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On the Controllability of fractional differential equations with generalized proportional-Caputo fractional derivative

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Abstract

In this paper, we study the controllability of generalized proportional-Caputo fractional differential equations (GPC-FDE). The main key in this investigation is the Krasnoselskii's fixed point theorem.

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1 Introduction

Fractional differential equations have been studied extensively in the literature because of their applications in various fields of engineering and science (see, for example, the monographs [17, 19, 20], and the cited therein references).

There are several definitions of fractional derivative operators, the most famous of which are Riemann-Liouville fractional derivative, Caputo fractional derivative, Hadamard fractional derivative (see [16, 21]), and others that were soon discovered, such as Atangana-Baleanu fractional derivative and Caputo-Fabrizio fractional derivative (see [5, 10]).

Recently, Khalil et al. [15] introduce a new definition of fractional derivative, called the conformable fractional derivative, with an obstacle that it does not tend to the original function as the order α tends to zero. The new

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definition has attracted good efforts of many researchers to establish some useful results (see, for example, [7, 8, 9, 18, 24]).

In control theory, a proportional derivative controller for controller output u at time t with two tuning parameters has the algorithm

$$u(t) = \kappa_P \mathcal{E}(t) + \kappa_d \frac{d}{dt}\mathcal{E}(t),$$

where κ_P and κ_d are the proportional control parameter and the derivative control parameter, respectively. The function $\mathcal{E}$ is the error between the state variable and the process variable. This control law enables Dawei et al. [11] to present the control of complex networks models.

Inspired by the above concept of the proportional derivative controller, Anderson et al. [4] were able to define the proportional (conformable) derivative of order α by

$${}_0^P D_t^\alpha g(t) = k_1(\alpha, t)g(t) + k_0(\alpha, t)g'(t),$$

where g is differentiable function and $k_0, k_1 : [0, 1] \times \mathbb{R} \to [0, \infty)$ are continuous functions of the variable t and the parameter $\alpha \in [0, 1]$ which satisfy the following conditions for all $t \in \mathbb{R}$:

$$\lim_{\alpha \to 0^+} k_0(\alpha, t) = 0, \quad \lim_{\alpha \to 1^-} k_0(\alpha, t) = 1, \quad k_0(\alpha, t) \neq 0, \ \alpha \in (0, 1], \tag{1.1}$$

$$\lim_{\alpha \to 0^+} k_1(\alpha, t) = 1, \quad \lim_{\alpha \to 1^-} k_1(\alpha, t) = 0, \quad k_1(\alpha, t) \neq 0, \ \alpha \in [0, 1). \tag{1.2}$$

This newly defined local derivative tends to the original function as the order α tends to zero and hence improved the conformable derivatives. In [14], Jarad et al. discussed a special case of the proportional derivatives when $k_1(\alpha, t) = 1 - \alpha$ and $k_0(\alpha, t) = \alpha$.

On the other hand, controllability is one of the fundamental notions of modern control theory, which enables one to steer the control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls where initial and final state may vary over the entire space. The problem of controllability of nonlinear systems represented by fractional differential equations has been extensively studied by several authors (see, for example, [1, 2, 3, 12, 13]).

In this paper, we study the controllabilty of the following GPC-FDE:

$$\begin{cases} {}_0^{PC} D_t^\alpha x(t) = f(t, x(t)) + Bu(t), \quad 0 < \alpha \leq 1, \ t \in J = [0, b], \ b < \infty, \\ x(0) = x_0, \end{cases}$$

(1.3)

where ${}^{PC}_{0}D^{\alpha}_{t}$ denotes the proportional-Caputo fractional derivative of order α, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, the control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions with U as a Banach space, B is a bounded linear operator from U to $\mathbb{R}$ and $x_0 \in \mathbb{R}$.

It is worth noting that the results of this paper are novel; this is the first paper dealing with the controllabilty of GPC-FDE.

