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NOTE ON λ-LINEAR FUNCTION ARISING FROM THE p-ADIC FERMIONIC INTEGRAL ON $\mathbb{Z}_p$

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Abstract. The p-adic Fermionic integral over $\mathbb{Z}_p$ is one of the important techniques among the many ways to investigate and construct generating functions for special polynomials and numbers.

In this paper, we consider two λ-linear functionals on $\mathbb{C}_p[x]$ arising from the p-adic fermionic integral on $\mathbb{Z}_p$ and $\mathbb{Z}_p^r = \mathbb{Z}_p \times \cdots \mathbb{Z}_p$, respectively, by using umbral calculus and λ-umbral calculus. In other word, we study to determine the linear functionals given by $P(x) \to \int_{\mathbb{Z}_p} P(x)d\mu_{-1}(x)$ and $P(x) \to \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} P(x_1+\cdots+x_r)d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$, respectively.

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1. Introduction

The p-adic integral has a very important role in physics, mathematics and engineering. Recently, many mathematicians have studied various degenerate versions of special polynomials and numbers and have yielded many interesting results. Kim constructed the p-adic q-Volkenborn integration [13]. When $q = -1$, it is called the p-adic Fermionic integral on $\mathbb{Z}_p$. The p-adic Fermionic integral over $\mathbb{Z}_p$ is one of the important techniques among the many ways to investigate and construct generating functions for special polynomials and numbers. In addition, the degenerate versions of special numbers and polynomials have been much studied so far and have important applications in a variety of natural and social sciences.

In this paper, first, we consider linear functional on $\mathbb{C}_p[x]$ arising from the p-adic fermionic integral on on $\mathbb{Z}_p$. We study to determine the linear functionals given by $P(x) \to \int_{\mathbb{Z}_p} P(x)d\mu_{-1}(x)$ which is given by the generating function of degenerate Euler polynomials. Second, we consider linear functional on $\mathbb{C}_p[x]$ arising from the mutivariate p-adic fermionic integral on $\mathbb{Z}_p^r = \mathbb{Z}_p \cdots \mathbb{Z}_p$. We study to determine the linear functionals given by y $P(x) \to \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} P(x_1 + \cdots + x_r)d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$ which is given by the generating function of degenerate Euler polynomials of order r. In other word, we show that λ-differentiations of any polynomial by such generating functions can be expressed by p-adic fermionic integrals on $\mathbb{Z}_p$ or $\mathbb{Z}_p^r$.

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For any $\lambda \in \mathbb{R}$,

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad \text{(see [5-9])}, \tag{1}$$

where $(x)_{0,\lambda} = 1$ and $(x)_{n,\lambda} = x(x-\lambda)\cdots\big(x-(n-1)\lambda\big)$, $(n \geq 1)$.

For $r \in \mathbb{N}$ and $\lambda, t \in \mathbb{Z}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$, the degenerate Euler polynomials $E_{n,\lambda}^{(r)}(x)$ of order r are defined by the generating function to be

$$\sum_{n=0}^{\infty} E_{n,\lambda}^{(r)}(x) \frac{t^n}{n!} = \left(\frac{2}{e_\lambda(t)+1}\right)^r e_\lambda^x(t), \quad \text{(see [5])}. \tag{2}$$

When $x = 0$, $E_{n,\lambda}^{(r)} = E_{n,\lambda}^{(r)}(0)$ are called the degenerate Euler numbers of order r.

When $r = 1$, the ordinary degenerate Euler polynomials are given by

$$\frac{2}{e_\lambda(t)+1} e_\lambda^x(t) = \sum_{n=0}^{\infty} E_{n,\lambda}(x) \frac{t^n}{n!}, \quad \text{(see [6])}. \tag{3}$$

Let p be chosen as an odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of an algebraic closure of $\mathbb{Q}_p$. Let$|\cdot|_p$ be the p-adic norm normalized as $|p|_p = \frac{1}{p}$.

