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ON WEAKLY SYMMETRIC GENERALIZED (k, μ)-SPACE FORMS

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ABSTRACT. In this paper, we study weakly symmetric, weakly Ricci symmetric properties of a generalized (k, μ)-space forms admitting a semi-symmetric metric connection. Also, we investigate the properties of special weakly Ricci symmetric and generalized Ricci recurrent of generalized (k, μ)-space forms.

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1. INTRODUCTION

In 1924, Friedman and Schouten [7] introduced the idea of semi-symmetric metric connection on a differentiable manifold. Hayden [8] initiated the notion of a metric connection with torsion. Semi-symmetric metric connections were also investigated by Amur and Pujar [1], Binh [2], De and Biswas [5], Prvanoic [12] and Maralabhavi et al. [14].

In [4] and [13], the authors have defined and studied a generalized (k, μ) space form, whose curvature tensor can be written as

$$R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_5R_5 + f_6R_6, \tag{1}$$

where $f_1, f_2, f_3, f_4, f_5, f_6$ are differentiable functions on M and R_1, R_2, R_3 are tensors defined as:

(2)

$$R_1(X_1, X_2)X_3 = g(X_2, X_3)X_1 - g(X_1, X_3)X_2$$
$$R_2(X_1, X_2)X_3 = g(X_1, \varphi X_3)\varphi X_2 - g(X_2, \varphi X_3)\varphi X_1 + 2g(X_1, \varphi X_2)\varphi X_3$$
$$R_3(X_1, X_2)X_3 = \eta(X_1)\eta(X_3)X_2 - \eta(X_2)\eta(X_3)X_1$$
$$+ g(X_1, X_3)\eta(X_2)\xi - g(X_2, X_3)\eta(X_1)\xi,$$

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and

(3)

$R_4(X_1,X_2)X_3 = g(X_2,X_3)hX_1 - g(X_1,X_3)hX_2 + g(hX_2,X_3)X_1 - g(hX_1,X_3)X_2$

$R_5(X_1,X_2)X_3 = g(hX_2,X_3)hX_1 - g(hX_1,X_3)hX_2 + g(\varphi hX_1,X_3)\varphi hX_2 - g(\varphi hX_2,X_3)\varphi hX_1$

$R_6(X_1,X_2)X_3 = \eta(X_1)\eta(X_3)hX_2 - \eta(X_2)\eta(X_3)hX_1 + g(hX_1,X_3)\eta(X_2)\xi - g(hX_2,X_3)\eta(X_1)\xi,$

for any vector fields X_1, X_2, X_3, where $2h = L_\xi\varphi$ and L is the usual Lie derivative. This manifold was denoted by $M(f_1, f_2, f_3, f_4, f_5, f_6)$.

A semi-symmetric connection in a Riemannian manifold is defined by Friedman and Schouten [7] as a connection ∇ whose torsion tensor T satisfies

$$T(X_1,X_2) = \eta(X_2)(X_1) - \eta(X_1)(X_2), \tag{4}$$

where $g(X_1,\xi) = \eta(X_1)$ is a 1-form and ξ is a (1,1) tensor field for all vector fields $X_1, X_2 \in \chi(M)$ where $\chi(M)$ is the set of all differentiable vector fields in M. In addition, if $\nabla g = 0$, then ∇ is known as a semi symmetric metric connection.

A non-flat Riemannian manifold M is called a weakly symmetric manifold if there exist 1-forms C_1, C_2, C_3, C_4 not simultaneously zero such that the curvature tensor R satisfies

$$\begin{aligned}(\nabla_{X_1}R)(X_2,X_3)X_4 = {} & C_1(X_1)R(X_2,X_3)X_4 + C_2(X_2)R(X_1,X_3)X_4 + C_3(X_3)R(X_2,X_1)X_4 \\ & + C_4(X_4)R(X_2,X_3)X_1 + g(R(X_2,X_3)X_4,X_1)\rho,\end{aligned} \tag{5}$$

for all $X_1, X_2, X_3, X_4 \in \chi(M)$ and ∇ is the operator of covariant differentiation with respect to the Riemannian metric g.

A Riemannian manifold M is known as a weakly Ricci symmetric manifold if there exist 1-forms A_1, A_2, A_3, A_4 not simultaneously zero such that the Ricci tensor S is not identically zero and satisfies

$$\begin{aligned}(\nabla_{X_1}S)(X_2,X_3) = {} & A_1(X_1)S(X_2,X_3) + A_2(X_2)S(X_1,X_3) + \\ & A_3(X_3)S(X_2,X_1),\end{aligned} \tag{6}$$

for all $X_1.X_2, X_3 \in \chi(M)$.

