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A CRITERION FOR A LIE GROUP TO ADMIT A CONTINUOUS EMBEDDING IN A MOTION GROUP

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Abstract. It is proved that a connected Lie group G admits a continuous embedding in a motion Lie group (a Lie extension of an Abelian connected Lie group by a compact connected Lie group) if and only if the Lie group G has an Abelian normal subgroup N such that the quotient group G/N is a direct product of a compact topological group L and an Abelian group A and this product $L \times A$ admits a compactification Q such that the following diagram of Lie groups with exact rows, continuous arrows, embeddings $f_{L\times A}$ and f_G and the identity mapping f_N is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{\iota} & G & \xrightarrow{\pi} & L\times A & \longrightarrow & 0\\ \downarrow & & f_N\downarrow & & f_G\downarrow & & f_{L\times A}\downarrow & & \downarrow \\ 0 & \longrightarrow & N & \xrightarrow{\widetilde{\iota}} & G' & \xrightarrow{\widetilde{\pi}} & Q & \longrightarrow & 0. \end{array}$$

Here f_G is an embedding of G in the motion group G'.

§ 1. Introduction

By a motion Lie group we mean a Lie extension of an Abelian Lie group by a compact connected Lie group. We consider connected Lie groups that can be continuously embedded in a motion Lie group.

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§ 2. Preliminaries

By the famous Freudenthal–Weil theorem [1], a connected locally compact group that can continuously be embedded in a compact topological group is a direct product of a compact connected group and a vector group. As was proved in [2], the continuity condition is not needed here.

If a connected Lie group G can be embedded continuously by a mapping θ in a motion Lie group H which is a Lie extension of a vector group V by a compact connected Lie group K, i.e., if we have an exact sequence of connected Lie groups and their continuous homomorphisms

$$0 \longrightarrow V \xrightarrow{\ i\ } H \xrightarrow{\ \rho\ } K \longrightarrow 0,$$

then either $\theta^{-1}(V) = \{e_G\}$, in which case there is an embedding of G in K, and hence the group G is a direct product of a connected compact group and a vector group, or $\theta^{-1}(V) = N \neq \{e_G\}$, where N is obviously Abelian since θ is an embedding. Note that N is closed as the preimage of the closed normal subgroup V. The composition of θ and the continuous canonical epimorphism ρ of H onto K defines a homomorphism of G into a compact connected group K with kernel C. Thus, the quotient Lie group G/N is isomorphic to the direct product of a compact connected group L and a vector group A and, therefore, the following exact sequence holds:

$$0 \longrightarrow N \xrightarrow{\ \iota\ } G \xrightarrow{\ \pi\ } L \times A \longrightarrow 0.$$

§ 3. Main result

Theorem. *A a connected Lie group G admits a continuous embedding in a motion Lie group (a Lie extension of an Abelian connected Lie group by a compact connected Lie group) if and only if the Lie group G has an Abelian normal subgroup N such that the quotient group G/N is a direct product of a compact topological group L and an Abelian group A and this product $L \times A$ admits a compactification Q such that the following diagram of Lie groups with exact rows, continuous arrows, embeddings $f_{L\times A}$ and f_G and the identity mapping f_N is commutative:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{\ \iota\ } & G & \xrightarrow{\ \pi\ } & L \times A & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle f_N} & & \downarrow{\scriptstyle f_G} & & \downarrow{\scriptstyle f_{L\times A}} & & \downarrow \\
0 & \longrightarrow & N & \xrightarrow{\ \widetilde{\iota}\ } & G' & \xrightarrow{\ \widetilde{\pi}\ } & Q & \longrightarrow & 0.
\end{array} \tag{1}$$

Here f_G is an embedding of G in the motion group G'.

Proof. If the embedding in question exists, then, as was shown above, the group G admits the desired structure. The very group H and the original embedding define the mappings that enter the commutative diagram we need.

Conversely, if the commutative diagram (1) takes place, then G' is the motion group in which G is thus embedded.

§ 4. Comments

Example. The structure condition (that a Lie group G has an Abelian normal subgroup N such that the quotient group G/N is a direct product of a compact topological group L and an Abelian group A) is insufficient for the existence of an embedding of a connected Lie group in a motion group. Indeed, the conjugacy classes in a motion group are obviously compact. Consider the solvable Lie group M of matrices $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$, $a \in \mathbb{C}^*$, $b \in \mathbb{C}$; it has, for example, the conjugacy classes $\{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) | b \in \mathbb{C}^*\}$ and $\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$. It is clear that the closure of the first class contains the other class for any compactification of the quotient group $\mathbb{C}^* = \{a\}$. Hence, such a compactification cannot admit an extension of the existing action of the diagonal subgroup of G on the subgroup of triangular matrices, and the group in question, having the desired structure, cannot be embedded in a motion group.

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