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STUDY ON SOME IDENTITIES ARISING FROM HIGHER ORDER CHANGHEE POLYNOMIALS BASES

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Abstract. In this paper, we study explicit formulas for expressing any polynomials as linear combinations of the higher order Changhee polynomials bases of the first and second kind. We use umbral calculus to get them and find interesting identities by applying these formulas to certain special polynomials. In addition, we show that the higher-order Euler polynomials and the higher-order Bernoulli polynomials are expressed as linear combinations of the higher-order Changhee polynomials, of both kinds.

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1. Introduction

For $r \in \mathbb{N}$, the Changhee polynomials $Ch_n^{(r)}(x)$ of the first kind of order r and those $\widehat{Ch}_n^{(r)}(x)$ of the second kind of order r are given by

$$\left(\frac{2}{2+t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(r)}(x)\frac{t^n}{n!}, \quad \text{(see [8, 11, 18, 20])}, \tag{1}$$

and

$$\left(\frac{2(1+t)}{2+t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} \widehat{Ch}_n^{(r)}(x)\frac{t^n}{n!}, \quad \text{(see [8, 11])}, \tag{2}$$

respectively.

When $x = 0$, $Ch_n^{(r)} = Ch_n^{(r)}(0)$ and $\widehat{Ch}_n^{(r)} = \widehat{Ch}_n^{(r)}(0)$ are the Changhee numbers of the first and second kind of order r, respectively.

When $r = 1$, $Ch_n(x) = Ch_n^{(r)}(x)$ and $\widehat{Ch}_n(x) = \widehat{Ch}_n^{(r)}(x)$ are called the Changhee polynomials of the first and second kind, respectively.

When $x = 0$, $Ch_n = Ch_n(0)$ and $\widehat{Ch}_n = \widehat{Ch}_n(0)$ are called the Changhee numbers of the first and second kind, respectively.

We note that $Ch_n^{(r)}(x)$, $\widehat{Ch}_n^{(r)}(x)$ are polynomials of degree n with rational coefficients for all nonnegative integer n. Thus, both $\{Ch_0^{(r)}(x), Ch_1^{(r)}(x)\cdots,$

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$Ch_n^{(r)}(x)\}$ and $\{\widehat{Ch}_0^{(r)}(x), \widehat{Ch}_1^{(r)}(x) \cdots, \widehat{Ch}_n^{(r)}(x)\}$ are bases for the $(n+1)$-dimensional space $\mathbb{P}_n(\mathbb{C}) = \{p(x) \in \mathbb{C}[x] | \deg p(x) \leq n\}$. Let $p(x) \in \mathbb{P}_n(\mathbb{C})$. Then, we can express $p(x)$ by

$$p(x) = \alpha_{0,r} Ch_0^{(r)}(x) + \alpha_{1,r} Ch_2^{(r)}(x) + \cdots + \alpha_{n,r} Ch_n^{(r)}(x),$$

and

$$p(x) = \widetilde{\alpha}_{0,r} \widehat{Ch}_0^{(r)}(x) + \widetilde{\alpha}_{1,r} \widehat{Ch}_2^{(r)}(x) + \cdots + \widetilde{\alpha}_{n,r} \widehat{Ch}_n^{(r)}(x).$$

In this paper, we study methods for computing $\alpha_{i,r}$ and $\widetilde{\alpha}_{i,r}$, $(i = 1, 2 \cdots, n)$, by using umbral calculus. Furthermore, by applying these formulas to certain special polynomials, we derive some interesting identities and properties. In addition, the higher-order Euler polynomials and the higher-order Bernoulli polynomials are shown to be expressed as linear combinations of the higher-order Changhee polynomials, of both kinds.

Let p be a prime number with $p \equiv 1 \pmod 2$. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let $|\cdot|_p$ be the p-adic norm with $|p|_p = \frac{1}{p}$.