A $(2n+1)$-dimensional generalized (k,μ) space-form (M,g) is called a special weakly Ricci-symmetric manifold $(SWRS)_n$ if

$$\begin{aligned}(\nabla_{X_1}S)(X_2,X_3) = {} & 2A(X_1)S(X_2,X_3) + 2A(X_2)S(X_1,X_3) + \\ & 2A(X_3)S(X_1,X_2).\end{aligned} \tag{7}$$

A non-flat Riemannian manifold is called generalized Ricci recurrent realizing the following relation:

$$(\nabla_{X_1}S)(X_2,X_3) = A(X_1)S(X_2,X_3) + B(X_1)g(X_2,X_3). \tag{8}$$

In [10], the authors studied the invariant and anti-invariant submanifolds of (κ, μ)-contact metric manifolds as Ricci solitons. In [17, 18], the authors studied some results on indefinite Sasakian manifold admitting quarter-symmetric metric connection and η-Ricci solitons of some curvature tensors and investigated certain curvature tensor on indefinite trans-Sasakian manifold. Recently in [19], the authors studied the slant submanifolds of generalized Sasakian space forms when structure tensor field ϕ is Killing. Also, obtained the conditions for anti-invariant submanifolds under some geometrical conditions such as $\nabla Q = 0$ and $\nabla T = 0$.

2. Preliminaries

A $(2n+1)$-dimension Riemannian manifold (M, g) is called an almost contact manifold if it admits a tensor field φ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying:

$$\varphi^2 X_1 = -X_1 + \eta(X_1)\xi,\ \eta\circ\varphi = 0,\ \varphi\xi = 0,\ \eta(\xi) = 1, \tag{9}$$

$$g(\varphi X_1, \varphi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2), \tag{10}$$

$$g(X_1, \varphi X_2) = -g(\varphi X_1, X_2), g(X_1, \xi) = \eta(X_1). \tag{11}$$

The tensor h is defined by $2h = L_\xi \varphi$ which is symmetric and satisfies the following relations.

$$h\xi = 0, h\varphi = -\varphi h, trh = 0, \eta\circ h = 0, \tag{12}$$

$$\nabla_{X_1}\xi = -\varphi X_1 - \varphi h X_1, \tag{13}$$

$$(\nabla_{X_1}\eta)X_2 = g(X_1 + hX_1, \varphi X_2). \tag{14}$$

In a $(2n+1)$-dimensional (k, μ)-contact metric manifold, we have [3]

$$h^2 = (k-1)\varphi^2, k \leq 1. \tag{15}$$

In a $(2n+1)$-dimensional generalized (k, μ) space-form, the following relations hold:

$$\begin{aligned} R(X_1, X_2)\xi = {} & (f_1 - f_3)[\eta(X_2)X_1 - \eta(X_1)X_2] + \\ & (f_4 - f_6)[\eta(X_2)hX_1 - \eta(X_1)hX_2], \end{aligned} \tag{16}$$

$$\begin{aligned} & S(X_1, X_2) = \\ & [2nf_1 + 3f_2 - f_3]g(X_1, X_2) + [(2n-1)f_4 - f_6]g(hX_1, X_2) - \\ & [3f_2 + (2n-1)f_3]\eta(X_1)\eta(X_2), \end{aligned} \tag{17}$$

$$S(X_1, \xi) = 2n(f_1 - f_3)\eta(X_1), \tag{18}$$

for any vector fields X_1, X_2, X_3, where S is the Ricci tensor of $M(f_1, \cdots, f_6)$.

A semi-symmetric connection $\tilde{\nabla}$ is called semi-symmetric metric connection, if it fuhrer satisfies $\tilde{\nabla} g = 0$.

A semi-symmetric metric connection $\tilde{\nabla}$ in a generalized (k, μ) space-form can be defined by

$$\tilde{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + \eta(X_2)X_1 - g(X_1, X_2)\xi, \tag{19}$$

where ∇ is the Levi-Civita connection on M (See [20, 16])

(20) $(\tilde{\nabla}_{X_1}\eta)(X_2) = -g(\varphi X_1, X_2) - g(\varphi h X_1, X_2) + g(X_1, X_2) - \eta(X_1)\eta(X_2).$

The curvature tensor $\tilde{R}$ and Ricci tensor $\tilde{S}$ of M with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is defined by (See [14]):

