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On r-Dynamic coloring on Mycielskian graphs

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Abstract

Mycielski graphs are based on transformation from a graph G to a new graph, say $\mu(G)$. It is well known that these graphs have same clique number as in the graph G, but the chromatic number is defined as $\chi(\mu(G)) = \chi(G)+1$. The Mycielskian graph G is denoted as $\mu(G)$. These Mycielski graphs are colored using r- dynamic coloring which is an proper vertex k-coloring and defined as $|c(Neigh(v))| \geq \min\ \{r, deg_G(v)\}$, for each $v \in V(G)$. It is denoted by $\chi_r(G)$. The r-dynamic chromatic number of a graph G is the minutest coloring k of G which is r-dynamic k-colorable. In this paper, the authors investigated the r-coloring of Mycielskian graph of Double Fan graph $\mu(F_{2,n})$, Friendship graph $\mu(F_n)$, Pan graph $\mu(P_n)$ and Cocktail party graph $\mu(CP_n)$.

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1 Introduction

In this work, all the graphs are considered to be connected, loop less, undirected and graphs without multiple edges. Graph coloring is widely applicable in various fields and researchs. Here, the coloring means the vertex coloring of a graph $G = (V, E)$ with vertices and edges. The k-coloring of G is defined by a map c from the vertex set $V(G)$ to the set of colors $\{1, 2, \cdots, k\}$. In this work, we investigated the r-dynamic coloring of different graphs that are erected from Mycielskian graphs.

The concept r-dynamic coloring was first introduced by Bruce Montgomery [11]. It is an proper vertex coloring such that no two vertices receives similar colors. A r-dynamic coloring is defined as a mapping $c : V(G) \to k$ which has to satisfy two condition. The first one is no two adjacent vertices should receives same colors i.e., $c(v) \neq c(w)$ and the second condition is $|c(Neigh(v))| \geq \min\ \{r, deg_G(v)\}$, the coloring of the neighborhood vertex $Neigh(v)$ receives the minimal coloring from the r-values and the degree of the vertex v which is denoted as $deg_G(v)$. Thus, the r-dynamic coloring is the minimum k-coloring of the graph G which is denoted as $\chi_r(G)$. When $r = 1$, the results of 1-dynamic chromatic number is same as the chromatic number of the graph G and in the case of $r = 2$, the results of 2-dynamic chromatic number is similar to the dynamic chromatic number. The one of the most famous lower bound for the r-dynamic coloring are

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Lemma 1. $\chi_r(G) \geq min\{r, \Delta(G)\} + 1$

The bounds for the r-dynamic chromatic number of different graphs and their exact values are explained in the following papers [3], [7], [8], [11], [14].

In order to construct a triangle-free graphs with large chromatic number and small clique number, Mycielski[12] introduced a graph transformation which transforms the graph G to a new graph $\mu(G)$ which can call as Mycielskian graph $\mu(G)$. It has same clique number as in G and the chromatic number will be $\chi(G)+1$. Apart from the properties of clique number and the chromatic number, Mycielskian graph has some other parameters which can be predicted. Larsen et.al [9] shows from the fractional chromatic number $\chi_f(\mu(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}$ for any graph G. Gerard J. Chang[4] proves that for any graph G, $\chi_c(\mu^2(G)) \leq \chi(\mu^2(G)) - \frac{1}{d}$. Fisher[6] used Mycielskian graph as an example for optimal fractional coloring which has large denominators. Also, many results of Mycielskian graphs are given in [4], [2].

2 Preliminaries

[12] **Mycielskian graph** is extracted from the given graph G. The vertex set of Mycielskian graph is $V(\mu(G)) = V \cup V' \cup \{x\}$ where $V' = \{u' : u \ \epsilon\ V\}$ and the edge set are $E(\mu(G)) = E \cup \{uv' : uv \ \epsilon\ E\} \cup \{v'x : v' \ \epsilon\ V'\}$. The vertex u' is called the *twin* of the vertex u (also u is the *twin* of u') and the vertex x is the *root* of $\mu(G)$.
[1] **Friendship graph** F_n which obtained from the n copies of cycle C_3 and these n copies are joined to a common vertex u. It has $2n+1$ vertices and $3n$ edges.
[10] **Double Fan graph** $F_{2,n}$ is the graph join of $\overline{K_m}$ and path P_n. we considered $m = 2$. It can also stated as $P_n + 2K_1$.
[13] **Pan graph** is an graph obtained by adding a singleton graph to any one of the vertex of cycle graph C_n and it is denoted as (P_n).
[5] **Cocktail Party graph** consists of two rows of paired vertices, where the vertices in each rows are complete. The edges are joined between every vertices except the paired one. It can be denoted as (CP_n).

In this work we investigated the r-dynamic coloring of Mycielskian graphs and obtained the exact results of some special graphs such as $\mu(F_{2,n})$, $\mu(F_n)$, $\mu(P_n)$ and $\mu(CP_n)$.