2 Preliminaries

In this section we collect some definitions, properties and propositions of the new generalized proportional-Caputo hybrid fractional derivative.

Definition 2.1. *[6] The proportional-Caputo hybrid fractional derivative of order $\alpha \in (0, 1)$ of a differentiable function $g(t)$ is given by*

$${}^{PC}_{0}D^{\alpha}_{t} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \big(k_1(\alpha,\tau)g(t) + k_0(\alpha,\tau)g'(t)\big)(t-\tau)^{-\alpha}\, d\tau, \quad (2.1)$$

where the function space domain is given by requiring that g is differentiable and both g and g' are locally L^1 functions on the positive reals.

Definition 2.2. *[6] The inverse operator of the proportional-Caputo hybrid fractional derivative of order is given by*

$${}^{PC}_{0}I^{\alpha}_{t} g(t) = \int_0^t \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) \frac{{}^{RL}_{0}D^{1-\alpha}_{u} g(u)}{k_0(\alpha,u)}\, du, \quad (2.2)$$

where ${}^{RL}_{0}D^{1-\alpha}_{u}$ denotes the Riemann-Liouville fractional derivative of order $1-\alpha$ and is given by

$${}^{RL}_{0}D^{1-\alpha}_{u} g(u) = \frac{1}{\Gamma(\alpha)} \frac{d}{du} \int_0^u (u-s)^{\alpha-1} g(s)\, ds. \quad (2.3)$$

For more details, we refer the reader to the book of Kilbas et al. [16].

Proposition 2.3. *[6] The following inversion relations:*

$${}^{PC}_{0}D^{\alpha}_{t}\, {}^{PC}_{0}I^{\alpha}_{t} g(t) = g(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \lim_{t\to 0} {}^{RL}_{0}I^{\alpha}_{t} g(t), \quad (2.4)$$

$${}^{PC}_{0}I^{\alpha}_{t}\, {}^{PC}_{0}D^{\alpha}_{t} g(t) = g(t) - \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) g(0) \quad (2.5)$$

are satisfied.

Proposition 2.4. *[6] The proportional-Caputo hybrid fractional derivative operator* ${}^{PC}_{0}D^{\alpha}_{t}$ *is non-local and singular.*

Remark 2.5. *[6] In the limiting cases* $\alpha \to 0$ *and* $\alpha \to 1$*, we recover the following special cases:*

$$\begin{aligned} \lim_{\alpha \to 0} {}^{PC}_{0}D^{\alpha}_{t} g(t) &= \int_0^t g(\tau)\, d\tau, \\ \lim_{\alpha \to 1} {}^{PC}_{0}D^{\alpha}_{t} g(t) &= g(t). \end{aligned}$$

Theorem 2.6. *(**Krasnoselskii's fixed point theorem** [23]) Let* Ω *be a closed convex and non-empty subset of a Banach space* $\mathbb{X}$*. Let* $\mathcal{P}_1$ *and* $\mathcal{P}_2$*, be two operators such that*

(i) $\mathcal{P}_1 x + \mathcal{P}_2 y \in \Omega$*, for all* $x, y \in \Omega$*;*

(ii) $\mathcal{P}_1$ *is compact and continuous;*

(iii) $\mathcal{P}_2$ *is a contraction mapping.*

Then there exists $z \in \Omega$ *such that* $z = \mathcal{P}_1 z + \mathcal{P}_2 z$*.*

3 Controllability Results

In this section, we employ the generalized proportional Caputo fractional derivative operator to discuss the controllability of the given GPC-FDE (1.3).

Let $C(J, \mathbb{R})$ be the Banach space of all real-valued continuous functions from J into $\mathbb{R}$ equipped by the norm $\|x\| = \sup_{t\in[0,T]} |x(t)|$.

The following lemma is considered the linear issue of the GPC-FDE (1.3).