For a $\mathbb{C}_p$-valued continuous function f on $\mathbb{Z}_p$, the p-adic Fermionic integral on $\mathbb{Z}_p$ is defined by Kim [14] as follows:

$$\begin{aligned} I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) &= \lim_{\mathbb{N}\to\infty} \sum_{x=0}^{p^{\mathbb{N}}-1} f(x) \mu_{-1}(x + p^{\mathbb{N}}\mathbb{Z}_p) \\ &= \lim_{\mathbb{N}\to\infty} \sum_{x=0}^{p^{\mathbb{N}}-1} f(x)(-1)^x, \quad \text{(see [5, 6, 12])}. \end{aligned} \tag{4}$$

Let $f_n(x) = f(x+n)$ for $n \in \mathbb{N}$. From (4), we observe that

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \quad \text{(see [5, 6, 12])}. \tag{5}$$

In (5), when $n = 1$, we have

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0). \tag{6}$$

Let $f(x) = e_\lambda^x(t)$. Then

$$\int_{\mathbb{Z}_p} e_\lambda^x(t) d\mu_{-1}(x) = \frac{2}{e_\lambda(t)+1} = \sum_{n=0}^{\infty} E_{n,\lambda} \frac{t^n}{n!}, \tag{7}$$

and

$$\int_{\mathbb{Z}_p} (x)_{n,\lambda} d\mu_{-1}(x) = E_{n,\lambda}. \tag{8}$$

Let $f(x)=e_\lambda^{x+y}(t)$. Then

(9)
$$\int_{\mathbb{Z}_p} e_\lambda^{x+y}(t)d\mu_{-1}(y)=\left(\frac{2}{e_\lambda(t)+1}\right)e_\lambda^x(t)=\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!},$$
and
$$\int_{\mathbb{Z}_p}(x+y)_{n,\lambda}d\mu_{-1}(x)=E_{n,\lambda}(x)\quad \text{(see [6]).}$$

2. Review of λ-Umbral calculus

In this section, we introduced λ-Sheffer sequences.

Let $\mathcal{F}=\left\{f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\middle|a_k\in\mathbb{C}_p\right\}$ be the algebra of all formal power series in t with coefficients in $\mathbb{C}_p$. Let $\mathbb{P}=\mathbb{C}_p[x]$ be the ring of all polynomials in x with coefficients in $\mathbb{C}_p$, and let $\mathbb{P}*$ denote the vector space of all linear functionals on $\mathbb{P}$ (see[9, 18, 19]). Let $\langle L|P(x)\rangle$ denote the action of the linear functional L on the polynomial $P(x)$.

For $f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\in\mathcal{F}$ and a fixed nonzero real number λ, each λ gives rise to the linear functional $\langle f(t)\mid\cdot\rangle_\lambda$ on $\mathbb{P}$, called λ-linear functional given by $f(t)$, which is defined by

(10)
$$\langle f(t)\mid(x)_{n,\lambda}\rangle_\lambda=a_n,\quad \text{for all }\ n\geq 0\qquad \text{(see [9]).}$$

In particular $\langle t^k\mid(x)_{n,\lambda}\rangle_\lambda=n!\delta_{n,k}$, for all $n,\ k\ \geq 0$, where $\delta_{n,k}$ is the Kronecker's symbol.

For $\lambda=0$, we observe that the linear functional $\langle f(t)\mid\cdot\rangle_0$ agrees with the one in $\langle f(t)\mid x^n\rangle=a_n,\ (k\geq 0)$.

For each $\lambda\in\mathbb{R}$ and each nonnegative integer k, the λ-differential operator t^k on $\mathbb{P}$ is defined by

(11)
$$(t^k)_\lambda(x)_{n,\lambda}=\begin{cases}(n)_k(x)_{n-k,\lambda}, & \text{if}\quad k\leq n,\\ 0 & \text{if}\quad k\geq n,\quad \text{(see [9]),}\end{cases}$$

and for any power series $f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\in\mathcal{F}$,
$$(f(t))_\lambda(x)_{n,\lambda}=\sum_{k=0}^{n}\binom{n}{k}a_k(x)_{n-k,\lambda},\quad (n\geq 0).$$

The order $o(f(t))$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. The series $f(t)$ is called invertible if $o(f(t))=0$ and such series has a multiplicative inverse $1/f(t)$ of $f(t)$. $f(t)$ is called a delta series if $o(f(t))=1$ and it has a compositional inverse $\overline{f}(t)$ of $f(t)$ with $\overline{f}(f(t))=f(\overline{f}(t))=t$.

Let $f(t)$ and $g(t)$ be a delta series and an invertible series, respectively. Then there exists a unique sequences $s_{n,\lambda}(x)$ such that the orthogonality conditions holds

(12)
$$\langle g(t)\big(f(t)\big)^k\mid s_{n,\lambda}(x)\rangle_\lambda=n!\delta_{n,k},\quad (n,k\geq 0)\qquad \text{(see [9]).}$$

The sequence $s_{n,\lambda}(x)$ is called the λ-Sheffer sequence for $(g(t), f(t))$, which are denoted by $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$.