(21) $$\tilde{R}(X_1, X_2)X_3 = R(X_1, X_2)X_3 - \alpha(X_2, X_3)X_1 + \alpha(X_1, X_3)X_2 - g(X_2, X_3)\beta(X_1) + g(X_1, X_3)\beta(X_2),$$

(22) $$\tilde{S}(X_2, X_3) = S(X_2, X_3) - (2n-1)\alpha(X_2, X_3) - trace(\alpha)g(X_2, X_3),$$

(23) $$\tilde{S}(X_1, \xi) = [2n(f_1 - f_3) - \frac{(2n-1)}{2} - trace(\alpha)]\eta(X_1),$$

where

(24)
$$\alpha(X_1, X_2) = -g(\varphi X_1, X_2) - g(\varphi h X_1, X_2) + \frac{3}{2}g(X_1, X_2) - \eta(X_1)\eta(X_2)$$
$$\beta(X_1) = -\varphi X_1 - \varphi h X_1 + X_1 - \eta(X_1)\xi.$$

3. Weakly Symmetric Generalized (k, μ) Space Forms admitting Semi-Symmetric Metric Connection

A non flat Generalized (k, μ) space forms M^n is called a weakly symmetric Generalized (k, μ) space forms with respect to the semi-symmetric metric connection $\tilde{\nabla}$ if there exists a 1-form C_1, C_2, C_3, C_4 not simultaneously zero and the curvature tensor $\tilde{R}$ with respect to the semi-symmetric metric connection $\tilde{\nabla}$ satisfies

(25) $$(\tilde{\nabla}_{X_1}\tilde{R})(X_2, X_3)X_4 = C_1(X_1)\tilde{R}(X_2, X_3)X_4 + C_2(X_2)\tilde{R}(X_1, X_3)X_4 + C_3(X_3)\tilde{R}(X_2, X_1)X_4 + C_4(X_4)\tilde{R}(X_2, X_3)X_1 + g(\tilde{R}(X_2, X_3)X_4, X_1)\rho,$$

for all vector fields $X_1, X_2, X_3, X_4 \in XM$.

Theorem 3.1. *In a weakly symmetric generalized (k, μ) space forms with respect to a semi-symmetric metric connection $\tilde{\nabla}$, the sum of the 1-form $C_1 + C_3 + C_4$ vanishes.*

Proof. Suppose equation (25) holds. Then, by contracting (25) with respect to X_2, we have

(26) $$(\tilde{\nabla}_{X_1}\tilde{S})(X_3, X_4) = C_1(X_1)\tilde{S}(X_3, X_4) + C_2\tilde{R}(X_1, X_3)X_4 + C_3(X_3)\tilde{S}(X_1, X_4) + C_4(X_4)\tilde{S}(X_3, X_1) + B(\tilde{R}(X_1, X_4)X_3),$$

where $B(X_1) = g(_{X_1}, \rho)$. Replacing X_4 by ξ and applying equations (21) and (23) in equation (26), we get

(27)
$$(\tilde{\nabla}_{X_1}\tilde{S})(X_3,\xi) = [2n(f_1-f_3) - \frac{(2n-1)}{2} - trace(\alpha)][C_1(X_1)\eta(X_3)+ C_3(X_3)\eta(X_1)] + C_2\{(f_1-f_3-\frac{1}{2})[\eta(X_3)X_1 - \eta(X_1)X_3]+ (f_4-f_6)[\eta(X_3)hX_1 - \eta(X_1)hX_3] - \eta(X_3)\beta(X_1)+ \eta(X_1)\beta(X_3)\} + B\{(f_1-f_3)[\eta(X_3)X_1 - g(X_1,X_3)\xi]+ (f_4-f_6)[\eta(X_3)hX_1 - g(hX_1,X_3)\xi] - \frac{1}{2}\eta(X_3)X_1+ \alpha(X_1,X_3)\xi - \beta(X_1)\eta(X_3) + \frac{1}{2}g(X_1,X_3)\xi\} + C_4(\xi)\tilde{S}(X_3,X_1).$$

Also,

$$(\tilde{\nabla}_{X_1}\tilde{S})(X_3,\xi) = \tilde{\nabla}_{X_1}\tilde{S}(X_3,\xi) - \tilde{S}(X_3,\tilde{\nabla}_{X_1}\xi) - \tilde{S}(\tilde{\nabla}_{X_1}X_3,\xi). \tag{28}$$

Using (11), (19) in (28), we obtain

(29)
$$(\tilde{\nabla}_{X_1}\tilde{S})(X_3,\xi) = \tilde{S}(X_3,\varphi X_1) + \tilde{S}(X_3,\varphi hX_1) - \tilde{S}(X_1,X_3) + \tilde{S}(X_3,\xi)\eta(X_1) - \tilde{S}(X_1,\xi)\eta(X_3) + g(X_1,X_3) + \tilde{S}(\xi,\xi).$$