Lemma 3.1. *Let* $0 < \alpha \leq 1$ *and* $h \in C(J, \mathbb{R})$*. Then the solution of the following linear fractional differential equation*

$$\begin{cases} {}^{PC}_{0}D^{\alpha}_{t} x(t) = h(t), \quad , \ t \in J, \\ x(0) = x_0, \end{cases} \tag{3.1}$$

is equivalent to the Volterra integral equation:

$$x(t) = \exp\left(-\int_0^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)} ds\right) x_0 + \frac{1}{\Gamma(\alpha - 1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)} ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha, u)} h(\tau)\, d\tau\, du. \tag{3.2}$$

Proof. Applying the operator ${}^{PC}_{0}I^{\alpha}_{t}(\cdot)$ on both sides of (3.1), we get

$${}^{PC}_{0}I^{\alpha}_{t}\,{}^{PC}_{0}D^{\alpha}_{t}x(t) = {}^{PC}_{0}I^{\alpha}_{t}h(t).$$

Using (2.2) and (2.3) together with the proposition 2.3, we get

$$x(t) - \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)x(0) = \int_0^t \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{{}^{RL}_{0}D^{1-\alpha}_{u}h(u)}{k_0(\alpha,u)}\,du.$$

In view of the following elementary relations between the Riemann-Liouville fractional derivative and fractional integral:

$$\begin{aligned} {}^{RL}_{0}D^{1-\alpha}_{u}h(u) &= {}^{RL}_{0}I^{-(1-\alpha)}_{u}h(u) \\ &= {}^{RL}_{0}I^{\alpha-1}_{u}h(u) \\ &= \frac{1}{\Gamma(\alpha-1)}\int_0^u (u-\tau)^{\alpha-2}h(\tau)\,d\tau, \end{aligned}$$

the formula is directly concluded (3.2).
The converse follows by direct computation. This completes the proof □

Definition 3.2. *The GPC-FDE (1.3) is said to be controllable on the interval J if, for every $x_0, x_1 \in \mathbb{R}$, there exists a control $u \in L^2(J,U)$ such that a solution x of Eq. (1.3) satisfies $x(b) = x_1$.*

We consider the following assumptions.

(**A1**) The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous.

(**A2**) There exists a constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \le L|x-y|, \quad \text{for all} \quad t \in J, \; x,y \in \mathbb{R}.$$

(**A3**) The linear operator $W : L^2(J,U) \to \mathbb{R}$, defined by

$$Wu = \frac{1}{\Gamma(\alpha-1)}\int_0^b\int_0^u \exp\left(-\int_u^b \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}Bu(\tau)\,d\tau\,du$$

has an induced inverse operator W^{-1} which takes values in $L^2(J,U)/\ker W$, where the kernel space of W is defined by $\ker W = \{x \in L^2(J,U) : Wx = 0\}$ and there exist constants $M_1, M_2 > 0$ such that $\|B\| \le M_1$ and $\|W^{-1}\| \le M_2$.

Now we formulate the main theorem of the paper.

Theorem 3.3. *If the assumptions* (**A1**) − (**A2**) *are satisfied. Then the GPC-FDE (1.3) is controllable on J provided that*

$$\frac{M_1 M_2 M_k^2 b^{2\alpha} L}{\Gamma^2(\alpha+1)} < 1. \tag{3.3}$$

Proof. Let us set $\sup_{t\in J}|f(t,0)| = M_f$, $\sup_{t\in J}\frac{1}{|k_0(\alpha,t)|} = M_k$ and define the two constants $\Lambda_1 > 0$ and $0 < \Lambda_2 < 1$ by

$$\Lambda_1 = |x_0| + \frac{M_k M_f b^\alpha}{\Gamma(\alpha+1)} + \frac{M_1 M_2 M_k b^\alpha}{\Gamma(\alpha+1)}\Big[|x_1| + |x_0| + \frac{M_k M_f b^\alpha}{\Gamma(\alpha+1)}\Big],$$

$$\Lambda_2 = \frac{M_k L b^\alpha}{\Gamma(\alpha+1)}\Big(1 + \frac{M_1 M_2 M_k b^\alpha}{\Gamma(\alpha+1)}\Big).$$

Consider the set $\mathcal{B}_r = \{x \in C(J,\mathbb{R}) : \|x\| \le r)\}$ with $r \ge \frac{\Lambda_1}{1-\Lambda_2}$.