The sequence $s_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ if and only if

$$\frac{1}{g\big(\overline{f}(t)\big)} e_\lambda^x\big(\overline{f}(t)\big) = \sum_{k=0}^{\infty} \frac{s_{k,\lambda}(x)}{k!} t^k \quad (n, k \geq 0) \qquad \text{(see [9])}. \tag{13}$$

Assume that for each $\lambda \in \mathbb{R}^*$ of the set of nonzero real numbers, $s_{n,\lambda}(x)$ is λ-Sheffer for $(g_\lambda(t), f_\lambda(t))$. Assume also that $\lim_{\lambda\to 0} f_\lambda(t) = f(t)$ and $\lim_{\lambda\to 0} g_\lambda(t) = g(t)$, for some delta series $f(t)$ and an invertible series $g(t)$. Then $\lim_{\lambda\to 0} \overline{f}_\lambda(t) = \overline{f}(t)$, where is the compositional inverse of $f(t)$ with $\overline{f}(f(t)) = f(\overline{f}(t)) = t$. Let $\lim_{\lambda\to 0} s_{k,\lambda}(x) = s_k(x)$.

The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$, which are denoted by $s_n(x) \sim (g(t), f(t))$.

The sequence $s_n(x) \sim (g(t), f(t))$ if and only if

$$\frac{1}{g\big(\overline{f}(t)\big)} e^x\big(\overline{f}(t)\big) = \sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k \quad (n, k \geq 0) \qquad \text{(see [18])}. \tag{14}$$

Let $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), g(t))$, $(n \geq 0)$. Then

(15)

$$s_n(x) = \sum_{k=0}^{n} z_{n,k} r_k(x), \quad (n \geq 0),$$

$$\text{where} \quad z_{n,k} = \frac{1}{k!}\left\langle \frac{h(\overline{f}(t))}{g(\overline{f}(t))}\big(l(\overline{f}(t))\big)^k \mid x^n \right\rangle, \quad (n, k \geq 0), \qquad \text{(see [18])}.$$

Let $u_\lambda(t) = \frac{1}{\lambda}(e^{\lambda t} - 1)$. Then the compositional inverse function of $u_\lambda(t)$ is

$$\overline{u}_\lambda(t) = \frac{1}{\lambda}\log(1 + \lambda t). \tag{16}$$

Let $f(t), g(t) \in F$ with $o(f(t)) = 1$ and $o(g(t)) = 0$. From (13) and (14), we observe that

$$\begin{aligned} &S_{n,\lambda}(x) \sim (g(u_\lambda(t)), f(u_\lambda(t))) \\ &\Leftrightarrow \sum_{n=0}^{\infty} S_{n,\lambda}(x)\frac{t^n}{n!} = \frac{1}{g(\bar{f}(t))} e^{x\bar{u}_\lambda(\bar{f}(t))} = \frac{1}{g(\bar{f}(t))} e_\lambda^x(\bar{f}(t)) \\ &\Leftrightarrow S_{n,\lambda}(x) \sim (g(t), f(t))_\lambda. \end{aligned} \tag{17}$$

From (17), we have

(18)

$$S_{n,\lambda}(x) \sim (g(t), f(t))_\lambda \Leftrightarrow S_{n,\lambda}(x) \sim (g(u_\lambda(t)), f(u_\lambda(t))) \qquad \text{(see [14])}.$$

Let $S_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$. Then from (12), we observe that

(19)
$$\langle g(t)f(t)^k|(f(t))_\lambda S_{n,\lambda}(x)\rangle_\lambda = \langle g(t)f(t)^{k+1}|S_{n,\lambda}(x)\rangle_\lambda = n!\delta_{n,k+1} = n(n-1)!\delta_{n-1,k} = \langle g(t)f(t)^k|nS_{n-1,\lambda}(x)\rangle_\lambda.$$

Thus, from (18) and (19), we have

$$(f(t))_\lambda S_{n,\lambda}(x) = nS_{n-1,\lambda}(x). \tag{20}$$

Since $\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!} = e_\lambda^x(t) = e^{x\bar{u}_\lambda(t)}$,

$$(x)_{n,\lambda} \sim (1, u_\lambda(t)). \tag{21}$$

From (11), (20) and (21), we note that $u_\lambda(t)(x)_{n,\lambda} = n(x)_{n-1,\lambda}$ and

$$(t^k)_\lambda(x)_{n,\lambda} = (n)_k(x)_{n-k,\lambda} = (u_\lambda(t))^k(x)_{n,\lambda}. \tag{22}$$