3 r-dynamic coloring of Mycielskian graph of Double Fan graph

Theorem 1. *For $m = 2$, $n \geq 4$, the r- dynamic coloring of Mycielskian graph of Double Fan graph $\mu(F_{2,n})$ are*

$$\chi_r[\mu(F_{2,n})] = \begin{cases} 4, & \textit{for } 1 \le r \le 2 \\ 5, & \textit{for } r = 3 \\ 6, & \textit{for } r = 4,\ n \equiv 1,3(\text{mod}4) \\ 7, & \textit{for } r = 4,\ n \not\equiv 1,3(\text{mod}4) \\ r+3, & \textit{for } r = 5 \\ r+4, & \begin{cases} \textit{for } r = 6,\ n = 5 \\ \textit{for } 6 \le r \le 7, n \ne 5 \end{cases} \\ r+5, & \begin{cases} \textit{for } 6 \le r \le \Delta,\ n = 4 \\ \textit{for } 7 \le r \le \Delta, n = 5 \\ \textit{for } 8 \le r \le \Delta, n \ne 4,5 \end{cases} \end{cases}$$

Proof : Let $V[\mu(F_{2,n})] = \{z\} \cup \{p_i, p'_i, q_j, q'_j : i\epsilon[1,n], j\epsilon[1,2]\}$, where p_i are the vertices of path graph and q_j are the vertices of complement of complete graph. The vertices z, q'_j, p'_i are added by using the operation Mycielski. The degrees of $\mu(F_{2,n})$ are $\delta = 4$ and $\Delta = 2n$. Then, $|V[\mu(F_{2,n})]| = 2n+5$ and $|E[\mu(F_{2,n})]| = \dfrac{19n + (n-2)}{2}$. Then, the r-dynamic coloring are as follows;

Case : 1 $1 \le r \le 2$

Based on the lemma the lower bound are $\chi_r[\mu(F_{2,n})] \ge 4$. To find the upper bound $\chi_r[\mu(F_{2,n})] \le 4$, define a bijection $c_1 : V[\mu(F_{2,n})] \to \{1, 2, 3, \cdots, |V[\mu(F_{2,n})]|\}$. Consider, $c_1(p_i)$ and $c_1(p'_i) = 1, 2$ for $1 \le i \le n$. Next, $c_1(q_j)$ and $c_1(q'_j) = 3$ for $1 \le j \le 2$.
Since, the vertex z is adjacent to p'_i and q'_j for $1 \le i \le n$ and $1 \le j \le 2$, we must include a new color. So, $c_1(z) = 4$. Hence, $\chi_r[\mu(F_{2,n})] \le 4$. Based on the lemma, it is clear that $\chi_r[\mu(F_{2,n})] = 4$.

Case : 2 $r = 3$

Based on the lemma the lower bound are $\chi_r[\mu(F_{2,n})] \ge 5$. To find the upper bound $\chi_r[\mu(F_{2,n})] \le 5$, consider a map $c_2 : V[\mu(F_{2,n})] \to \{1, 2, 3, \cdots, |V[\mu(F_{2,n})]|\}$ such that
$c_2(p_i)$and $c_2(p'_i) = 1, 2, 3$ for $1 \le i \le n$. Then, $c_2(q_j) = 4, 5$ for $1 \le j \le 2$.
Next, color the vertices q'_1 and q'_2 with color 4. So the vertices p'_i and q'_j has $\{1, 2, 3, 4\}$ colors for $1 \le i \le n$ and $1 \le j \le 2$. Thus, $c_2(z) = 5$, which implies $\chi_r[\mu(F_{2,n})] \le 5$. Thence, $\chi_r[\mu(F_{2,n})] = 5$.

Case : 3 $r = 4$

From the lemma the lower bound are given as $\chi_r[\mu(F_{2,n})] \ge 6$. To get the exact values of r-dynamic coloring of $\mu(F_{2,n})$ we need to calculate the upper bound as $\chi_r[\mu(F_{2,n})] \le 6$. So, define a map $c_3 : V[\mu(F_{2,n})] \to \{1, 2, 3, \cdots, k\}$. Here, the results are followed from two subcases.

- When $n \equiv 1, 3(mod\ 4)$, color the vertices $c_3(p_i) = 1, 2, 3, 4$ for $1 \le i \le n$, so that the vertex p_n has either color 1 or color 2. Next, $c_3(q_j)$ and $c_3(q'_j) = 5, 6$ for $1 \le j \le 2$. Then, $c_3(p'_i) = 4, 1, 2$ for $1 \le i \le n$ according to r-adjacency condition.

Finally, the last vertex are colored as $c_3(z) = 3$, so that vertex z also satisfy the 4-adjacency condition. Thence, $\chi_r[\mu(F_{2,n})] \leq 6$. Therefore, $\chi_r[\mu(F_{2,n})] = 6$.

- When $n \not\equiv 1, 3(mod\ 4)$, we need one more color to satisfy the lower bound, so $\chi_r[\mu(F_{2,n})] \geq 7$. Thus, the r-coloring are as follows:
  Color the vertices p_i, q_j and q'_j as given in $n \equiv 1, 3(mod\ 4)$. Next, $c_3(p'_i) = 4, 3, 1, 2$ for $1 \leq i \leq n$ in order to 4-adjacency condition. Finally, the vertex z need one more color to satisfy the condition. Thus, $c_3(z) = 7$. Thence, $\chi_r[\mu(F_{2,n})] \leq 7$. Therefore, it is easy to check that $\chi_r[\mu(F_{2,n})] = 7$.

Case : 4 $r = 5$
Observing the lemma the lower bound are $\chi_r[\mu(F_{2,n})] \geq r + 3$. To find the upper bound $\chi_r[\mu(F_{2,n})] \leq r + 4$, we define a map $c_4 : V[\mu(F_{2,n})] \rightarrow \{1, 2, 3, \cdots, r + 3\}$ such that
$c_4(p_i) = \{1, 2, 3, 4\}$ for $1 \leq i \leq n$ and $c_4(q_1)$ and $c_4(q_2) = 5, 6$.
Next, $c_4(q'_j) = \{r+1, r+2\}$ for $1 \leq j \leq 2$. Then, $c_4(q'_2) = c_4(p'_1)$ and color the remaining vertices of p'_i with the colors from the set $\{1, 2, 3, 4\}$. But, the color of the vertices p'_{n-1} and p'_n depends on the color of the vertices p_{n-1} and p_n, so that the vertices p'_{n-1} and p'_n get any of the colors from the set $\{1, 2, 3, 4\}$. At the end, $c_4(z) = r+3$. Therefore, $\chi_r[\mu(F_{2,n})] \leq r+3$. Hence, $\chi_r[\mu(F_{2,n})] = r + 3$.