We define the control $u_x(t)$ by

$$\begin{aligned} u_x(t) &= W^{-1}\Bigg[x_1 - \exp\Big(-\int_0^b \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\Big)\,x_0 \\ &+ \frac{1}{\Gamma(\alpha-1)}\int_0^b\int_0^u \exp\Big(-\int_u^b \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\Big)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}f(\tau,x(\tau))\,d\tau\,du\Bigg](t),\ t\in J. \end{aligned}$$

Later, we shall use the following two estimations.

$$\begin{aligned} \|u_x\| &= \sup_{t\in J}|u_x(t)| \\ &\le M_2\sup_{t\in J}\Bigg\{|x_1| + |x_0| \\ &\quad + \frac{1}{\Gamma(\alpha-1)}\int_0^b\int_0^u \frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|}|(f(\tau,x(\tau)) - f(\tau,0)) + f(\tau,0)|\,d\tau\,du\Bigg\} \\ &\le M_2\sup_{t\in J}\Bigg\{|x_1| + |x_0| \\ &\quad + \frac{1}{\Gamma(\alpha-1)}\int_0^b\int_0^u \frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|}\big(|f(\tau,x(\tau)) - f(\tau,0)| + |f(\tau,0)|\big)\,d\tau\,du\Bigg\} \\ &\le M_2\sup_{t\in J}\Bigg\{|x_1| + |x_0| + \frac{1}{\Gamma(\alpha-1)}\int_0^b\int_0^u \frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|}\big(L|x(\tau)| + |f(\tau,0)|\big)\,d\tau\,du\Bigg\} \\ &\le M_2\Big[|x_1| + |x_0| + \frac{M_k b^\alpha}{\Gamma(\alpha+1)}\big(L\|x\| + M_f\big)\Big]. \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
\|u_x - u_y\| &= \sup_{t\in J} |u_x(t) - u_y(t)| \\
&\le M_2 \sup_{t\in J} \left\{ \frac{1}{\Gamma(\alpha-1)} \int_0^b \int_0^u \frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|} |f(\tau,x(\tau)) - f(\tau,y(\tau))| \, d\tau \, du \right\} \\
&\le \frac{M_2 M_k L}{\Gamma(\alpha-1)} \sup_{t\in J} \left\{ \int_0^b \int_0^u (u-\tau)^{\alpha-2} |x(\tau) - y(\tau)| \, d\tau \, du \right\} \\
&\le \frac{M_2 M_k L b^\alpha}{\Gamma(\alpha+1)} \|x-y\|. \qquad (3.5)
\end{aligned}
$$

Using the control $u_x(t)$, we define the operators $\mathcal{P}_1, \mathcal{P}_2$ on $\mathcal{B}_r$ as

$$
\begin{aligned}
(\mathcal{P}_1 x)(t) &= \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) x_0 \\
&+ \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} f(\tau,x(\tau)) \, d\tau \, du, \\
(\mathcal{P}_2 x)(t) &= \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} B u_x(\tau) \, d\tau \, du.
\end{aligned}
$$

Clearly, One can notice that $(\mathcal{P}_1 x + \mathcal{P}_2 x)(b) = x_1$. This means that u_x steers the GPC-FDE (1.3) from x_0 to x_1 in finite time b, which implies that the GPC-FDE (1.3) is controllable on J.

The proof is divided into three main steps.