For a $\mathbb{C}_p$-valued continuous function f on $\mathbb{Z}_p$, Kim [13, 14] introduced the p-adic fermionic integral on $\mathbb{Z}_p$ as follows:

(3)

$$\begin{aligned} I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) &= \lim_{\mathbb{N}\to\infty} \sum_{x=0}^{p^{\mathbb{N}}-1} f(x) \mu_{-1}(x + p^{\mathbb{N}}\mathbb{Z}_p) \\ &= \lim_{\mathbb{N}\to\infty} \sum_{x=0}^{p^{\mathbb{N}}-1} f(x)(-1)^x, \quad (\text{see } [3-8, 11-18]). \end{aligned}$$

Let $f_n(x) = f(x+n)$ for $n \in \mathbb{N}$. From (3), we observe that

(4)

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \qquad (\text{see } [3-8, 11-18]).$$

In (4) when $n = 1$, we have

(5)
$$I_{-1}(f_1) + I_{-1}(f) = 2f(0).$$

From (1), (2) and (5), for $r \in \mathbb{N}$, Kim-Kim introduced the Changhee numbers $Ch_n^{(r)}$ and $\widehat{Ch}_n^{(r)}$ of the first and second kind of order r, respectively, as follows:

(6)

$$Ch_n^{(r)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r)_n d\mu_{-1}(x_1) \cdots d_{\mu_{-1}}(x_r), \quad (\text{see } [8, 11, 18]),$$

and

(7)

$$\widehat{Ch}_n^{(r)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - \cdots - x_r)_n d_{\mu_{-1}}(x_1) \cdots d\mu_{-1}(x_r) \quad (\text{see } [8, 11]).$$

From (6) and (7), the generating function of Changhee polynomials of the first and second kind of order r, respectively, are given by

(8)
$$\sum_{n=0}^{\infty} Ch_n^{(r)}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{x_1+\cdots+x_r+x}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$$
$$= \left(\frac{2}{2+t}\right)^r(1+t)^x, \qquad \text{(see [8, 11, 18])},$$

and

$$\sum_{n=0}^{\infty} \widehat{Ch}_n^{(r)}\frac{t^n}{n!} = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{-x_1-\cdots-x_r-x}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$$
(9)
$$= \left(\frac{2(1+t)}{2+t}\right)^r(1+t)^x, \qquad \text{(see [8, 11])}.$$

For $n \geq 0$, the Stirling numbers of second kind are defined by

(10)
$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad \text{and} \quad \frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \qquad \text{(see [8, 9, 11])},$$

where $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$, $(n \geq 1)$, and $(x)_0 = 1$.

The Bernoulli polynomials of order r are given by

(11)
$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}, \quad (r \in \mathbb{N}), \qquad \text{(see [7, 9, 10])}.$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$, which are called the Bernoulli numbers of order r.

When $r = 1$, $B_n(x) = B_n^{(1)}(x)$ are the ordinary Bernoulli polynomials (see [1, 6, 9]).

The Euler polynomials of order r are defined by

(12)
$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!}, \quad (r \in \mathbb{N}), \qquad \text{(see [5, 8, 19])}.$$

When $x = 0$, $E_n^{(r)} = E_n^{(r)}(0)$, which are called the Euler numbers of order r. When $r = 1$, $E_n(x) = E_n^{(1)}(x)$ are the ordinary Euler polynomials (see [3, 4, 16]).

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \;\middle|\; a_k \in \mathbb{C} \right\}.$$

Let $\mathbb{P} = \mathbb{C}[x]$ and $\mathbb{P}^*$ be the vector space all linear functionals on $\mathbb{P}$.

$$\mathbb{P}_n = \{\ P(x) \in \mathbb{C}[x] \mid degP(x) \leq n\}, \quad (n \geq 0).$$

Then $\mathbb{P}_n$ is an $(n+1)$-dimensional vector space over $\mathbb{C}$.

Each $f(t)=\sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$ gives rise to the linear functional $\langle f(t) \mid \cdot \rangle$ on $\mathbb{P}$, called the linear functional given by $f(t)$, which is defined by

$$\langle f(t) \mid x^n \rangle = a_n, \quad \text{for all } \ n \geq 0 \qquad (\text{see } [10, 21]). \tag{13}$$

For each nonnegative integer k, the differential operator t^k on $\mathbb{P}$ is defined by

$$(t^k)x^n = \begin{cases} (n)_k x^{n-k}, & \text{if} \quad k \leq n, \\ 0 & \text{if} \quad k \geq n, \end{cases} \quad (\text{see } [10, 21]). \tag{14}$$

In particular,

$$\langle t^k \mid x^n \rangle = n!\delta_{n,k}, \quad (n, \ k \ \geq 0), \tag{15}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

From (15), we get

$$\langle e^{yt} \mid p(x) \rangle = p(y). \tag{16}$$

For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$f(t)=\sum_{k=0}^{\infty} \langle f(t)|x^k \rangle \frac{t^k}{k!}, \qquad p(x)=\sum_{k=0}^{\infty} \langle t^k|p(x) \rangle \frac{x^k}{k!}. \tag{17}$$