Case : 5 $r + 4$-coloring
Observing from the lemma the lower bound are $\chi_r[\mu(F_{2,n})] \geq r + 4$. To find the upper bound $\chi_r[\mu(F_{2,n})] \leq r + 4$, we define a map $c_5 : V[\mu(F_{2,n})] \rightarrow \{1, 2, 3, \cdots, r + 4\}$. The $r + 4$-coloring are given in the following two cases;

- When $r = 6$ and $n = 5$, $c_5(p_i) = \{1, 2, \cdots, 5\}$ for $1 \leq i \leq n$.
  $c_5(q_j) = 6, 7$ and $c_5(q'_j) = 8, 9$ for $1 \leq j \leq 2$.
  Next, $c_5(q'_2) = c_5(p'_1)$ and the remaining vertices of p'_i are colored from the set $\{1, 2, 3, 4\}$. Atlast, $c_5(z) = r + 4$, since the vertices p'_i and q'_j are with colors from the set $\{1, 2, \cdots, r + 3\}$. Thus, $\chi_r[\mu(F_{2,n})] \leq r + 4$. Therefore, $\chi_r[\mu(F_{2,n})] = r + 4$.
- When $6 \leq r \leq 7$ and $n \neq 5$, $c_5(p_i) = \{1, 2, \cdots, r - 1\}$ for $1 \leq i \leq n$. Next, $c_5(q_j) = r, r + 1$ and $c_5(q'_j) = r + 2, r + 3$ for $1 \leq j \leq 2$.
  Then, $c_5(p'_1) = r+3$ and the leftover vertices of p'_i are with colors from the set $\{1, 2, \cdots, r-1\}$ with r-adjacency condition. Thus, $\chi_r[\mu(F_{2,n})] \leq r+4$. Therefore, $\chi_r[\mu(F_{2,n})] = r + 4$.

Case : 6 $r + 5$-coloring
From the lemma the lower bound are $\chi_r[\mu(F_{2,n})] \geq r + 5$. To find the upper bound $\chi_r[\mu(F_{2,n})] \leq r+5$, we define a map $c_6 : V[\mu(F_{2,n})] \rightarrow \{1, 2, 3, \cdots, r+5\}$. The $r + 5$-coloring are consider in the following cases:

- When $6 \leq r \leq \Delta$ and $t = 4$, $c_6(p_i) = \{1, 2, \cdots, n\}$ for $1 \leq i \leq n$. Then, $c_6(q_j) = n+1, n+2$ for $1 \leq j \leq 2$ and $c_6(q'_j) = n+3, n+4$ for $1 \leq j \leq 2$.

Next, $c_6(p'_i) = \{n+5, n+6, \cdots, r+4, 1, 2\}$ in accordance with the r-adjacency condition. Atlast, $c_6(z) = r+5$. i.e., $\chi_r[\mu(F_{2,n})] \leq r+5$. Therefore, $\chi_r[\mu(F_{2,n})] = r+5$, for $6 \leq r \leq \Delta$ and $t = 4$.