Step 1. $\mathcal{P}_1 x + \mathcal{P}_2 y \in \mathcal{B}_r, \quad \forall x, y \in \mathcal{B}_r$

For each $t \in J$ and $x, y \in \mathcal{B}_r$, using (3.4), we have

$$
\begin{aligned}
\|\mathcal{P}_1 x + \mathcal{P}_2 y\| &= \sup_{t\in J} |(\mathcal{P}_1 x)(t) + (\mathcal{P}_2 y)(t)| \\
&\leq \sup_{t\in J} \Bigg\{ \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) |x_0| \\
&+ \frac{1}{\Gamma(\alpha-1)} \int_0^b \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) \\
&\quad \times \frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|} |(f(\tau,x(\tau)) - f(\tau,0)) + f(\tau,0)|\, d\tau\, du \\
&+ \frac{1}{\Gamma(\alpha-1)} \int_0^b \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) \frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|} |Bu_y(\tau)|\, d\tau\, du \Bigg\} \\
&\leq |x_0| + \frac{M_k}{\Gamma(\alpha-1)} \int_0^b \int_0^u (u-\tau)^{\alpha-2} \big(L\|x\| + M_f\big)\, d\tau\, du \\
&+ \frac{M_k}{\Gamma(\alpha-1)} \int_0^b \int_0^u (u-\tau)^{\alpha-2} \|B\| \|u_y\|\, d\tau\, du \\
&\leq |x_0| + \frac{M_k b^\alpha}{\Gamma(\alpha+1)} \big(Lr + M_f\big) + \frac{M_1 M_k b^\alpha}{\Gamma(\alpha+1)} M_2 \Big[|x_1| + |x_0| + \frac{M_k b^\alpha}{\Gamma(\alpha+1)} \big(Lr + M_f\big) \Big] \\
&\leq |x_0| + \frac{M_k M_f b^\alpha}{\Gamma(\alpha+1)} + \frac{M_1 M_2 M_k b^\alpha}{\Gamma(\alpha+1)} \Big[|x_1| + |x_0| + \frac{M_k M_f b^\alpha}{\Gamma(\alpha+1)} \Big] \\
&+ \frac{M_k L b^\alpha}{\Gamma(\alpha+1)} \Big(1 + \frac{M_1 M_2 M_k b^\alpha}{\Gamma(\alpha+1)} \Big) r \\
&= \Lambda_1 + \Lambda_2 r \leq r.
\end{aligned}
$$

Thus, we conclude that $\mathcal{P}_1 x + \mathcal{P}_2 y \in \mathcal{B}_r$.

Step 2. $\mathcal{P}_1$ is compact and continuous.

First, we show that $\mathcal{P}_1$ is continuous.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ as $n \to \infty$ in $\mathcal{B}_r$. Thus, for each $t \in J$, we have

$$
\begin{aligned}
\|\mathcal{P}_1 x_n - \mathcal{P}_1 x\| &= \sup_{t\in J} |(\mathcal{P}_1 x_n)(t) - (\mathcal{P}_1 x)(t)| \\
&\leq \frac{M_k}{\Gamma(\alpha-1)} \int_0^t \int_0^u (u-\tau)^{\alpha-2} \|(f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))\|\, d\tau\, du.
\end{aligned}
$$

Therefore, the continuity of f implies that $\mathcal{P}_1$ is continuous.

Next, we show that $\mathcal{P}_1$ is uniformly bounded on $\mathcal{B}_r$.

For each $t \in J$ and $x \in \mathcal{B}_r$, we have

$$\begin{aligned} \|\mathcal{P}_1 x\| &= \sup_{t\in J} |(\mathcal{P}_1 x)(t)| \\ &\le \sup_{t\in J} \Bigg\{ \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)|x_0| + \frac{1}{\Gamma(\alpha-1)}\int_0^b\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right) \\ &\quad \times \frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|}|(f(\tau,x(\tau))-f(\tau,0))+f(\tau,0)|\,d\tau\,du \Bigg\} \\ &\le |x_0| + \frac{M_k b^\alpha}{\Gamma(\alpha+1)}\big(Lr+M_f\big), \end{aligned}$$

which implies that $\mathcal{P}_1$ is uniformly bounded on $\mathcal{B}_r$.