Comparing (28) and (29), we have

$$\begin{aligned} &\tilde{S}(X_3,\varphi X_1) + \tilde{S}(X_3,\varphi hX_1) - \tilde{S}(X_1,X_3) \\ &+ \tilde{S}(X_3,\xi)\eta(X_1) - \tilde{S}(X_1,\xi)\eta(X_3) + g(X_1,X_3) + \tilde{S}(\xi,\xi) \\ &= [2n(f_1-f_3) - \frac{(2n-1)}{2} - trace(\alpha)][C_1(X_1)\eta(X_3) + C_3(X_3)\eta(X_1)] \\ &+ C_2\{(f_1-f_3-\frac{1}{2})[\eta(X_3)X_1 - \eta(X_1)X_3] + (f_4-f_6)[\eta(X_3)hX_1- \\ &\eta(X_1)hX_3] - \eta(X_3)\beta(X_1) + \eta(X_1)\beta(X_3)\} + B\{(f_1-f_3)[\eta(X_3)X_1- \\ &g(X_1,X_3)\xi] + (f_4-f_6)[\eta(X_3)hX_1 - g(hX_1,X_3)\xi] - \frac{1}{2}\eta(X_3)X_1+ \\ &\alpha(X_1,X_3)\xi - \beta(X_1)\eta(X_3) + \frac{1}{2}g(X_1,X_3)\xi\} + C_4(\xi)\tilde{S}(X_3,X_1). \end{aligned} \tag{30}$$

Setting $X_1 = X_2 = \xi$ in (30), it gives:

$$C_1(\xi) + C_2(\xi) + C_3(\xi) = 0. \tag{31}$$

In (26), consider $X_3 = \xi$, we get:

(32)
$$\begin{aligned}(\tilde{\nabla}_{X_1}\tilde{S})(\xi, X_4) = &[2n(f_1 - f_3) - \frac{(2n-1)}{2} - trace(\alpha)]\\ &[C_1(X_1)\eta(X_4) + C_4(X_4)\eta(X_1)]\\ &+ C_2\{(f_1 - f_3)[\eta(X_4)X_1 - g(X_1, X_4)\xi]+\\ &(f_4 - f_6)[\eta(X_4)hX_1 - g(hX_1, X_4)\xi]-\\ &\frac{1}{2}\eta(X_4)(X_1) + \alpha(X_1, X_4)\xi - \eta(X_4)\beta(X_1) + \frac{1}{2}g(X_1, X_4)\xi\}\\ &+ B\{(f_1 - f_3 - \frac{1}{2})[\eta(X_4)X_1 - \eta(X_1)X_4)]+\\ &(f_4 - f_6)[\eta(X_4)hX_1 - \eta(X_1)hX_4-\\ &\eta(X_4)\beta(X_1) + \beta(X_4)\eta(X_1)]\} + C_3(\xi)\tilde{S}(X_1, X_4).\end{aligned}$$

Also

(33)
$$\begin{aligned}(\tilde{\nabla}_{X_1}\tilde{S})(\xi, X_4) = &\tilde{S}(X_4, \varphi X_1) + \tilde{S}(X_4, \varphi hX_1) - \tilde{S}(X_1, X_4)\\ &+ \tilde{S}(X_4, \xi)\eta(X_1) - \tilde{S}(X_1, \xi)\eta(X_4) + g(X_1, X_4) + \tilde{S}(\xi, \xi).\end{aligned}$$

Using (32) and (33), we obtain

(34)
$$\begin{aligned}&[2n(f_1 - f_3) - \frac{(2n-1)}{2} - trace(\alpha)][C_1(X_1)\eta(X_4) + C_4(X_4)\eta(X_1)]\\ &+ C_2\{(f_1 - f_3)[\eta(X_4)X_1 - g(X_1, X_4)\xi] + (f_4 - f_6)[\eta(X_4)hX_1-\\ &g(hX_1, X_4)\xi] - \frac{1}{2}\eta(X_4)(X_1) + \alpha(X_1, X_4)\xi - \eta(X_4)\beta(X_1) + \frac{1}{2}g(X_1, X_4)\xi\}\\ &+ B\{(f_1 - f_3 - \frac{1}{2})[\eta(X_4)X_1 - \eta(X_1)X_4)] + (f_4 - f_6)[\eta(X_4)hX_1-\\ &\eta(X_1)hX_4 - \eta(X_4)\beta(X_1) + \beta(X_4)\eta(X_1)]\} + C_3(\xi)\tilde{S}(X_1, X_4) = \tilde{S}(X_4, \varphi X_1)+\\ &\tilde{S}(X_4, \varphi hX_1) - \tilde{S}(X_1, X_4) + \tilde{S}(X_4, \xi)\eta(X_1) - \tilde{S}(X_1, \xi)\eta(X_4) + g(X_1, X_4) + \tilde{S}(\xi, \xi).\end{aligned}$$