- When $7 \leq r \leq \Delta$ and $n = 5$, color the vertices p_i, q_j and q'_j as given above for $1 \leq i \leq n$ and $1 \leq j \leq 2$. At $r = 7$, $c_6(p'_1)$ and $c_6(p'_2) = n+5, n+6$. Then, the leftover vertices of p'_i are with colors from the set $\{1, 2, \cdots, n\}$ for $3 \leq i \leq n$ with r-adjacency condition and $c_6(z) = r+5$.
  At $r = 8$, $c_6(p'_i) = \{n+5, n+6, n+7\}$ for $1 \leq i \leq 3$ and the remaining vertices of p'_i are with any of the colors from $\{1, 2, \cdots, n\}$ for $4 \leq i \leq n$ and $c_6(z) = r+5$. Similarly, when $r = \Delta$, $c_6(p'_i) = \{n+5, n+6, \cdots, r+4\}$ for $1 \leq i \leq n$ and $c_6(z) = r+5$. Thus, $\chi_r[\mu(F_{2,n})] \leq r+5$. Therefore, $\chi_r[\mu(F_{2,n})] = r+5$, for $7 \leq r \leq \Delta$ and $n = 5$.
- When $8 \leq r \leq \Delta$ and $n \neq 4,\ 5$
  1. For, $6 \leq n \leq 7$, we color the vertices p_i, q_j and q'_j as given above for $1 \leq i \leq n$ and $1 \leq j \leq 2$.
     - At $n = 6$ and $r = 8$, $c_6(p'_i) = \{r+3, r+4\}$ for $1 \leq i \leq 2$ and the remaining vertices of p'_i are with colors from the set $\{1, 2, \cdots, n\}$ for $3 \leq i \leq n$ and then, $c_6(z) = r+5$. At $r = 9$, we use $c_6(p'_i) = \{n+5, n+6, r+4\}$ for $1 \leq i \leq 3$ and the other vertices of p'_i are with colors from the set $\{1, 2, \cdots, n\}$ for $4 \leq i \leq n$. Then, $c_6(z) = r+5$. Continuing this process, $c_6(p'_i) = \{n+5, n+6, \cdots, r+4\}$ for $1 \leq i \leq n$ and $c_6(z) = r+5$. at $r = \Delta$.
     - At $n = 7$ and $r = 8$, $c_6(p'_i) = \{r+4, n-1, n, 1, 2, \cdots, 4\}$ for $1 \leq i \leq n$ and $c_6(z) = r+5$. At $r = 9$, $c_6(p'_i) = \{n+5, n+6, n, 1, 2, \cdots, r+4\}$ for $1 \leq i \leq n$ and $c_6(z) = r+5$. Continuing by this way, $c_6(p'_i) = \{n+5, n+6, \cdots, r+4\}$ for $1 \leq i \leq n$. Finally, we need one more color to satisfy the our condition, so $c_6(z) = r+5$. Thus, $\chi_r[\mu(F_{2,n})] \leq r+5$. Therefore, $\chi_r[\mu(F_{2,n})] = r+5$, for $6 \leq n \leq 7$ and $8 \leq r \leq \Delta$.
  2. For, $n \geq 8$, consider the following two cases
     - When $8 \leq r \leq n$, $c_6(p_i) = \{1, 2, \cdots, r-1\}$ for $1 \leq i \leq r-1$ and the remaining vertices of p_i may starts again with colors from the set $\{1, 2, \cdots, r-1\}$.
       Then, $c_6(q_j) = r,\ r+1$ for $1 \leq j \leq 2$ and $c_6(q'_j) = r+2,\ r+3$ for $1 \leq j \leq 2$. Next, $c_6(p'_1) = r+4$ and the leftover vertices of p'_i are with colors from the set $\{1, 2, \cdots, r-1\}$ in order to the r-adjacency condition. At the end, $c_6(z) = r+5$. We may continue this process till the r-value reaches n. Thus, $\chi_r[\mu(F_{2,n})] \leq r+5$. Therefore, $\chi_r[\mu(F_{2,n})] = r+5$, for $8 \leq r \leq n$ and $n \geq 8$.
     - When $n+1 \leq r \leq \Delta$, $c_6(p_i) = \{1, 2, \cdots, n\}$ for $1 \leq i \leq n$. Next, $c_6(q_j) = n+1,\ n+2$ for $1 \leq j \leq 2$ and $c_6(q'_j) = n+3,\ n+4$ for $1 \leq j \leq 2$.
       At $r = n+1$, $c_6(p'_1) = n+5$ and the other vertices of p'_i are with colors from the set $\{1, 2, \cdots, n\}$ for $2 \leq i \leq n$. Then,

$c_6(z) = r + 5$.
At $r = n+2$, only the vertices p'_i receives different colors whereas the other vertices z, p_i, p'_i, q_j, q'_j has the same color as mentioned for $r = n + 1$. So, $c_6(p'_1)$ and $c_6(p'_2) = n + 5, n + 6$ and the other vertices are colored from the set $\{1, 2, \cdots, t\}$.
By continuing this process at $r = \Delta$, $c_6(p'_i) = \{n+5, n+6, \cdots, r+4\}$ for $1 \leq i \leq n$. Thus, $\chi_r[\mu(F_{2,n})] \leq r + 5$. Therefore, $\chi_r[\mu(F_{2,n})] = r + 5$, for $n + 1 \leq r \leq \Delta$ and $n \geq 8$.

4 r-dynamic coloring of Mycielskian graph of Friendship graph

Theorem 2. *For $n \geq 2$, the r- dynamic coloring of Mycielskian graph of Friendship graph $\mu(F_n)$ are*

$$\chi_r[\mu(F_n)] = \begin{cases} 4, & \text{for } 1 \leq r \leq 2 \\ 5, & \text{for } r = 3 \\ r + 3, & \text{for } 4 \leq r \leq \Delta \end{cases}$$

Proof : Let $V[\mu(F_n)] = \{u, u', w\} \cup \{u_i, u'_i : i\epsilon[1, 2n]\}$
$E[\mu(F_n)] = \{uu_i : i\epsilon[1, 2n]\} \cup \{uu'_i : i\epsilon[1, 2n]\} \cup \{u'_i u_{i+1} : i\epsilon[1, 2n],\ i \text{ is odd}\} \cup \{u'_{i+1} u_i : i\epsilon[1, 2n],\ i \text{ is odd}\} \cup \{u' u_i : i\epsilon[1, 2n]\} \cup \{wu', wu'_i : i\epsilon[1, 2n]\}$. The minimum and maximum degree are $\delta = 3$ and $\Delta = 4n$. The order of the $\mu(F_n)$ are $|V[\mu(F_n)]| = 4n+3$. The r-dynamic coloring of Mycielskian graph of Friendship graph are described in the below cases:

Case : 1 $1 \leq r \leq 2$

From the lemma, lower bound are $\chi_r[\mu(F_n)] \geq 4$. To find the upper bound $\chi_r[\mu(F_n)] \leq 4$, define a map $c_1 : V[\mu(F_n)] \to \{1, 2, 3, \cdots, |V[\mu(F_n)]|\}$ such that
$c_1(u_i)$ and $c_1(u'_i) = \{1, 2\}$ for $1 \leq i \leq 2n$ and the remaining vertices $c_1(u)$ and $c_1(u') = 3$. Atlast the leftover vertex w are colored with a new color in order to satisfy the 1, 2-adjacency condition so $c_1(w) = 4$. Thus, $\chi_r[\mu(F_n)] \leq 4$. Thence, $\chi_r[\mu(F_n)] = 4$.

