lemma 4.1. *If G and F are two complete graphs, then*

$$\rho(G + F) = 1.$$

Proof. Since G and F are both complete, $G + F$ is also complete, so by Lemma 2.1, the assertion follows. □

Lemma 4.2. *For two graphs G and F with $|V(G)| = 1$ and F is a complete graph,*

$$\rho(G \circ F) = 1.$$

Proof. The condition on G and F implies that $G \circ F$ is also a complete graph. So by Lemma 2.1, we have $\rho(G \circ F) = 1$. □

Lemma 4.3. *Let $G \cong K_{p_1}$ with $p_1 \geqslant 3$ and F be a graph with order $p_2 = 1$. Then*

$$\rho(G \circ F) = p_1 - 1.$$

Proof. $h(G \circ F) = p_1$ by Theorem 1.7. Since G is a complete graph, removing $p_1 - 1$ edges incident with one vertex from G say v, leaves the resulting graph disconnected. Let E_1 denote the set of desired edges. So by definition of the corona product of two graphs, the graph $(G \circ F) - E_1$ is disconnected and consists of two components L_1, L_2

where $L_1 \cong K_2$ and $L_2 \cong ((G-v) \circ F)$. According to the order of G, the component L_2 is not a complete graph and $h(L_2) = p_1 - 1$. Therefore $h((G \circ F) - E_1) = 2 + p_1 - 1 = p_1 + 1 > h(G \circ F)$ which completes the proof. □

We now deal with the other possibility of corona product of two graphs with conditions different from the conditions in the preceding proposition:

Theorem 4.1. *Let G and F be two graphs such that $p_1, p_2 \geqslant 2$ where p_1 is the order of a connected non-complete graph G. Then*

$$\rho(G \circ F) = 1.$$

Proof. Theorem 1.7 implies that $h(G \circ F) = p_1$. So $H = V(G)$ is the minimum hub-set of $G \circ F$. Let $v, w \in V(G)$ and $u_1, u_2 \in F^v$. By the definition of the corona product, v is in every $u_1 - z$ path and $u_2 - z$ path where $z \in F^w$. Hence if we remove the edge vu_1, then there is no $H-$path between u_1 and z in the graph $((G \circ F) - vu_1)$. Consequently, either u_1 should be in the minimum hub set of the resulting graph and so $h((G \circ F) - vu_1) = p_1 + 1$ or u_1 must be replaced by v in H so that there is no path between u_2 and z in the resulting graph. This reveals that all vertices of F^v should be included in any minimum hub set of the resulting graph. Therefore, in both cases, $h(((G \circ F) - vu_1)) > h(G \circ F)$. This concludes the proof. □

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