Taking $X_4 = \xi$ in (34), we have

$$\begin{aligned}&[2n(f_1 - f_3) - \frac{(2n-1)}{2} - trace(\alpha)][C_1(X_1) + C_4(\xi)\eta(X_1)+\\ &C_3(\xi)\eta(X_1)] + C_2\{(f_1 - f_3)[X_1 - \eta(X_1)\xi] + (f_4 - f_6)(hX_1)-\\ &\frac{1}{2}X_1 + \frac{1}{2}\eta(X_1)\xi - \beta(X_1) + \frac{1}{2}\eta(X_1)\xi\} + B\{(f_1 - f_3 - \frac{1}{2})[X_1-\\ &\eta(X_1)\xi] + (f_4 - f_6)(hX_1) - \beta(X_1) + \beta(\xi)\eta(X_1)\} = 0.\end{aligned} \tag{35}$$

Take $X_1 = \xi$ in (34)

$$\begin{aligned}&[2n(f_1 - f_3) - \frac{(2n-1)}{2} - trace(\alpha)][C_1(\xi)\eta(X_4) + C_4(X_4)+\\ &C_3(\xi)\eta(X_4)] + B\{(f_1 - f_3 - \frac{1}{2})[\eta(X_4)\xi - X_4]-\\ &(f_4 - f_6)(hX_4) + \beta(X_4) - \frac{1}{2}\eta(X_4)\xi\} = 0.\end{aligned} \tag{36}$$

Replace X_4 by X_1 in (36), it becomes

$$\begin{aligned}&[2n(f_1 - f_3) - \frac{(2n-1)}{2} - trace(\alpha)][C_1(\xi)\eta(X_1) + C_4(X_1)\\ &+ C_3(\xi)\eta(X_1)] + B\{(f_1 - f_3 - \frac{1}{2})[\eta(X_1)\xi - X_1] -\\ &(f_4 - f_6)(hX_4) + \beta(X_1) - \frac{1}{2}\eta(X_1)\xi\} = 0.\end{aligned} \tag{37}$$

Adding (35) and (37) and using equation (31), it gives

$$\begin{aligned}&[2n(f_1 - f_3) - \frac{(2n-1)}{2} - trace(\alpha)][C_1(X_1) + C_3(\xi)\eta(X_1) +\\ &C_4(X_1)] + C_2\{(f_1 - f_3 - \frac{1}{2})[X_1 - \eta(X_1)\xi] + (f_4 - f_6)(hX_1)\\ &- \beta(X_1) + \frac{1}{2}\eta(X_1)\xi\} = 0.\end{aligned} \tag{38}$$

Substitute $X_1 = \xi$ in (30), we obtain

$$\begin{aligned}&[2n(f_1 - f_3) - \frac{(2n-1)}{2} - trace(\alpha)][C_1(\xi)\eta(X_3) + C_3(X_3)\\ &+ C_4(\xi)\eta(X_3)] + C_2\{(f_1 - f_3 - \frac{1}{2})[\eta(X_3)\xi - X_3]\\ &- (f_4 - f_6)(hX_3) + \beta(X_3) - \frac{1}{2}\eta(X_3)\xi\} = 0.\end{aligned} \tag{39}$$

Replace X_3 by X_1 in (39), it becomes

$$\begin{aligned}&[2n(f_1 - f_3) - \frac{(2n-1)}{2} - trace(\alpha)][C_1(\xi)\eta(X_1) + C_3(X_1)\\ &+ C_4(\xi)\eta(X_1)] + C_2\{(f_1 - f_3 - \frac{1}{2})[\eta(X_1)\xi - X_1] -\\ &(f_4 - f_6)(hX_1) + \beta(X_1) - \frac{1}{2}\eta(X_1)\xi\} = 0.\end{aligned} \tag{40}$$