Case : 2 $r = 3$

Define a map $c_2 : V[\mu(F_n)] \to \{1, 2, 3, \cdots, 5\}$ such that $c_2(u_i) = \{1, 2\}$ for $1 \leq i \leq 2n - 1$ and the vertex $u_{2n} = 3$. Next, the remaining vertices are colored as $c_2(u'_i) = 3$ for $1 \leq i \leq 2n - 2$ and the leftover vertices $c_2(u'_{2n-1})$ and $c_2(u'_{2n}) = 2$. Then, the other vertices are colored as $c_2(u)$ and $c_2(u') = 4$ and the last vertex are colored as $c_2(w) = 5$ since, in order to satisfy the 3-adjacency condition. Thus, $\chi_r[\mu(F_n)] \leq 5$. Thence, it is easy check that $\chi_r[\mu(F_n)] = 5$.

Case : 3 $4 \leq r \leq \Delta$

Consider a function $c_3 : V[\mu(F_n)] \to \{1, 2, 3, \cdots, r + 3\}$ such that
color the vertices as $c_3(u) = r + 1$ and $c_3(u') = r + 2$ and the other vertex as $c_3(w) = r + 3$.

- When $4 \leq r \leq 2n$, follow the above coloring for the vertices of $\mu(F_n)$ except u_i and u'_i. So, $c_3(u_i) = \{1, 2, \cdots, r\}$ for $1 \leq i \leq 2n$. From the coloring of the vertices u_i, the vertices u'_i are colored from the set $\{1, 2, \cdots, r\}$ with respect to r-adjacency condition. We continue the same process, upto $r = 2n$, but with not affecting our r-adjacency condition.
- When $2n + 1 \leq r \leq \Delta$, the vertices u, u' and w are colored as given above depending on the r-value. The vertices u_i are colored as $c_3(u_i) = \{1, 2, \cdots, 2n\}$ for $1 \leq i \leq 2n$. Only the coloring of the vertices u'_i may vary for all the r-values. At $r = 2n + 1$, the vertices u'_i are colored as $c_3(u'_i) = \{2n + 1, 4, 5, \cdots, 2n, 1, 2, \cdots, r\}$. Then, when $r = 2n + 2$, $c_3(u'_i) = \{2n + 1, 2n + 2, 5, \cdots, 2n, 1, 2, \cdots, r\}$. Thus, by continuing this process at $r \geq \Delta$, the vertices $c_3(u'_i) = \{2n + 1, 2n + 2, \cdots, r\}$. Hence, $\chi_r[\mu(F_n)] \leq r + 3$. Therefore, it is easy check that $\chi_r[\mu(F_n)] = r + 3$.

5 r-dynamic coloring of Mycielskian graph of Pan graph

Theorem 3. *For $n \geq 6$, the r- dynamic coloring of Mycielskian graph of pan graph $\mu(P_n)$ are*

$$\chi_r[\mu(P_n)] = \begin{cases} 3, & \text{for } r = 1,\ n \text{ is even} \\ 4, & \text{for } r = 1,\ n \text{ is odd} \\ 4, & \text{for } r = 2,\ n \equiv 2 (\text{mod } 3) \\ 5, & \text{for } r = 2, n \not\equiv 2 (\text{mod } 3) \\ 5, & \text{for } r = 3,\ n \equiv 0 (\text{mod } 3) \\ 6, & \text{for } r = 3,\ n \not\equiv 0 (\text{mod } 3) \\ r + 3, & \text{for } 4 \leq r \leq 5,\ n \not\equiv 0 (\text{mod } 4),\ n \equiv 0 (\text{mod } 3, 4) \\ r + 4, & \text{for } 4 \leq r \leq 5,\ n \equiv 0 (\text{mod } 4) \\ r + 3, & \text{for } 6 \leq r \leq \Delta,\ n \equiv 1 (\text{mod } 3) \\ r + 4, & \text{for } 6 \leq r \leq \Delta,\ n \not\equiv 1 (\text{mod } 3) \end{cases}$$

Proof : Let $V[\mu(P_n)] = \{p_i, p'_i, q, q', s : i\epsilon[1, n]\}$. The vertices p_i forms an cycle of order n and the vertex q are adjacent to p_1. Here, $Neigh(p'_i) = Neigh(p_i)$, where $Neigh$ is neighborhood of the vertex. $E[\mu(P_n)] = \{p_i p'_{i+1} : i\epsilon[1, n-1]\} \cup \{p_n p_1, p_1 q, p_1 q'\} \cup \{p'_i p_{i-1}, p_i p'_{i-1} : i\epsilon[2, n]\} \cup \{p'_1 p_n, p_1 p'_n\} \cup \{sq', sp'_i : i\epsilon[1, n]\}$.
The order of the graph $|V[\mu(P_n)]| = 2n + 3$ and the size is $|E[\mu(P_n)]| = \dfrac{8(n+1)}{2}$.
The degrees are $\delta(\mu(P_n)) = 2$ and $\Delta(\mu(P_n)) = n + 1$. The r-dynamic coloring of $\mu(P_n)$ are considered in the following cases:

Case : 1 $r = 1$

From the lemma, lower bound are $\chi_r[\mu(P_n)] \geq 3$. To find the upper bound $\chi_r[\mu(P_n)] \leq 3$, define a map $c_1 : V[\mu(P_n)] \to \{1, 2, 3, \cdots, |V[\mu(P_n)]|\}$.
When n is even,
$c_1(p_i)$ and $c_1(p'_i) = \{2, 1\}$ for $1 \leq i \leq n$.
$c_1(q)$ and $c_1(q') = 1$ and finally, $c_1(s) = 3$. Therefore, $\chi_r[\mu(P_n)] \leq 3$ and

hence, $\chi_r[\mu(P_n)] = 3$.
When n is odd, $c_1(p_1)$ and $c_1(p'_1) = 3$, $c_1(p_i)$ and $c_1(p'_i) = \{1, 2\}$ for $2 \leq i \leq n$. $c_1(q)$ and $c_1(q') = 1$ and finally, $c_1(s) = 4$. Therefore, $\chi_r[\mu(P_n)] \leq 4$ and hence, $\chi_r[\mu(P_n)] = 4$.