Thus, by (14) and (16), we can write

$$p^{(k)}(0)=\langle t^k|p(x)\rangle = \langle 1|p^{(k)}(x)\rangle, \qquad \text{where } \ p^{(k)}(0)=\left.\frac{d^k p(x)}{dx^k}\right|_{x=0}. \tag{18}$$

From (14), we have

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}. \tag{19}$$

The order $o(f(t))$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. The series $f(t)$ is called invertible if $o(f(t))=0$ and such series has a multiplicative inverse $1/f(t)$ of $f(t)$. $f(t)$ is called a delta series if $o(f(t))=1$ and it has a compositional inverse $\overline{f}(t)$ of $f(t)$ with $\overline{f}(f(t))=f(\overline{f}(t))=t$.

Let $f(t)$ and $g(t)$ be a delta series and an invertible series, respectively. Then there exists a unique sequences $s_n(x)$ such that the orthogonality conditions hold

$$\left\langle g(t)\big(f(t)\big)^k \mid s_n(x) \right\rangle = n!\delta_{n,k}, \quad (n,k \geq 0) \qquad (\text{see } [10, 22]). \tag{20}$$

The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$, which is denoted by $s_n(x) \sim (g(t), f(t))$.

The sequence $s_n(x) \ \sim \ (g(t), f(t))$ if and only if

$$\frac{1}{g\big(\overline{f}(t)\big)} e^{x\overline{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k, \quad (n,k \geq 0), \qquad (\text{see } [10, 21]). \tag{21}$$

Let $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), l(t))$, $(n \geq 0)$. Then

(22)
$$s_n(x) = \sum_{k=0}^{n} a_{n,k} r_k(x), \quad (n \geq 0),$$
$$\text{where} \quad a_{n,k} = \frac{1}{k!}\left\langle \frac{h(\overline{f}(t))}{g(\overline{f}(t))}\big(l(\overline{f}(t))\big)^k \mid x^n \right\rangle, \quad (n,k \geq 0), \qquad (\text{see } [10, 21]).$$

We note that

(23)
$$s_n(x) \sim (g(t), f(t)) \quad \text{if and only if} \quad n s_{n-1}(x) = f(t) s_n(x), \quad (\text{see } [10, 22]).$$

Let $p(x) \in \mathbb{P}_n(\mathbb{C})$ with $p(x) = \sum_{k=0}^{n} \alpha_k q_k(x)$, where $q_n(x) \sim (g(t),\ f(t))$. Then, by (20), we have

$$\langle g(t) f(t)^k | p(x) \rangle = \sum_{j=0}^{n} \alpha_j \langle g(t) f(t)^k | q_j(x) \rangle = \sum_{j=0}^{n} \alpha_j j! \delta_{j,k} = k! \alpha_k. \tag{24}$$

By (24), we have

$$\alpha_k = \frac{1}{k!}\langle g(t) f(t)^k | p(x) \rangle, \quad (k \geq 0). \tag{25}$$

Example.

Let us consider for the ordinary Euler polynomials.
Then we have $E_n(x) \sim (\frac{e^t+1}{2}, t)$.
We observe that

$$\alpha_0 = \left\langle \frac{e^t+1}{2} t^0 \middle| p(x) \right\rangle = \frac{1}{2}\{p(1) + p(0)\},$$

and

$$\alpha_k = \frac{1}{k!}\left\langle \frac{e^t+1}{2} t^k \middle| p(x) \right\rangle = \frac{1}{2k!}\langle (e^t+1) t^k | p(x) \rangle = \frac{1}{2k!}\langle e^t + 1 | p^{(k)}(x) \rangle$$
$$= \frac{1}{2k!}(p^{(k)}(1) + p^{(k)}(0)), \quad \text{for } k = 1, 2, 3, \cdots, n.$$

2. Changhee polynomials and numbers of the first and second kind of order r

We derive formulas for the coefficients when a polynomial is expressed in terms of the higher-order Changhee polynomials of both kinds by using umbral calculus. Further, we determine some interesting identities by applying these to some special polynomials.