Adding (38) and (40), we have

$$C_1(X_1) + C_3(X_1) + C_4(X_1) = 0. \tag{41}$$

□

4. Weakly Ricci Symmetric Generalized (k, μ) Space Forms Admitting Semi-Symmetric Metric Connection

A non-flat Generalized (k, μ) space forms M^n is said to be weakly Ricci symmetric with respect to the semi-symmetric metric connection $\tilde{\nabla}$ if there exists a 1-form A_1, A_2, A_3 not simultaneously zero and the Ricci tensor $\tilde{S}$ with respect to the semi-symmetric metric connection is not identically zero and satisfies

$$(\tilde{\nabla}_{X_1}\tilde{S})(X_2, X_3) = A_1(X_1)\tilde{S}(X_2, X_3) + A_2(X_2)\tilde{S}(X_1, X_3) + A_3(X_3)\tilde{S}(X_2, X_1), \tag{42}$$

for all $X_1, X_2, X_3 \in \chi(M)$.

Theorem 4.1. *The necessary condition for a generalized (k, μ) space forms with respect to the semi-symmetric metric connection $\tilde{\nabla}$ to be weakly symmetric with respect to $\tilde{\nabla}$ is that the sum of the associated 1-forms A_1, A_2, A_3 vanishes every where.*

Proof. On replacing X_3 by ξ in (42), we get

$$(\tilde{\nabla}_{X_1}\tilde{S})(X_2, \xi) = A_1(X_1)\tilde{S}(X_2, \xi) + A_2(X_2)\tilde{S}(X_1, \xi) + A_3(\xi)\tilde{S}(X_2, X_1), \tag{43}$$

which implies on using equation (29)

$$\begin{aligned} &\tilde{S}(X_3, \varphi X_1) + \tilde{S}(X_3, \varphi h X_1) - \tilde{S}(X_1, X_3) + \tilde{S}(X_3, \xi)\eta(X_1) - \tilde{S}(X_1, \xi)\eta(X_3) \\ &+ g(X_1, X_3) + \tilde{S}(\xi, \xi) = [2n(f_1 - f_3) - \frac{(2n-1)}{2} - trace(\alpha)][A_1(X_1)\eta(X_2) \\ &+ A_2(X_2)\eta(X_1)] + A_3(\xi)\tilde{S}(X_2, X_1). \end{aligned} \tag{44}$$

In (44) setting $X_1 = X_2 = \xi$, we obtain

$$A_1(\xi) + A_2(\xi) + A_3(\xi) = 0. \tag{45}$$

Now substituting $X_2 = \xi$, equation (45) becomes:

$$A_1(X_1) = A_1(\xi)\eta(X_1), \tag{46}$$

$$A_2(X_2) = A_2(\xi)\eta(X_2), \tag{47}$$

and

$$A_3(X_3) = A_3(\xi)\eta(X_3). \tag{48}$$

Taking the sum of (46), (47) and (48), we obtain

$$A_1(X_1) + A_2(X_1) + A_3(X_1) = 0. \tag{49}$$

□

5. Special Weakly Ricci-Symmetric Generalized (k, μ) Space Forms Admitting Semi-Symmetric Metric Connection

A non-flat Generalized (k, μ) space forms M^n is said to be special weakly Ricci symmetric with respect to the semi-symmetric metric connection $\tilde{\nabla}$ if

$$(\tilde{\nabla}_{X_1}\tilde{S})(X_2, X_3) = 2A(X_1)\tilde{S}(X_2, X_3) + 2A(X_2)\tilde{S}(X_1, X_3) + 2A(X_3)\tilde{S}(X_2, X_1), \tag{50}$$

where A is a 1-form and is defined by $A(X_1) = g(X_1, \rho)$ for associated vector field ρ (See [9, 15]).

Theorem 5.1. *In a generalized (k, μ) space forms with respect to a semi-symmetric metric connection $\tilde{\nabla}$ can not be a special weakly Ricci-symmetric manifold.*

Proof. Taking $X_3 = \xi$ in (50) and using (9) and (23)

$$\begin{aligned} &(\tilde{\nabla}_{X_1}\tilde{S})(X_2, \xi) = \\ &[2n(f_1 - f_3) - \frac{(2n-1)}{2} - trace(\alpha)][2A(X_1)\eta(X_2) + \\ &A(X_2)\eta(X_1)] + A(\xi)\tilde{S}(X_2, X_1), \end{aligned} \tag{51}$$

we know that

$$(\tilde{\nabla}_{X_1}\tilde{S})(X_2,\xi) = \tilde{\nabla}_{X_1}\tilde{S}(X_2,\xi) - \tilde{S}(X_2,\tilde{\nabla}_{X_1}\xi) - \tilde{S}(\tilde{\nabla}_{X_1}X_2,\xi). \tag{52}$$

