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Subclasses of Analytic Functions Defined with Generalized Sãlãgean Operator

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Abstract: The present investigation deals with certain subclasses of analytic-univalent functions in the open unit disc $E=\{z:|z|<1\}$. The coefficient estimates, distortion theorem, argument theorem and relation of these classes with some other classes have been studied and the results so obtained generalize the results of several earlier works.
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1. Introduction

Let A denote the class of functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \tag{1}$$

which are analytic in the unit disc $E=\{z:|z|<1\}$ and further normalized specifically by $f(0)=f'(0)-1=0$.

By S, we denote the subclass of A consisting of functions of the form (1) and which are univalent in E.

Let U be the class of Schwarzian functions

$$w(z)=\sum_{k=1}^{\infty}c_k z^k$$

which are regular in the unit disc E and satisfying the conditions

$$w(0)=0 \text{ and } |w(z)|<1.$$

For the functions f and g analytic in E, we say that f is subordinate to g, symbolically $f \prec g$, if a Schwarzian function $w(z)\in U$ can be found for which $f(z)=g(w(z))$.

For $\delta \geq 1$, Al-Oboudi [2] introduced the following differential operator:

$$D_\delta^0 f(z)=f(z),$$

$$D_\delta^1 f(z)=(1-\delta)f(z)+\delta z f'(z)$$

and in general

$$D_\delta^p f(z)=D\left(D_\delta^{p-1}f(z)\right)$$

$$= z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^{p} a_k z^k , p \in N_0 = N \cup \{0\}$$

with $D_\delta^0 f(0) = 0$.

For $\delta = 1$, the operator $D_1^p f(z) \equiv D^p f(z)$, the well known Sãlãgean operator introduced by G. Sãlãgean [16]. The operator $D_\delta^p f(z)$ is named as Generalized Sãlãgean operator.

$S^*(\alpha)$ and $K(\alpha)$ are respectively the classes of starlike and convex functions of order α ($\alpha \geq 0$). In particular $S^*(0) \equiv S^*$, the well known class of starlike functions and $K(0) \equiv K$, the class of convex functions. Goel and Mehrok [4] studied the classes $S^*(A, B)$ and $K(A, B)$, the subclasses of starlike and convex functions respectively. In particular $S^*(1\text{-}2\alpha,\text{-}1) \equiv S^*(\alpha)$, $S^*(1,\text{-}1) \equiv S^*$, $K(1\text{-}2\alpha,\text{-}1) \equiv K(\alpha)$ and $K(1,\text{-}1) \equiv K$.

Kaplan [6] introduced the class C of close-to-convex functions. After that various subclasses of close-to-convex functions such as C_1, $C(A, B)$, $C_1(A, B)$ were studied respectively by Abdel Gawad and Thomas [1], Mehrok [7] and Mehrok and Singh [8]. In particular $C(1,\text{-}1)$ $\equiv C$

and $C_1(1,\text{-}1) \equiv C_1$. Again the classes $C(A,B;C,D)$ and $C_1(A,B;C,D)$ were studied by Singh and Mehrok [18]. Particularly $C(1,\text{-}1;C,D) \equiv C(C,D)$ and $C_1(1,\text{-}1;C,D) \equiv C_1(C,D)$.

The concept of close-to-star functions was established by Reade [14] and this class is denoted by CS^*. Further various subclasses of close-to-star functions such as CS_1^*, $CS^*(A,B)$, $CS_1^*(A,B)$ and $CS^*(A,B;C,D)$ were studied by Mehrok et al.[10], Mehrok et al.[11] and Mehrok and Singh [9] respectively. Specifically $CS^*(1,\text{-}1) \equiv CS^*$, $CS_1^*(1,\text{-}1) \equiv CS_1^*$ and $CS^*(1,\text{-}1; C, D) \equiv CS^*(C, D)$

$S(p,\alpha) = \left\{ f : f \in A, \mathrm{Re}\left(\frac{D^{p+1} f(z)}{D^p f(z)} \right) > \alpha, 0 \leq \alpha < 1, z \in E \right\}$, the class introduced by Sãlãgean [16] and studied further by Kadioğlu [5].