Combining (1), (2) with (21), we get the two Sheffer sequences as follows:

$$Ch_n^{(r)}(x) \ \sim \ \left(\left(\frac{1+e^t}{2}\right)^r,\ e^t - 1\right), \tag{26}$$

and

$$\widehat{Ch}_n^{(r)}(x) \ \sim \ \left(\left(\frac{1+e^t}{2e^t}\right)^r,\ e^t - 1\right). \tag{27}$$

Theorem 2.1. *For $n \geq 0$, we have*

$$Ch_n(x) = \frac{2}{n+1}\left((x)_{n+1} - \sum_{j=0}^{n+1}\binom{n+1}{j}(-1)^j j! 2^{-j}(x)_{n-j+1}\right).$$

Proof. From (16) and (23), we observe that

$$Ch_n(x+1) - Ch_n(x) = (e^t - 1)Ch_n(x) = nCh_{n-1}(x), \quad (n \geq 1). \tag{28}$$

On the other hand, by (1), we get

$$\begin{aligned}&\sum_{n=0}^{\infty}(Ch_n(x+1) - Ch_n(x))\frac{t^n}{n!}\\ &\qquad = \frac{2}{t+2}((1+t)^{x+1} - (1+t)^x) = \frac{2t}{t+2}(1+t)^x.\end{aligned} \tag{29}$$

By (29), we have

$$Ch_n(x+1) - Ch_n(x) = \frac{d^n}{dt^n}\left(\frac{2t}{t+2}(1+t)^x\right)\Big|_{t=0}. \tag{30}$$

We observe that

$$\begin{aligned}\left(\frac{2t}{t+2}(1+t)^x\right)' &= \frac{d}{dt}\left(2(1+t)^x - 4\frac{1}{t+2}(1+t)^x\right)\\ &= 2x(1+t)^{x-1} - 4\frac{d}{dt}\left\{\left(\frac{1}{t+2}\right)(1+t)^x\right\},\end{aligned}$$

$$\begin{aligned}\left(\frac{2t}{t+2}(1+t)^x\right)^{(2)} =& 2x(x-1)(1+t)^{x-2}\\ &- 4\left\{\sum_{j=0}^{2}\binom{2}{j}\left(\left(\frac{d}{dt}\right)^j\frac{1}{t+2}\right)\left(\left(\frac{d}{dt}\right)^{2-j}(1+t)^x\right)\right\},\end{aligned}$$

Continuing this process, we have

$$\begin{aligned}&\left(\left(\frac{2t}{t+2}\right)(1+t)^x\right)^{(n)}\\ &= 2(x)_n(1+t)^{x-n} - 4\left(\sum_{j=0}^{n}\binom{n}{j}\left(\left(\frac{d}{dt}\right)^j\frac{1}{t+2}\right)\left(\frac{d}{dt}\right)^{n-j}(1+t)^x\right)\\ &= 2(x)_n(1+t)^{x-n} - 4\left(\sum_{j=0}^{n}\binom{n}{j}(-1)^j j!\left(\frac{1}{t+2}\right)^{j+1}(x)_{n-j}(1+t)^{x-n+j}\right).\end{aligned} \tag{31}$$

Combining (28), (30) with (31), we have

$$\begin{aligned}Ch_n(x) &= \frac{1}{n+1}\left(2(x)_{n+1} - 4\sum_{j=0}^{n+1}\binom{n+1}{j}(-1)^j j! 2^{-j-1}(x)_{n-j+1}\right)\\ &= \frac{2}{n+1}\left((x)_{n+1} - \sum_{j=0}^{n+1}\binom{n+1}{j}(-1)^j j! 2^{-j}(x)_{n-j+1}\right).\end{aligned} \tag{32}$$

By (32), we get the desired result. □

Theorem 2.2. *Let $p(x) \in \mathbb{P}_n(\mathbb{C})$ with $p(x) = \sum_{k=0}^{n} \alpha_{k,r} Ch_k^{(r)}(x)$. Then we have*

$$\alpha_{k,r} = \frac{1}{2^r k!} \sum_{l=0}^{k} \sum_{j=0}^{r} \binom{k}{l} \binom{r}{j} (-1)^{k-l} p(l+j).$$

Proof. Let $p(x) = \sum_{k=0}^{n} \alpha_{k,r} Ch_k^{(r)}(x)$. From (25) and (26), we have

$$\begin{aligned} \alpha_{k,r} &= \frac{1}{k!} \left\langle \left(\frac{e^t+1}{2} \right)^r (e^t-1)^k \middle| p(x) \right\rangle \\ &= \frac{1}{2^r k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \sum_{j=0}^{r} \binom{r}{j} \left\langle e^{(j+l)t} \middle| p(x) \right\rangle \\ &= \frac{1}{2^r k!} \sum_{l=0}^{k} \sum_{j=0}^{r} \binom{k}{l} \binom{r}{j} (-1)^{k-l} p(l+j). \end{aligned} \tag{33}$$