In sequence of (19) and (23), (52) becomes

$$\begin{aligned}(\tilde{\nabla}_{X_1}\tilde{S})(X_2,X_3) = [2n(f_1-f_3) - \frac{(2n-1)}{2} - trace(\alpha)]\\ [(\tilde{\nabla}_{X_1}\eta)X_2 + \eta(X_1)\eta(X_2)]+\\ \tilde{S}(X_2,\phi X_1) + \tilde{S}(X_2,\phi h X_1) - \tilde{S}(X_2,X_1).\end{aligned} \tag{53}$$

Now (53) with equations (9), (19), (23), (51) and $X_1 = \xi$ gives

$$[2n(f_1-f_3) - \frac{(2n-1)}{2} - trace(\alpha)][3A(\xi)\eta(X_2) + A(X_2)]. \tag{54}$$

Setting $X_2 = \xi$ in (54) and using (9),we have

$$[2n(f_1-f_3) - \frac{(2n-1)}{2} - trace(\alpha)]A(\xi) = 0, \tag{55}$$

which implies

$$A(\xi) = 0. \tag{56}$$

In view of (54), (56) yields

$$A(X_2) = 0, \tag{57}$$

which is inadmissible. □

Theorem 5.2. *If a generalized (k,μ) space forms with respect to a semi-symmetric metric connection $\tilde{\nabla}$ with cyclic Ricci tensor, then M^n can not be a special weakly Ricci-symmetric manifold.*

Proof. Now taking cyclic sum of (50), we have

$$\begin{aligned}(\tilde{\nabla}_{X_1}\tilde{S})(X_2,X_3) + (\tilde{\nabla}_{X_2}\tilde{S})(X_3,X_1) + (\tilde{\nabla}_{X_3}\tilde{S})(X_1,X_2) =\\ 4[A(X_1)\tilde{S}(X_2,X_3) + A(X_2)\tilde{S}(X_1,X_3) + A(X_3)\tilde{S}(X_2,X_1)].\end{aligned} \tag{58}$$

If (M,g) admits a cyclic Ricci tensor, that is:
$(\tilde{\nabla}_{X_1}\tilde{S})(X_2,X_3) + (\tilde{\nabla}_{X_2}\tilde{S})(X_3,X_1) + (\tilde{\nabla}_{X_3}\tilde{S})(X_1,X_2) = 0,$
then (58) reduces to

$$A(X_1)\tilde{S}(X_2,X_3) + A(X_2)\tilde{S}(X_1,X_3) + A(X_3)\tilde{S}(X_2,X_1) = 0. \tag{59}$$

Setting $Z = \xi$ in (59) and using (23), we obtain

$$[2n(f_1-f_3) - \frac{(2n-1)}{2} - trace(\alpha)][A(X_1)\eta(X_2) + A(X_2)\eta(X_1)] + A(X_3)\tilde{S}(X_2,X_1). \tag{60}$$

Also setting $X_1 = \xi$ in (60) and using (9) and (23), we have

$$[2n(f_1-f_3) - \frac{(2n-1)}{2} - trace(\alpha)][2A(\xi)\eta(X_2) + \eta(X_2)] = 0. \tag{61}$$

Again setting $X_2 = \xi$ in (61) and using (9), we get

$$A(\xi) = 0. \tag{62}$$

In consequence of (62), (61) gives $A(X_2) = 0$, which is a contradiction. □

6. Generalized Ricci-Reccurent Generalized (k,μ) Space Forms with respect to the Semi-Symmetric Metric Connection

A non-flat Generalized (k,μ) space forms M^n is said to be Ricci-recurrent manifold [6] with respect to the semi-symmetric metric connection $\tilde{\nabla}$, if its Ricci tensor S satisfies the condition

$$(\tilde{\nabla}_{X_1}\tilde{S})(X_2,X_3) = A(X_1)\tilde{S}(X_2,X_3) + B(X_1)g(X_2,X_3), \tag{63}$$

where $\tilde{\nabla}$ is the Riemannian connection of the Riemannian metric g and A, B are 1-forms associated with the vector fields P_1, P_2 respectively on M, that is:

$$A(X_1) = g(X_1,P_1); B(X_1) = g(X_1,P_2), \tag{64}$$

for arbitrary vector fields X_1, X_2 and X_3. If the 1-form B vanishes identically, the manifold M^n reduces to the well known Ricci recurrent manifold (See [11]).