For any $f(t) \in F$ and $P(x) \in \mathbb{P} = \mathbb{C}_p[x]$, by (22), we have

$$(f(t))_\lambda P(x) = f(u_\lambda(t))P(x). \tag{23}$$

Thus, from (20) and (23), we have

$$(f(t))_\lambda S_{n,\lambda}(x) = f(u_\lambda(t))S_{n,\lambda}(x) = nS_{n-1,\lambda}(x) \quad \text{(see [14])}. \tag{24}$$

3. The λ-linear functional associated with degenerate Euler numbers

In this section, we introduce the λ-linear functional $f(t)$ that satisfies

$$\langle f(t)|P(x)\rangle_\lambda = \int_{\mathbb{Z}_p} p(x)d\mu_{-1}(x), \tag{25}$$

for all polynomials $P(x) \in \mathbb{P} = \mathbb{C}_p[x]$.

Theorem 3.1. *For $P(x) \in \mathbb{P}$, we have*

$$\int_{\mathbb{Z}_p} P(x)d\mu_{-1}(x) = \left\langle \int_{\mathbb{Z}_p} e_\lambda^x(t)d\mu_{-1}(x)\middle|P(x)\right\rangle_\lambda = \left\langle \frac{2}{e_\lambda(t)+1}\middle|P(x)\right\rangle_\lambda.$$

Proof. From (1) and (25), we observe that

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty}\frac{\langle f(t)|(x)_{k,\lambda}\rangle_\lambda}{k!}t^k = \sum_{k=0}^{\infty}\int_{\mathbb{Z}_p}(x)_{k,\lambda}d\mu_{-1}(x)\frac{t^k}{k!} \\ &= \int_{\mathbb{Z}_p}\sum_{k=0}^{\infty}(x)_{k,\lambda}\frac{t^k}{k!}d\mu_{-1}(x) = \int_{\mathbb{Z}_p}e_\lambda^x(t)d\mu_{-1}(x) = \frac{2}{e_\lambda(t)+1}. \end{aligned} \tag{26}$$

Therefore, by (26), we obtain the desired result. □

Theorem 3.2. *For any $P(x) \in \mathbb{P}$, we have*

$$\int_{\mathbb{Z}_p} P(x+y)d\mu_{-1}(y) = \left(\frac{2}{e_\lambda(t)+1}\right)_\lambda P(x) = \left(\frac{2}{e^t+1}\right)P(x).$$

Proof. First, for $P(x) = (x)_{n,\lambda}$, from (8) and (10), we observe that

$$E_{n,\lambda} = \left\langle \int_{\mathbb{Z}_p} e_\lambda^y(t) d\mu_{-1}(y) \middle| (x)_{n,\lambda} \right\rangle_\lambda . \tag{27}$$

From (3) and (17), we note that

$$E_{n,\lambda}(x) \sim \left(\frac{e_\lambda(t)+1}{2}, t \right)_\lambda \text{ and } E_{n,\lambda}(x) \sim \left(\frac{e^t+1}{2}, t \right). \tag{28}$$

From (13) and (28), we get

$$\left(\frac{e_\lambda(t)+1}{2} \right)_\lambda E_{n,\lambda}(x) \sim (1,t)_\lambda. \tag{29}$$

Moreover, we note that the compositional inverse $\overline{u}_\lambda(t) = \frac{1}{\lambda}\log(1+\lambda t)$ of $u_\lambda(t) = \frac{1}{\lambda}(e^{\lambda t}-1)$.

Since $\sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} = e_\lambda^x(t) = e^{x\bar{u}_\lambda(t)}$, from (13) and (14), we get

$$(x)_{n,\lambda} \sim (1,t)_\lambda \text{ and } (x)_{n,\lambda} \sim (1, u_\lambda(t)). \tag{30}$$

From (22) and (30),

$$(t)_\lambda E_{n,\lambda}(x) = u_\lambda(t) E_{n,\lambda}(x) = n E_{n-1,\lambda}(x). \tag{31}$$

From (23), (29) and (31), the uniqueness of the Sheffer sequences and noting that $g(u_\lambda(t)) = \frac{e^t+1}{2}$, we obtain

(32)

$$\begin{aligned}\int_{\mathbb{Z}_p} (x+y)_{n,\lambda} d\mu_{-1}(y) = E_{n,\lambda}(x) &= \left(\frac{2}{e_\lambda(t)+1} \right)_\lambda \left(\frac{e_\lambda(t)+1}{2} \right)_\lambda E_{n,\lambda}(x) \\ &= \left(\frac{2}{e_\lambda(t)+1} \right)_\lambda (x)_{n,\lambda} = \left(\frac{2}{e^t+1} \right) (x)_{n,\lambda}.\end{aligned}$$

Therefore, from (32), we obtain the desired result. □

Examples.