From (33), we obtain the desired result. □

Theorem 2.3. *Let $p(x) \in \mathbb{P}_n(\mathbb{C})$ with $p(x) = \sum_{k=0}^{n} \tilde{\alpha}_{k,r} \widehat{Ch}_k^{(r)}(x)$. Then we have*

$$\tilde{\alpha}_{k,r} = \frac{1}{2^r k!} \sum_{l=0}^{k} \sum_{j=0}^{r} \binom{k}{l} \binom{r}{j} (-1)^{k-l} p(l+j-r).$$

Proof. Let $p(x) = \sum_{k=0}^{n} \alpha_{k,r} Ch_k^{(r)}(x)$. From (25) and (27), we have

$$\begin{aligned} \widetilde{\alpha}_{k,r} &= \frac{1}{k!} \left\langle \left(\frac{e^t+1}{2e^t} \right)^r (e^t-1)^k \middle| p(x) \right\rangle \\ &= \frac{1}{2^r k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \sum_{j=0}^{r} \binom{r}{j} \left\langle e^{(j+l-r)t} \middle| p(x) \right\rangle \\ &= \frac{1}{2^r k!} \sum_{l=0}^{k} \sum_{j=0}^{r} \binom{k}{l} \binom{r}{j} (-1)^{k-l} p(l+j-r). \end{aligned} \tag{34}$$

From (34), we obtain the desired result. □

In [2], Gessel showed a short proof of Miki's identity for Bernoulli numbers,

$$\sum_{i=1}^{n-1} \frac{1}{i(n-i)} B_i B_{n-i} = \sum_{i=2}^{n-2} \binom{n}{i} \frac{1}{i(n-i)} B_i B_{n-i} + 2H_n \frac{B_n}{n}, \quad (n \geq 4),$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ are the harmonic numbers.

In [4, 6], Kim and Kim showed

$$\sum_{i=1}^{n-1}\frac{1}{i(n-i)}B_i(x)B_{n-i}(x)$$
$$=\frac{2}{n}\sum_{i=0}^{n-2}\binom{n}{i}\frac{1}{n-i}B_{n-i}B_i(x)+\frac{2}{n}H_{n-1}B_n(x),\quad (n\geq 2),$$

where $H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$ are the harmonic numbers.

The identity obtained by substituting $x=0$ in (34) is a slightly different version of the well-known Miki's identity ([2, 19]).

Miki's proof uses a formula for the Fermat quotient $\frac{a^p-a}{p}$ (mod p^2), Gessels proof is based on two different expressions for Stirling numbers of the second kind $S_2(n,k)$ [2], Shiratani-Yokoyama's proof employs p-adic analysis [22], and Dunne-Schubert exploits the asymptotic expansion of some special polynomials coming from the quantum field theory computations [1].

From Theorem 2 and Theorem 3, we obtain the following identities.

Applications.

(a) Let $p(x)=\sum_{k=1}^{n-1}\frac{1}{k(n-k)}B_k(x)B_{n-k}(x)\in\mathbb{P}_n(\mathbb{C})$. Then, we have

$$\sum_{k=1}^{n-1}\frac{1}{k(n-k)}B_k(x)B_{n-k}(x)$$
$$=\frac{1}{2^r}\sum_{k=0}^{n}\frac{1}{k!}\sum_{m=1}^{n-1}\Bigg(\sum_{l=0}^{k}\sum_{j=0}^{r}\binom{k}{l}\binom{r}{j}(-1)^{k-l}$$
$$\times\frac{1}{m(n-m)}B_m(l+j)B_{n-m}(l+j)\Bigg)Ch_k^{(r)}(x)$$
$$=\frac{1}{2^r}\sum_{k=0}^{n}\frac{1}{k!}\sum_{m=1}^{n-1}\Bigg(\sum_{l=0}^{k}\sum_{j=0}^{r}\binom{k}{l}\binom{r}{j}(-1)^{k-l}$$
$$\times\frac{1}{m(n-m)}B_m(l+j-r)B_{n-m}(l+j-r)\Bigg)\widehat{Ch}_k^{(r)}(x).$$