$C(p;\alpha;\beta) = \left\{ f : f \in A, \left| \arg \frac{D^p f(z)}{g(z)} \right| < \frac{\beta\pi}{2} \; or \; \frac{D^p f(z)}{g(z)} \prec \left(\frac{1+z}{1-z} \right)^{\beta}, g \in S^*(\alpha), 0 \leq \alpha < 1, 0 < \beta \leq 1, z \in E \right\}$,

the class introduced and studied by Porwal [13].

In particular

(i) $C(0,\alpha,\beta) \equiv CS^*(\alpha,\beta)$ and $C(1,\alpha,\beta) \equiv C(\alpha,\beta)$, the classes studied by Mishra [12]

(ii) $C(0,0,\beta) \equiv CS^*(\beta)$, the class studied by Reade [14].

(iii) $C(1,0,\beta) \equiv C(\beta)$, the class studied by Kaplan [6].

$C(p;\beta;A,B;C,D) = \left\{ f : f \in A, \frac{D^p f(z)}{g(z)} \prec \left(\frac{1+Cz}{1+Dz} \right)^{\beta}, g \in S^*(A,B), -1 \leq D \leq B < A \leq C \leq 1, 0 < \beta \leq 1, z \in E \right\}$, the class introduced and studied by Singh and Singh [17]. In particular, $C(p,\beta;1-2\alpha,-1;1,-1) \equiv C(p,\alpha,\beta)$.

To avoid repetition, it is laid down once for all that $0 < \beta \leq 1, -1 \leq D \leq B < A \leq C \leq 1$, $z \in E$.

Motivated by the above work, we introduce the following subclasses of analytic univalent functions defined with generalized Sãlãgean operator:

Definition 1.1 $S(\delta;p;\alpha)$ be the class of functions in A of the form (1) which satisfy the condition

$$\operatorname{Re}\left(\frac{D_\delta^{p+1}f(z)}{D_\delta^p f(z)}\right) > \alpha, 0 \le \alpha < 1.$$

Specifically, $S(\delta;p;0) \equiv S(\delta,p)$ and $S(1;p,\alpha) \equiv S(p,\alpha)$.

Definition 1.2 Let $C(\delta;p;\beta;A,B;C,D)$ denote the class of functions $f(z)$ of the form (1) and satisfying the condition that

$$\frac{D_\delta^p f(z)}{g(z)} \prec \left(\frac{1+Cz}{1+Dz}\right)^\beta \text{ where } g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in S^*(A,B).$$

We have the following observations:

(i) $C(\delta;p;1;A,B;C,D) \equiv C(\delta;p;A,B;C,D)$.

(ii) $C(1;p;\beta;A,B;C,D) \equiv C(p;\beta;A,B;C,D)$.

(iii) $C(1;p;\beta;1-2\alpha,-1;1,-1) \equiv C(p,\alpha,\beta)$.

(iv) $C(1;1;1;A,B;C,D) \equiv C(A;B;C,D)$.

(v) $C(1;0;1;1,-1;C,D) \equiv CS^*(C,D)$.

(vi) $C(1;0;1;1,-1;1,-1) \equiv CS^*$.

(vii) $C(1;1;1;1,-1;C,D) \equiv C(C,D)$.

(viii) $C(1;1;1;1,-1;1,-1) \equiv C$.

Definition 1.3 Let $C_1(\delta;p;\beta;A,B;C,D)$ denote the class of functions $f(z)$ of the form (1) and satisfying the condition that

$$\frac{D_\delta^p f(z)}{h(z)} \prec \left(\frac{1+Cz}{1+Dz}\right)^\beta \text{ where } h(z) = z + \sum_{k=2}^{\infty} d_k z^k \in K(A,B).$$

The following observations are obvious:

(i) $C_1(\delta;p;1;A,B;C,D) \equiv C_1(\delta;p;A,B;C,D)$.

(ii) $C_1(1;p;\beta;A,B;C,D) \equiv C_1(p;\beta;A,B;C,D)$.

(iii) $C_1(1;1;1;A,B;C,D) \equiv C_1(A;B;C,D)$.

(iv) $C_1(1;0;1;1,-1;C,D) \equiv CS_1^*(C,D)$.

(v) $C_1(1;0;1;1,-1;1,-1) \equiv CS_1^*$.