It remains to show that $\mathcal{P}_1$ is equicontinuous.

For each $t_1, t_2 \in J,\ t_1 < t_2$ and $x \in \mathcal{B}_r$, we have

$$\begin{aligned} &\|(\mathcal{P}_1 x)(t_2) - (\mathcal{P}_1 x)(t_1)\| \\ &\le \left\| \exp\left(-\int_0^{t_2} \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)x_0 - \exp\left(-\int_0^{t_1} \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)x_0 \right\| \\ &+ \frac{1}{\Gamma(\alpha-1)}\Bigg\| \int_0^{t_2}\int_0^u \left[\exp\left(-\int_0^{t_2} \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right) - \exp\left(-\int_0^{t_1} \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\right] \\ &\quad \times \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}(f(\tau,x(\tau))\,d\tau\,du \Bigg\| \\ &+ \frac{1}{\Gamma(\alpha-1)}\left\| \int_{t_1}^{t_2}\int_0^u \exp\left(-\int_0^{t_1} \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}(f(\tau,x(\tau))\,d\tau\,du \right\| \\ &= \left\| \frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)}x_0 \exp\left(-\int_0^{\xi} \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)(t_2-t_1) \right\| \\ &+ \frac{1}{\Gamma(\alpha-1)}\left\| \int_0^{t_2}\int_0^u \frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)}\exp\left(-\int_0^{\xi} \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)(t_2-t_1)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}(f(\tau,x(\tau))\,d\tau\,du \right\| \\ &+ \frac{1}{\Gamma(\alpha-1)}\left\| \int_{t_1}^{t_2}\int_0^u \exp\left(-\int_0^{t_1} \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}(f(\tau,x(\tau))\,d\tau\,du \right\| \\ &\le \left| \frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)}x_0 \right|(t_2-t_1) + \frac{\bar{f}}{\Gamma(\alpha+1)}\left|\frac{k_1(\alpha,\xi)}{k_0^2(\alpha,\xi)}\right| t_2(t_2-t_1) + \frac{\bar{f}}{\Gamma(\alpha+1)}\left|\frac{1}{k_0(\alpha,\xi)}\right|(t_2^\alpha - t_1^\alpha), \end{aligned}$$

where $\bar{f} = \sup_{t\in J\times\mathcal{B}_r} |f(t,x(t)|,\ \xi \in (t_1,t_2)$.
As $t_1 \to t_2$, the right hand side of the above inequality tends to zero independently of $x \in \mathcal{B}_r$. As a consequence of the Arzelà-Ascoli theorem, we

deduce that $\mathcal{P}_1$ is compact on $\mathcal{B}_r$.

Step 3. $\mathcal{P}_2$ is a contraction on $\mathcal{B}_r$.

For each $t \in J$ and $x, y \in \mathcal{B}_r$, using (3.5), we have

$$\begin{aligned}
\|\mathcal{P}_2 x - \mathcal{P}_2 y\| &= \sup_{t\in J} |(\mathcal{P}_2 x)(t) - (\mathcal{P}_2 y)(t)| \\
&= \sup_{t\in J} \left\{ \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) \right. \\
&\quad \times \left. \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} B(u_x(\tau) - u_y(\tau))\, d\tau\, du \right\} \\
&\leq \frac{M_1 M_k b^\alpha}{\Gamma(\alpha+1)} \|u_x - u_y\| \\
&\leq \frac{M_1 M_2 M_k^2 b^{2\alpha} L}{\Gamma^2(\alpha+1)} \|x - y\|.
\end{aligned}$$

In view of the condition (3.3), we conclude that $\mathcal{P}_2$ is a contraction mapping. Therefore, all the assumptions of Krasnoselskii's fixed point theorem (Theorem 2.6) are satisfied. Hence, the GPC-FDE (1.3) is controllable on J. This completes the proof. □

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