(b) Let $p(x)=\sum_{k=1}^{n-1}\frac{1}{k(n-k)}E_k(x)E_{n-k}(x)\in\mathbb{P}_n(\mathbb{C})$. Then, we have

$$\sum_{k=1}^{n-1}\frac{1}{k(n-k)}E_k(x)E_{n-k}(x)$$
$$=\frac{1}{2^r}\sum_{k=0}^{n}\frac{1}{k!}\sum_{m=1}^{n-1}\Bigg(\sum_{l=0}^{k}\sum_{j=0}^{r}\binom{k}{l}\binom{r}{j}(-1)^{k-l}$$
$$\times\frac{1}{m(n-m)}E_m(l+j)E_{n-m}(l+j)\Bigg)Ch_k^{(r)}(x)$$
$$=\frac{1}{2^r}\sum_{k=0}^{n}\frac{1}{k!}\sum_{m=1}^{n-1}\Bigg(\sum_{l=0}^{k}\sum_{j=0}^{r}\binom{k}{l}\binom{r}{j}(-1)^{k-l}$$
$$\times\frac{1}{m(n-m)}E_m(l+j-r)E_{n-m}(l+j-r)\Bigg)\widehat{Ch}_k^{(r)}(x).$$

(c) Let $p(x)=\sum_{k=1}^{n-1}\frac{1}{k(n-k)}B_k(x)E_{n-k}(x)\in\mathbb{P}_n(\mathbb{C})$. Then, we have

$$\begin{aligned}
&\sum_{k=1}^{n-1}\frac{1}{k(n-k)}B_k(x)E_{n-k}(x)\\
&=\frac{1}{2^r}\sum_{k=0}^{n}\frac{1}{k!}\sum_{m=1}^{n-1}\Bigg(\sum_{l=0}^{k}\sum_{j=0}^{r}\binom{k}{l}\binom{r}{j}(-1)^{k-l}\\
&\qquad\times\frac{1}{m(n-m)}B_m(l+j)E_{n-m}(l+j)\Bigg)Ch_k^{(r)}(x)\\
&=\frac{1}{2^r}\sum_{k=0}^{n}\frac{1}{k!}\sum_{m=1}^{n-1}\Bigg(\sum_{l=0}^{k}\sum_{j=0}^{r}\binom{k}{l}\binom{r}{j}(-1)^{k-l}\\
&\qquad\times\frac{1}{m(n-m)}B_m(l+j-r)E_{n-m}(l+j-r)\Bigg)\widehat{Ch}_k^{(r)}(x).
\end{aligned}$$

(d) Let $p(x)=\sum_{k=0}^{n}B_k^{(r_1)}(x)x^{n-k}\in\mathbb{P}_n(\mathbb{C})$. Then, we have

$$\begin{aligned}
&\sum_{k=0}^{n}B_k^{(r_1)}(x)x^{n-k}\\
&=\frac{1}{2^{r_2}}\sum_{k=0}^{n}\frac{1}{k!}\sum_{m=0}^{n}\Bigg(\sum_{l=0}^{k}\sum_{j=0}^{r_2}\binom{k}{l}\binom{r_2}{j}(-1)^{k-l}\\
&\qquad\times B_m^{(r_1)}(l+j)(l+j)^{n-m}\Bigg)Ch_k^{(r_2)}(x)\\
&=\frac{1}{2^{r_2}}\sum_{k=0}^{n}\frac{1}{k!}\sum_{m=0}^{n}\Bigg(\sum_{l=0}^{k}\sum_{j=0}^{r_2}\binom{k}{l}\binom{r_2}{j}(-1)^{k-l}\\
&\qquad\times B_m^{(r_1)}(l+j-r_2)(l+j-r_2)^{n-m}\Bigg)\widehat{Ch}_k^{(r_2)}(x).
\end{aligned}$$

(e) Let $p(x)=\sum_{k=0}^{n}E_k^{(r_1)}(x)x^{n-k}\in\mathbb{P}_n(\mathbb{C})$. Then, we have

$$\begin{aligned}
&\sum_{k=0}^{n}E_k^{(r_1)}(x)x^{n-k}\\
&=\frac{1}{2^{r_2}}\sum_{k=0}^{n}\frac{1}{k!}\sum_{m=0}^{n}\Bigg(\sum_{l=0}^{k}\sum_{j=0}^{r_2}\binom{k}{l}\binom{r_2}{j}(-1)^{k-l}\\
&\qquad\times E_m^{(r_1)}(l+j)(l+j)^{n-m}\Bigg)Ch_k^{(r_2)}(x)\\
&=\frac{1}{2^{r_2}}\sum_{k=0}^{n}\frac{1}{k!}\sum_{m=0}^{n}\Bigg(\sum_{l=0}^{k}\sum_{j=0}^{r_2}\binom{k}{l}\binom{r_2}{j}(-1)^{k-l}\\
&\qquad\times E_m^{(r_1)}(l+j-r_2)(l+j-r_2)^{n-m}\Bigg)\widehat{Ch}_k^{(r_2)}(x).
\end{aligned}$$