(vi) $C_1(1;1;1;1,-1;C,D) \equiv C_1(C,D)$.

(vii) $C_1(1;1;1;1,-1;1,-1) \equiv C_1$.

The paper in hand studies the classes $C(\delta;p;\beta;A,B;C,D)$ and $C_1(\delta,p;\beta;A,B;C,D)$ focusing on coefficient estimates, distortion theorems, argument theorems and the relation of these subclasses with some other classes. The results already proved by various authors follow as special cases.

2. Preliminary Lemmas

Lemma 2.1[15] If $P(z)=\left(\frac{1+Cw(z)}{1+Dw(z)}\right)^{\beta}=1+\sum_{k=1}^{\infty}p_k z^k$, then

$$|p_n|\leq\beta(C-D), n\geq 1.$$

Lemma 2.2[4] If $g(z)\in S^*(A,B)$, then for $A-(n-1)B\geq(n-2), n\geq 3$,

$$|b_n|\leq\frac{1}{(n-1)!}\prod_{j=2}^{n}(A-(j-1)B).$$

Lemma 2.3[4] If $g(z)\in S^*(A,B)$, then for $|z|=r<1$,

$$r(1-Br)^{\frac{A-B}{B}}\leq|g(z)|\leq r(1+Br)^{\frac{A-B}{B}}, B\neq 0;$$

$$re^{-Ar}\leq|g(z)|\leq re^{Ar}, B=0.$$

Lemma 2.4[4] If $g(z)\in S^*(A,B)$, then for $|z|=r<1$,

$$\left|\arg\frac{g(z)}{z}\right|\leq\frac{(A-B)}{B}\sin^{-1}(Br), B\neq 0;$$

$$\left|\arg\frac{g(z)}{z}\right|\leq Ar, B=0.$$

Lemma 2.5[18] If $h(z)\in K(A,B)$, then for $A-(n-1)B\geq(n-2), n\geq 3$,

$$|d_n|\leq\frac{1}{n!}\prod_{j=2}^{n}(A-(j-1)B).$$

Lemma 2.6[18] If $h(z)\in K(A,B)$, then for $|z|=r<1$,

$$\frac{1}{A}\left[1-(1-Br)^{\frac{A}{B}}\right]\leq|h(z)|\leq\frac{1}{A}\left[(1+Br)^{\frac{A}{B}}-1\right], B\neq 0;$$

$$\frac{1}{A}\left[1-e^{-Ar}\right]\leq|h(z)|\leq\frac{1}{A}\left[e^{Ar}-1\right], B=0.$$

Lemma 2.7[18] If $h(z)\in K(A,B)$, then for $|z|=r<1$,

$$\left|\arg\frac{h(z)}{z}\right|\leq\frac{A}{B}\sin^{-1}(Br), B\neq 0;$$

$$\left|\arg\frac{h(z)}{z}\right|\leq Ar, B=0.$$

3. The class $C(\delta;p;\beta;A,B;C,D)$

Theorem 3.1. If $f(z)\in C(\delta;p;\beta;A,B;C,D)$, then

$$|a_n|\leq\frac{1}{[1+(n-1)\delta]^p}\left\{\frac{1}{(n-1)!}\prod_{j=2}^{n}(A-(j-1)B)+\beta(C-D)\left[1+\sum_{k=2}^{n-1}\frac{1}{(k-1)!}\prod_{j=2}^{k}(A-(j-1)B)\right]\right\},\quad n\geq 2. \tag{2}$$

The bounds are sharp.

Proof. In Definition 1.2, using Principle of subordination, we have

$$D_\delta^p f(z)=\left(\frac{1+Cw(z)}{1+Dw(z)}\right)^\beta g(z),\ \ w(z)\in U. \tag{3}$$

On expanding (3), it yields

$$z+\sum_{k=2}^{\infty}\left[1+(k-1)\delta\right]^p a_k z^k=\left(z+\sum_{k=2}^{\infty}b_k z^k\right)\left(1+\sum_{k=1}^{\infty}p_k z^k\right). \tag{4}$$

Equating the coefficients of z^n in (4), we have

$$\left[1+(n-1)\delta\right]^p a_n=b_n+p_1 b_{n-1}+p_2 b_{n-2}+\ldots+p_{n-1}. \tag{5}$$

Applying triangle inequality and Lemma 2.1 in (5), it gives

$$\left[1+(n-1)\delta\right]^p |a_n|\le |b_n|+\beta(C-D)\left[|b_{n-1}|+|b_{n-2}|+\ldots+|b_2|+1\right]. \tag{6}$$

Using Lemma 2.2 in (6), the result (2) is obvious.