Case : 2 $r = 2$
Based on the lemma lower bound are $\chi_r[\mu(P_n)] \geq 4$. To prove $\chi_r[\mu(P_n)] \leq 4$, define a map $c_2 : V[\mu(P_n)] \to \{1, 2, 3, \cdots, k\}$.

- When $n \equiv 2(\text{mod } 3)$, $c_2(p_1) = 2$ and $c_2(p'_1) = 3$.
  Then, $c_2(p_i)$ and $c_2(p'_i) = \{1, 2, 3\}$ for $2 \leq i \leq n$ and the vertices $c_2(q) = 1$ and $c_2(q') = 3$. Atlast the vertex $c_2(s) = 4$. Thus, from the coloring of the above vertices, we get $\chi_r[\mu(P_n)] \leq 4$. Hence it is clear that, $\chi_r[\mu(P_n)] = 4$.
- When $n \not\equiv 2(\text{mod } 3)$, the r- dynamic coloring of Mycielskian graph of pan graph splits into two subcases:
  1. When $n \equiv 0(\text{mod } 3)$, then $c_2(p_i) = \{3, 1, 2\}$ for $1 \leq i \leq n$ and the singleton vertices are colored as $c_2(q) = 1$ and $c_2(q') = 1$. Then, $c_2(p'_i) = 4$ for $1 \leq i \leq n - 1$ and the last vertex $c_2(p'_n) = 2$. Atlast, the vertex $c_2(s) = 5$. Thus, from the coloring of the above vertices, we get $\chi_r[\mu(P_n)] \leq 5$. Hence it is clear that, $\chi_r[\mu(P_n)] = 5$.
  2. When $n \equiv 1(\text{mod } 3)$, then $c_2(p_1) = 4$ and the remaining vertices $c_2(p_i) = \{1, 2, 3\}$ for $2 \leq i \leq n$. The other vertices are colored as $c_2(p'_1) = 2$ and the leftover vertices $c_2(p'_i) = \{1, 2, 3\}$ for $2 \leq i \leq n$. Then, $c_2(q) = 1$ and $c_2(q') = 1$ and finally, $c_2(s) = 5$. Thus, $\chi_r[\mu(P_n)] \leq 5$. Hence it is clear that, $\chi_r[\mu(P_n)] = 5$.

Case : 3 $r = 3$
Based on the lemma lower bound are $\chi_r[\mu(P_n)] \geq 5$. To prove $\chi_r[\mu(P_n)] \leq 5$, define a map $c_3 : V[\mu(P_n)] \to \{1, 2, 3, \cdots, k\}$. The results of 3-coloring are exhibit in the following cases;

- When $n \equiv 0(\text{mod } 3)$, the 3-coloring of pan graph is same as given in case-2 in $n \equiv 0(\text{mod } 3)$. Hence, the result is, $\chi_r[\mu(P_n)] = 5$.
- We have mentioned that lower bound to be $\chi_r[\mu(P_n)] \geq 5$ but in the case of $n \not\equiv 0(\text{mod } 3)$ it is 6-colorable. So, we need extra one color. Thus, $\chi_r[\mu(P_n)] \geq 6$. Consider the following r-coloring to find the upper bound $\chi_r[\mu(P_n)] \leq 6$.
  $c_3(p_i) = \{1, 2, 3, 4\}$ for $1 \leq i \leq n$ and the vertex $c_3(q) = 3$ and $c_3(q') = 1$. The vertices p'_i are colored from the set $\{1, 2, 3, 4\}$ for $1 \leq i \leq n$ according to the r-adjacency condition and also depending on the coloring of the vertices p_i. Finally, color the vertex $c_3(s) = 6$. Thence, $\chi_r[\mu(P_n)] \leq 6$. Hence it is checked that, $\chi_r[\mu(P_n)] = 6$.

Case : 4 $4 \leq r \leq 5$
Based on the lemma lower bound are $\chi_r[\mu(P_n)] \geq r+3$. To prove $\chi_r[\mu(P_n)] \leq r + 3$, define a map $c_4 : V[\mu(P_n)] \to \{1, 2, 3, \cdots, r + 3\}$ such that

- When $n \not\equiv 0(\text{mod } 4)$, color the vertices $c_4(p_i) = \{1, 2, 3, 4\}$ for $1 \leq i \leq n$. Based on the coloring of the vertices p_i, the other vertices such as p'_i, q and q' for $1 \leq i \leq n$ are colored in accordance to the r-adjacency condition. Thus, the vertices are colored from the set $\{1, 2, \cdots, r+2\}$. Finally, $c_4(s) = r+3$. Thus, $\chi_r[\mu(P_n)] \leq r+3$. Hence it is proved that, $\chi_r[\mu(P_n)] = r+3$.
- When $n \equiv 0(\text{mod } 3, 4)$, color the vertices $c_4(p_i) = \{1, 2, 3\}$ for $1 \leq i \leq n$. Then, $c_4(q) = 5$ and $c_4(q') = 3$. Next, color the leftover vertices $c_4(p'_i) = \{4, 5, \cdots, r+2\}$ for $1 \leq i \leq n$ according the r-adjacency condition. Atlast, $c_4(s) = r+3$. Hence, $\chi_r[\mu(P_n)] \leq r+3$. Therefore, it is proved that, $\chi_r[\mu(P_n)] = r+3$.
- We have mentioned that lower bound to be $\chi_r[\mu(P_n)] \geq r+3$ but in the case of $n \equiv 0(\text{mod } 4)$ it is $r+4$-colorable. So, we need extra one color. Thus, $\chi_r[\mu(P_n)] \geq r+4$. When $n \equiv 0(\text{ mod } 4)$, color the vertices $c_4(p_i) = \{4, 1, 2, 3\}$ for $1 \leq i \leq n$. Next, $c_4(q) = 2$ and $c_4(q') = 3$. Similarly, the remaining vertices p'_i are colored from the set $\{5, 6, \cdots, r+2\}$ for $1 \leq i \leq n$ and lastly $c_4(s) = r+4$.