Theorem 2.4. *For $n \geq 0$ and $r_1,\ r_2 \in \mathbb{N}$, we have*

$$E_n^{(r_1)}(x) = \frac{1}{2^{r_2}} \sum_{k=0}^{n} \left(\sum_{l=k}^{n} \sum_{j=0}^{n-l} \binom{n}{l} \binom{n-l}{j} \times S_2(l,k) E_j^{(r_1)} \sum_{d=0}^{r_2} \binom{r_2}{d} d^{n-l-j} \right) Ch_k^{(r_2)}(x).$$

Proof. From (12), (21) and (26), we have two Sheffer sequences as follows :

(35)

$$E_n^{(r_1)}(x) \sim \left(\left(\frac{e^t+1}{2} \right)^{r_1},\ t \right) \quad \text{and} \quad Ch_n^{(r_2)}(x) \sim \left(\left(\frac{1+e^t}{2} \right)^{r_2},\ e^t - 1 \right).$$

By (22) and (35), we get

$$E_n^{(r_1)}(x) = \sum_{k=0}^{n} z_k Ch_k^{(r_2)}(x), \tag{36}$$

where, by (10), (12) and (16), we get

$$\begin{aligned} z_k &= \frac{1}{k!} \left\langle \left(\frac{1+e^t}{2} \right)^{r_2} \left(\frac{2}{e^t+1} \right)^{r_1} (e^t-1)^k \middle| x^n \right\rangle \\ &= \sum_{l=k}^{n} S_2(l,k) \binom{n}{l} \left\langle \left(\frac{1+e^t}{2} \right)^{r_2} \left(\frac{2}{e^t+1} \right)^{r_1} \middle| x^{n-l} \right\rangle \\ &= \sum_{l=k}^{n} \binom{n}{l} S_2(l,k) \sum_{j=0}^{n-l} E_j^{(r_1)} \binom{n-l}{j} \left\langle \left(\frac{1+e^t}{2} \right)^{r_2} \middle| x^{n-l-j} \right\rangle \\ &= \frac{1}{2^{r_2}} \sum_{l=k}^{n} \sum_{j=0}^{n-l} \binom{n}{l} \binom{n-l}{j} S_2(l,k) E_j^{(r_1)} \sum_{d=0}^{r_2} \binom{r_2}{d} \langle e^{dt} | x^{n-l-j} \rangle \\ &= \frac{1}{2^{r_2}} \sum_{l=k}^{n} \sum_{j=0}^{n-l} \binom{n}{l} \binom{n-l}{j} S_2(l,k) E_j^{(r_1)} \sum_{d=0}^{r_2} \binom{r_2}{d} d^{n-l-j}. \end{aligned} \tag{37}$$

Combining (36) with (37), we have the desired result.

□

Theorem 2.5. *For $n \geq 0$ and $r_1,\ r_2 \in \mathbb{N}$, we have*

$$E_n^{(r_1)}(x) = \frac{1}{2^{r_2}} \sum_{k=0}^{n} \left(\sum_{l=k}^{n} \sum_{j=0}^{n-l} \sum_{d=0}^{r_2} \binom{n}{l} \binom{n-l}{j} \binom{r_2}{d} \times S_2(l,k) E_j^{(r_1)} (d-r_2)^{n-l-j} \right) \widehat{Ch}_k^{(r_2)}(x).$$

Proof. From (12), (21) and (27), we get two Sheffer sequences as follows :

(38)

$$E_n^{(r_1)}(x) \sim \left(\left(\frac{e^t+1}{2} \right)^{r_1},\ t \right) \quad \text{and} \quad \widehat{Ch}_n^{(r_2)}(x) \sim \left(\left(\frac{1+e^t}{2e^t} \right)^{r_2},\ e^t - 1 \right).$$

By (22) and (38), we have

$$E_n^{(r_1)}(x) = \sum_{k=0}^{n} \widetilde{z}_k \widehat{Ch}_k^{(r_2)}(x), \tag{39}$$

where, by (10), (12) and (16), we get

(40)

$$\begin{aligned}
\widetilde{z}_k &= \frac{1}{k!}\left\langle \left(\frac{1+e^t}{2e^t}\right)^{r_2} \left(\frac{2}{e^t+1}\right)^{r_1} (e^t-1)^k \middle| x^n \right\rangle \\
&= \frac{1}{2^{r_2}} \sum_{l=k}^{n} \binom{n}{l} S_2(l,k) \sum_{j=0}^{n-l} E_j^{(r_1)} \binom{n-l}{j} \left\langle \sum_{d=0}^{r_2} \binom{r_2}{d} e^{(d-r_2)t} \middle| x^{n-l-j} \right\rangle \\
&= \frac{1}{2^{r_2}} \sum_{l=k}^{n} \sum_{j=0}^{n-l} \sum_{d=0}^{r_2} \binom{n}{l}\binom{n-l}{j}\binom{r_2}{d} S_2(l,k) E_j^{(r_1)} (d-r_2)^{n-l-j}.
\end{aligned}$$