(a) Choosing $P(x) = x^n$ in Theorem 2 , we get

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = \left(\frac{2}{e_\lambda(t)+1} \right)_\lambda x^n = \left(\frac{2}{e^t+1} \right) x^n.$$

(b) Let $P(x) = \sum_{k=0}^{n} \binom{n}{k} x^k + x^n$ in Theorem 2. Then, we have

$$\begin{aligned}\int_{\mathbb{Z}_p} P(x+y) d\mu_{-1}(y) &= \sum_{k=0}^{n} \binom{n}{k} E_k(x) + E_n(x) \\ &= 2x^n = E_n(x+1) + E_n(x).\end{aligned}$$

(c)Let $P(x) = \sum_{k=0, k\neq 1}^{n} \binom{n}{k} B_k x^{n-k}$ in Theorem 2, then we obtain

$$\int_{\mathbb{Z}_p} P(x+y) d\mu_{-1}(y) = \sum_{\substack{k=0 \\ k\neq 1}}^{n} \binom{n}{k} B_k E_{n-k}(x) = B_n(x).$$

4. The λ-linear functional associated with higher-order degenerate Euler numbers

For $r \in \mathbb{N}$ and $\lambda, t \in \mathbb{Z}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$, the degenerate Euler polynomials $E_{n,\lambda}^{(r)}(x)$ of order r are defined by the generating function to be

$$\sum_{n=0}^{\infty} E_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} = \left(\frac{2}{e_\lambda(t)+1}\right)^r e_\lambda^x(t), \quad \text{(see [4])}. \tag{33}$$

When $x = 0$, $E_{n,\lambda}^{(r)} = E_{n,\lambda}^{(r)}(0)$ are called the degenerate Euler numbers of order r.

From (9) and (33), we have

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e_\lambda^{x_1+\cdots+x_r+x}(t)d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1+\cdots+x_r+x)_{n,\lambda} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} E_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}, \quad \text{(see [4])}. \end{aligned} \tag{34}$$

By (34), for $n \geq 0$, we have

$$E_{n,\lambda}^{(r)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1+\cdots+x_r+x)_{n,\lambda} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r), \tag{35}$$

and

$$E_{n,\lambda}^{(r)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1+\cdots+x_r)_{n,\lambda} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r). \tag{36}$$

By the degenerate binomial expansion (see [17]), we note that

$$\begin{aligned} E_{n,\lambda}^{(r)}(x) &= \sum_{j=0}^{n}\binom{n}{j}(x)_{n-j,\lambda}\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1+\cdots+x_r)_{j,\lambda}\, d\mu_{-1}(x_1)\cdots\, d\mu_{-1}(x_r) \\ &= \sum_{j=0}^{n}\binom{n}{j}(x)_{n-j,\lambda}E_{j,\lambda}^{(r)}. \end{aligned} \tag{37}$$

From (22), we note that

$$(t)_\lambda E_{n,\lambda}^{(r)}(x) = u_\lambda(t)E_{n,\lambda}^{(r)}(x) = nE_{n-1,\lambda}^{(r)}(x). \tag{38}$$

We consider the λ-linear functional $f^{(r)}(t)$ that satisfies

$$\langle f^{(r)}(t)|P(x)\rangle_\lambda = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} P(x_1+\cdots+x_r)d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r), \tag{39}$$

for all polynomials $P(x) \in \mathbb{P} = \mathbb{C}_p[x]$.