Case : 5 $6 \leq r \leq \Delta$

Based on the lemma lower bound are $\chi_r[\mu(P_n)] \geq r+3$. To prove $\chi_r[\mu(P_n)] \leq r+3$, define a function $c_5 : V[\mu(P_n)] \to \{1, 2, 3, \cdots, r+3\}$ such that

- When $n \equiv 1(\text{mod } 3)$, color the vertices $c_5(p_i) = \{1, 2, 3, 4\}$ for $1 \leq i \leq n$. Then, $c_5(p'_i) = \{5, 6, \cdots, r+2\}$ for $1 \leq i \leq n$ according the r-adjacency condition. The coloring of the vertices q and q' may choosen from the set $\{1, 2, \cdots, r+2\}$ depending the r-adjacency condition. Atlast, $c_5(s) = r+3$. Thus, $\chi_r[\mu(P_n)] \leq r+3$. Hence it is proved that, $\chi_r[\mu(P_n)] = r+3$.
- When $n \not\equiv 1(\text{ mod } 3)$, we have mentioned that lower bound to be $\chi_r[\mu(P_n)] \geq r+3$ but in the case of $n \not\equiv 1(\text{mod } 3)$, $r+3$ colors are not sufficient. So, we need extra one color. Thus, $\chi_r[\mu(P_n)] \geq r+4$. These coloring are given in two subcases;
  1. When $n \equiv 0(\text{mod } 3)$, color the vertices $c_5(p_i) = \{1, 2, 3\}$ for $1 \leq i \leq n$ and the vertices $c_5(p'_i) = \{4, 5, \cdots, r+3\}$ for $1 \leq i \leq n$ in accordance to r-adjacency condition. Similarly, the vertices $c_5(q)$ and $c_5(q')$ are colored from anyone of the colors $\{1, 2, \cdots, r+3\}$. Atlast, $c_5(s) = r+4$. Thus, $\chi_r[\mu(P_n)] \leq r+4$. Hence it is proved that, $\chi_r[\mu(P_n)] = r+4$.
  2. When $n \equiv 2(\text{mod } 3)$, color the vertices $c_5(p_i) = \{1, 2, 3, 4\}$ for $1 \leq i \leq n$ and the vertices $c_5(p'_i) = \{5, 6, \cdots, r+3\}$ for $1 \leq i \leq n$ in accordance to r-adjacency condition. Then, the vertices q and q' are colored from the set $\{1, 2, \cdots, r+3\}$ depending on the coloring of p'_i and r-adjacency condition. Atlast, $c_5(s) = r+4$. Thus, $\chi_r[\mu(P_n)] \leq r+4$. Hence it is cleared that, $\chi_r[\mu(P_n)] = r+4$.

6 r-dynamic coloring of Mycielskian graph of Cocktail party graph

Lemma 2. *Let CP_n be the cocktail party graph. The lower bound for r-dynamic chromatic number of Mycielskian graph of cocktail party graph $\mu(CP_n)$ are*

$$\chi_r[\mu(CP_n)] = \begin{cases} n+1, & for\ 1 \le r < n \\ r+1, & for\ n \le r \le \Delta \end{cases}$$

Proof : Let $V[\mu(CP_n)] = \{s'', s_i, s'_j, q_i, q'_j : i, j \epsilon [1, n]\}$.
$E[\mu(CP_n)] = \{s_i s'_j : i, j \epsilon [1, n], i \neq j\} \cup \{q_i s''\} \cup \{q'_j s''\} \cup \{q_i s_i, q_i s'_j : i, j \epsilon [1, n]\} \cup \{q'_j s_i, q'_j s'_j : i, j \epsilon [1, n]\}$ where q_i are adjacent to all other vertices of CP_n where s_i are adjacent. Similarly, the vertices q'_j are adjacent to the remaining vertices where s'_j are adjacent.
The degrees are $\delta(\mu(CP_n)) = 2n - 1$ and $\Delta(\mu(CP_n)) = 4n - 4$. The order of the graph $|V[\mu(CP_n)]| = 4n + 1$.
For $1 \le r < n$, the vertices $V = \{s_i, s'_j\}$ persuade a clique of order n in $\mu(CP_n)$. Thus, $\chi_r[\mu(CP_n)] \ge n$. Hence, the lower bound is $\chi_r[\mu(CP_n)] \ge n+1$, for $1 \le r < n$. Thus for $n \le r \le \Delta$ based on Lemma 1, we have $\chi_r[\mu(CP_n)] \ge min\{r, \Delta[\mu(CP_n)]\} + 1 = r + 1$. Thus, it completes the proof.