Combining (39) with (40), we get the desired identity. □

Theorem 2.6. *For $n \in \mathbb{N} \cup \{0\}$ and $r_1,\ r_2 \in \mathbb{N}$, we have*

$$\begin{aligned}
B_n^{(r_1)}(x) = \frac{1}{2^{r_2}} \sum_{k=0}^{n} \Bigg(\sum_{l=k}^{n} \sum_{j=0}^{n-l} \sum_{d=0}^{r_2} \binom{n}{l} B_j^{(r_1)} \binom{n-l}{j} \binom{r_2}{d} \\
\times\, d^{n-l-j} S_2(l,k) \Bigg) Ch_k^{(r_2)}(x).
\end{aligned}$$

Proof. From (17), (21) and (26), we have two Sheffer sequences as follows :

(41)

$$B_n^{(r_1)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^{r_1},\ t \right) \quad \text{and} \quad Ch_n^{(r)}(x) \sim \left(\left(\frac{1+e^t}{2}\right)^{r_2},\ \ e^t-1 \right).$$

By (22) and (41), we have

$$B_n^{(r_1)}(x) = \sum_{k=0}^{n} z_k Ch_k^{(r_2)}(x), \tag{42}$$

where, from (10), (11) and (16), we have

$$\begin{aligned}
z_k &= \frac{1}{k!}\left\langle \left(\frac{1+e^t}{2}\right)^{r_2} \left(\frac{t}{e^t-1}\right)^{r_1} (e^t-1)^k \middle| x^n \right\rangle \\
&= \frac{1}{2^{r_2}} \sum_{l=k}^{n} \binom{n}{l} S_2(l,k) \sum_{j=0}^{n-l} \binom{n-l}{j} B_j^{(r_1)} \sum_{d=0}^{r_2} \binom{r_2}{d} d^{n-l-j}.
\end{aligned} \tag{43}$$

Combining (42) with (43), we have the desired result. □

Theorem 2.7. *For $n \in \mathbb{N} \cup \{0\}$ and $r_1,\ r_2 \in \mathbb{N}$, we have*

$$\begin{aligned}
B_n^{(r_1)}(x) = \frac{1}{2^{r_2}} \sum_{k=0}^{n} \Bigg(\sum_{l=k}^{n} \sum_{j=0}^{n-l} \sum_{d=0}^{r_2} \binom{n}{l} \binom{n-l}{j} \binom{r_2}{d} \\
\times\, S_2(l,k) B_j^{(r_1)} (d-r_2)^{n-l-j} \Bigg) \widehat{Ch}_k^{(r_2)}(x).
\end{aligned}$$

Proof. From (11), (21) and (36), we get two Sheffer sequences as follows :

(44)
$$B_n^{(r_1)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^{r_1},\ t\right) \quad \text{and} \quad \widehat{Ch}_n^{(r)}(x) \sim \left(\left(\frac{1+e^t}{2e^t}\right)^{r_2},\ e^t-1\right).$$

By (22) and (44), we have

$$B_n^{(r_1)}(x) = \sum_{k=0}^{n} \widetilde{z}_k \widehat{Ch}_k^{(r_2)}(x), \tag{45}$$

where, by (10), (11) and (16), we get

$$\begin{aligned} \widetilde{z}_k &= \frac{1}{k!}\left\langle \left(\frac{1+e^t}{2e^t}\right)^{r_2}\left(\frac{t}{e^t-1}\right)^{r_1}(e^t-1)^k \middle| x^n \right\rangle \\ &= \frac{1}{2^{r_2}} \sum_{l=k}^{n}\sum_{j=0}^{n-l}\sum_{d=0}^{r_2} \binom{n}{l}\binom{n-l}{j}\binom{r_2}{d} S_2(l,k) B_j^{(r_1)}(d-r_2)^{n-l-j}. \end{aligned} \tag{46}$$

Combining (45) with (46), we attain the desired result. □

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