Theorem 4.1. *For any* $P(x) \in \mathbb{P} = \mathbb{C}_p[x]$*, we have*

$$\begin{aligned}\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} & P(x_1 + \cdots + x_r) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= \left\langle \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e_\lambda^{x_1+\cdots+x_r}(t) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \middle| P(x) \right\rangle_\lambda \\ &= \left\langle \left(\frac{2}{e_\lambda(t)+1} \right)^r \middle| P(x) \right\rangle_\lambda .\end{aligned}$$

Proof. From (34) and (39), we note that

(40)

$$\begin{aligned}f^{(r)}(t) &= \sum_{k=0}^{\infty} \frac{\langle f^{(r)}(t) | (x)_{k,\lambda} \rangle_\lambda}{k!} t^k \\ &= \sum_{k=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r)_{k,\lambda} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \frac{t^k}{k!} \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e_\lambda^{x_1+\cdots+x_r}(t) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \left(\frac{2}{e_\lambda(t)+1} \right)^r .\end{aligned}$$

Therefore, by (40), we attain the desired result. □

Theorem 4.2. *For any* $P(x) \in \mathbb{P}$*, we have*

$$\begin{aligned}\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} & P(x_1 + \cdots + x_r + x) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= \left(\left(\frac{2}{e_\lambda(t)+1} \right)^r \right)_\lambda P(x) = \left(\frac{2}{e^t+1} \right)^r P(x).\end{aligned}$$

Proof. For $P(x) = (x)_{n,\lambda}$, from (10) and (36) we get

$$E_{n,\lambda}^{(r)} = \left\langle \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e_\lambda^{x_1+\cdots+x_r}(t) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \middle| (x)_{n,\lambda} \right\rangle_\lambda . \tag{41}$$

From Theorem, (23), and (35) and noting that $g(u_\lambda(t)) = \frac{e^t+1}{2}$, we obtain

(42)

$$\begin{aligned}\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} & (x_1 + \cdots + x_r + x)_{n,\lambda} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = E_{n,\lambda}^{(r)}(x) \\ &= \left(\left(\frac{2}{e_\lambda(t)+1} \right)^r \right)_\lambda \left(\left(\frac{e_\lambda(t)+1}{2} \right)^r \right)_\lambda E_{n,\lambda}^{(r)}(x) = \left(\frac{2}{e^t+1} \right)^r (x)_{n,\lambda}.\end{aligned}$$

Therefore, by linear extension, we obtain the following identity.

$$\begin{aligned}\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} & P(x_1 + \cdots + x_r + x) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= \left(\int_{\mathbb{Z}_p} e_\lambda^{x_1+\cdots+x_r}(t) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \right)_\lambda P(x) \\ &= \left(\left(\frac{2}{e_\lambda(t)+1} \right)^r \right)_\lambda P(x) = \left(\frac{2}{e^t+1} \right)^r P(x).\end{aligned} \tag{43}$$

From (43), we get the desired result. $\square$

Examples.
(1) Choosing $P(x) = x^n$ in Theorem 4, we get

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \left(\frac{2}{e^t+1}\right)^r x^n = E_n^{(r)}(x),$$

where $E_n^{(r)}(x)$ are the Euler polynomials of order r given by $(\frac{2}{e^t+1})^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!}$.

(2) By putting $P(x) = (x)_n$ in Theorem 4, we obtain

$$\begin{aligned}
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} &(x_1 + \cdots + x_r + x)_n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \left(\left(\frac{2}{e_\lambda(t)+1}\right)^r\right)_\lambda (x)_n \\
&= \sum_{k=0}^{n} S_{1,\lambda}(n,k)\left(\left(\frac{2}{e_\lambda(t)+1}\right)^r\right)_\lambda (x)_{k,\lambda} \\
&= \sum_{k=0}^{n} S_{1,\lambda}(n,k) E_{k,\lambda}^{(r)}(x).
\end{aligned}$$

where $S_{1,\lambda}(n,k)$, given by $(x)_n = \sum_{k=0}^{n} S_{1,\lambda}(n,k)(x)_{k,\lambda}$, are the degenerate Stirling numbers of the first kind.

(3) Recall that the Bernoulli polynomials of order r are given by $(\frac{t}{e^t-1})^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}$.
Let $P(x) = B_n^{(r)}(x)$ in Theorem , we have

$$\begin{aligned}
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} &B_n^{(r)}(x_1 + \cdots + x_r + x) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \left(\left(\frac{2}{e_\lambda(t)+1}\right)^r\right)_\lambda B_n^{(r)}(x) = \left(\frac{2}{e^t+1}\right)^r B_n^{(r)}(x) \\
&= \left(\frac{2}{e^t+1}\right)^r \left(\frac{t}{e^t-1}\right)^r x^n = \left(\frac{2t}{e^{2t}-1}\right)^r x^n = 2^n B_n^{(r)}\left(\frac{x}{2}\right).
\end{aligned}$$

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