Theorem 4. *For $n \ge 3$, the r- dynamic coloring of Mycielskian graph of cocktail party graph $\mu(CP_n)$ are*

$$\chi_r[\mu(CP_n)] = \begin{cases} n+1, & for\ 1 \le r < n \\ r+3, & for\ n \le r \le \delta \\ r+4, & for\ \delta + 1 \le r \le \Delta - n + 1 \\ r+5, & for\ \Delta - n + 2 \le r \le \Delta \end{cases}$$

Proof : The r-dynamic coloring of $\mu(CP_n)$ are considered in the following cases:

Case : 1 $1 \le r < n$

Based on the lemma, $\chi_r[\mu(CP_n)] \ge n + 1$. To prove $\chi_r[\mu(CP_n)] \le n + 1$, define a function $c_1 : V[\mu(CP_n)] \to \{1, 2, 3, \cdots, n + 1\}$ such that $c_1(s_i)$ and $c_1(s'_j) = \{1, 2, \cdots, n\}$ for $1 \le i \le n$ and also the vertices $c_1(q_i)$ and $c_1(q'_j) = \{1, 2, \cdots, n\}$ for $1 \le i \le n$. Since, these vertices form an complete graph, we used n different colors.
The common vertex $c_1(s'') = n + 1$, since this vertex is adjacent to other vertices which are n colors, we used $n + 1$ color for the common vertex s''. Thus, from the above coloring, we get $\chi_r[\mu(CP_n)] \le n + 1$. Hence, it is clear that, $\chi_r[\mu(CP_n)] = n + 1$.

Case : 2 $n \le r \le \delta$

Define a map $c_2 : V[\mu(CP_n)] \to \{1, 2, 3, \cdots, r + 3\}$ such that

- At, $n \le r \le \delta - 1$, the coloring of the vertices are generalised as, when $r = n$, $c_2(s_i)$ and $c_2(q_i) = \{1, 2, \cdots, r\}$ for $1 \le i \le n$ and the other vertices $c_2(s'_j)$ and $c_2(q'_j) = \{r + 1, r + 2, 3, 4, \cdots, r\}$ for $1 \le j \le n$.

Atlast, the vertex $c_2(s'') = r + 3$ since, the vertex s'' are adjacent to all other vertices which are already colored with $r + 2$ colors.
At $r = n+1$, $c_2(s_i)$ and $c_2(q_i) = \{1, 2, \cdots, n\}$ for $1 \leq i \leq n$ and the other vertices $c_2(s'_j)$ and $c_2(q'_j) = \{r, r+1, r+2, 3, 4, \cdots, n\}$ for $1 \leq j \leq n$. Finally, the common vertex $c_2(s'') = r + 3$.
By continuing this process, at $r = \delta - 1$, the r-coloring are, $c_2(s_i)$ and $c_2(q_i) = \{1, 2, \cdots, n\}$ for $1 \leq i \leq n$ and the other vertices $c_2(s'_j)$ and $c_2(q'_j) = \{n+1, n+2, n+3, \cdots, 2n\}$ for $1 \leq j \leq n$. At the end, the common vertex $c_2(s'') = r + 3$. Hence, from the above lemma, it is clear that, $\chi_r[\mu(CP_n)] = r + 3$.

- At, $r = \delta$, consider the below r-dynamic coloring, $c_2(s_i) = \{1, 2, \cdots, n\}$ for $1 \leq i \leq n$ and the remaining vertices $c_2(s'_j)$ and $c_2(q'_j) = \{n+1, n+2, \cdots, 2n\}$ for $1 \leq j \leq n$. Then, the vertices $c_2(q_i) = \{2n+1, 2n+1, 3, \cdots, n\}$ for $1 \leq i \leq n$. At the end, the common vertex $c_2(s'') = r+3$. Here, in coloring of the vertices q_i the $2n + 1$ color is repeated twice for the vertex q_1 and q_2 since, r-dynamic is an minimal coloring, the repeated values will not affect the r-adjacency condition. Hence, from the above lemma, it is clear that, $\chi_r[\mu(CP_n)] = r + 3$.

Case : 3 $\delta + 1 \leq r \leq \Delta - n + 1$

We define a map $c_3 : V[\mu(CP_n)] \to \{1, 2, 3, \cdots, r + 4\}$ such that color the vertices s_i, s'_j and q'_j as given in case-2 at $r = \delta$.
When $r = \delta + 1$, colored the vertices $c_3(q_i) = \{2n+1, \cdots, r+3, 5, \cdots, n\}$ for $1 \leq i \leq n$ and then color the vertex $c_3(s'') = r+4$. By continuing this process, at the case of $r = \Delta - n + 1$, the vertices $c_3(q_i) = \{2n+1, 2n+2, \cdots, r+3\}$ for $1 \leq i \leq n$ and finally color the vertex $c_3(s'') = r + 4$. Hence, from the above lemma, it is clear that, $\chi_r[\mu(CP_n)] = r + 4$.

Case : 4 $\Delta - n + 2 \leq r \leq \Delta$

We define a map $c_4 : V[\mu(CP_n)] \to \{1, 2, 3, \cdots, r + 5\}$ such that the coloring of the vertices s_i, s'_j and q_j are same as given in case-3 at $\Delta - n + 1$. Only the coloring of the vertices q'_j and s'' varies depending on the r-adjacency condition.
When $r = \Delta - n + 2$, color the vertices $c_4(q'_j) = \{r+3, r+4, 9, 10, \cdots, 2n\}$ for $1 \leq j \leq n$ and the leftover vertex $c_4(s'') = r + 5$. Then, at $r = \Delta - n + 3$, the coloring are, $c_4(q'_j) = \{3n+1, 3n+2, \cdots, r+4, 10, \cdots, 2n\}$ for $1 \leq j \leq n$ and the last vertex $c_4(s'') = r + 5$. By extending the above r-coloring, at $r \geq \Delta$ the coloring are $c_4(q'_j) = \{3n+1, 3n+2, \cdots, r+4\}$ for $1 \leq j \leq n$ and the leftover vertex $c_4(s'') = r + 5$. Hence, from the above lemma, it is clear that, $\chi_r[\mu(CP_n)] = r + 5$.

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