Theorem 6.1. *If a generalized Ricci-recurrent generalized (k,μ) space form with respect to the semi-symmetric metric connection, then the associated vector fields of the 1-forms A and B are in the opposite direction.*

Proof. Let M^n be a generalized Ricci-recurrent generalized (k,μ) space form with respect to the semi-symmetric metric connection. It is known that

$$(\tilde{\nabla}_{X_1}\tilde{S})(X_2,X_3) = \tilde{\nabla}_{X_1}\tilde{S}(X_2,X_3) - \tilde{S}(X_2,\tilde{\nabla}_{X_1}X_3) - \tilde{S}(\tilde{\nabla}_{X_1}X_2,X_3), \tag{65}$$

for arbitrary vector fields X_1, X_2 and X_3. From equations (63) and (65), we get

$$\begin{aligned} &A(X_1)\tilde{S}(X_2,X_3) + B(X_1)g(X_2,X_3) = \tilde{\nabla}_{X_1}\tilde{S}(X_2,X_3) - \\ &\tilde{S}(X_2,\tilde{\nabla}_{X_1}X_3) - \tilde{S}(\tilde{\nabla}_{X_1}X_2,X_3). \end{aligned} \tag{66}$$

Replacing Z by ξ in above equation and using (9), (19) and (23), we have

$$\begin{aligned} &[2n(f_1-f_3) - \frac{(2n-1)}{2} - trace(\alpha)]A(X_1)\eta(X_2) + B(X_1)\eta(X_2) = \\ &[2n(f_1-f_3) - \frac{(2n-1)}{2} - trace(\alpha)][(\tilde{\nabla}_{X_1})(Y) + \eta(X_1)\eta(X_2)] \\ &+ \tilde{S}(X_2,\phi X_1) + \tilde{S}(X_2,\phi hX_1) - \tilde{S}(X_2,X_1). \end{aligned} \tag{67}$$

In consequence of (20), (67) becomes:

$$\begin{aligned} &[2n(f_1-f_3) - \frac{(2n-1)}{2} - trace(\alpha)]A(X_1)\eta(X_2) + B(X_1)\eta(X_2) \\ &= [2n(f_1-f_3) - \frac{(2n-1)}{2} - trace(\alpha)] \\ &[-g(X_2,\phi X_1) - g(X_2,\phi hX_1) + g(X_2,X_1)] \\ &+ \tilde{S}(X_2,\phi X_1) + \tilde{S}(X_2,\phi hX_1) - \tilde{S}(X_2,X_1). \end{aligned} \tag{68}$$

Putting $Y=\xi$ in (68) and using (9), we obtain

$$[2n(f_1-f_3) - \frac{(2n-1)}{2} - trace(\alpha)]A(X_1) + B(X_1) = 0. \tag{69}$$

□

Theorem 6.2. *If a generalized Ricci-recurrent generalized (k,μ) space form with respect to the semi-symmetric metric connection admits a cyclic Ricci tensor, then manifold is an Einstein manifold, provided $A(\xi) \neq 0$.*

Proof. Let us consider that a generalized Ricci-recurrent generalized (k,μ) space form M_n admits a cyclic Ricci tensor S, that is:

$$(\tilde{\nabla}_{X_1}\tilde{S})(X_2,X_3) + (\tilde{\nabla}_{X_2}\tilde{S})(X_3,X_1) + (\tilde{\nabla}_{X_3}\tilde{S})(X_1,X_2) = 0, \tag{70}$$

for arbitrary vector fields X_1, X_2 and X_3. In view of (63), (70) follows that

$$\begin{aligned} &A(X_1)\tilde{S}(X_2,X_3) + B(X_1)g(X_2,X_3) + A(X_2)\tilde{S}(X_3,X_1) \\ &+ B(X_2)g(X_3,X_1) + A(X_3)\tilde{S}(X_1,X_2) + B(X_3)g(X_1,X_2) = 0. \end{aligned} \tag{71}$$

Replacing Z by ξ in (71) and using (10) and (23), we have

$$\begin{aligned} &\{[2n(f_1 - f_3) - \frac{(2n-1)}{2} - trace(\alpha)]A(X_1) + B(X_1)\}\eta(X_2) \\ &+ \{[2n(f_1 - f_3) - \frac{(2n-1)}{2} - trace(\alpha)]A(X_2) + B(X_2)\}\eta(X_1) \\ &+ A(\xi)\tilde{S}(X_1,X_2) + B(\xi)g(X_1,X_2) = 0. \end{aligned} \tag{72}$$

In view of (69), (72) gives

$$\tilde{S}(X_1,X_2) = -\frac{B(\xi)}{A(\xi)}g(X_1,X_2), \tag{73}$$

where $B(\xi) \neq 0$ and $A(\xi) \neq 0$. □

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