For $n=2$, equality sign in (2) hold for the functions $f_n(z)$ defined by

$$D_\delta^p f_n\ (z)=\left(\frac{1+C\delta_1 z}{1+D\delta_1 z}\right)^\beta (1+B\delta_2 z)^{\frac{(A-B)}{B}},|\delta_1|=1,|\delta_2|=1. \tag{7}$$

On putting $\beta=1$, Theorem 3.1 gives the following result:

Corollary 3.1.1 If $f(z)\in C(\delta;p;A,B;C,D)$, then

$$|a_n|\le\frac{1}{\left[1+(n-1)\delta\right]^p}\left\{\frac{1}{(n-1)!}\prod_{j=2}^{n}(A-(j-1)B)+(C-D)\left[1+\sum_{k=2}^{n-1}\frac{1}{(k-1)!}\prod_{j=2}^{k}(A-(j-1)B)\right]\right\},\ \ n\ge 2.$$

For $\delta=1,p=0,\beta=1,A=1,B=-1$, Theorem 3.1 yields the following result due to Mehrok et al. [10]:

Corollary 3.1.2. Let $f(z)\in CS^*(C,D)$, then

$$|a_n|\le n\left[1+\frac{(n-1)(C-D)}{2}\right].$$

For $\delta=1,p=0,\beta=1,A=1,B=-1,C=1,D=-1$, Theorem 3.1 gives the following result due to Reade [14]:

Corollary 3.1.3. Let $f(z)\in CS^*$, then

$$|a_n|\le n^2.$$

For $\delta=1,p=1,\beta=1,A=1,B=-1$, Theorem 3.1 agrees with the following result due to Mehrok [7]:

Corollary 3.1.4. Let $f(z)\in C(C,D)$, then

$$|a_n|\le 1+\frac{(n-1)(C-D)}{2}.$$

For $\delta=1,p=1,\beta=1,A=1,B=-1,C=1,D=-1$, Theorem 3.1 gives the following result due to Reade [14]:

Corollary 3.1.5. Let $f(z)\in C$, then

$$|a_n|\le n.$$

Theorem 3.2. If $f(z)\in C(\delta;p;\beta;A,B;C,D)$, then for $|z|=r$, $0<r<1$, we have

$$r(1-Br)^{\frac{A-B}{B}}\left(\frac{1-Cr}{1-Dr}\right)^\beta\le\left|D_\delta^p f(z)\right|\le r(1+Br)^{\frac{A-B}{B}}\left(\frac{1+Cr}{1+Dr}\right)^\beta,B\ne 0; \tag{8}$$

$$re^{-Ar}\left(\frac{1-Cr}{1-Dr}\right)^{\beta} \le \left|D_{\delta}^{p} f(z)\right| \le re^{-Ar}\left(\frac{1+Cr}{1+Dr}\right)^{\beta}, B=0. \tag{9}$$

Estimates are sharp.

Proof. From (3), we have

$$\left|D_{\delta}^{p} f(z)\right| = \left|\frac{1+Cw(z)}{1+Dw(z)}\right|^{\beta} \left|g(z)\right|, \quad w(z) \in U. \tag{10}$$

It is easy to show that the transformation

$$\frac{D_{\delta}^{p} f(z)}{g(z)} = \frac{1+Cw(z)}{1+Dw(z)}$$

maps $|w(z)| \le r$ onto the circle

$$\left|\frac{D_{\delta}^{p} f(z)}{g(z)} - \frac{1-CDr^2}{1-D^2r^2}\right| \le \frac{(C-D)r}{(1-D^2r^2)}, \quad |z| = r.$$

This implies that

$$\frac{1-Cr}{1-Dr} \le \left|\frac{1+Cw(z)}{1+Dw(z)}\right| \le \frac{1+Cr}{1+Dr},$$

which implies that

$$\left(\frac{1-Cr}{1-Dr}\right)^{\beta} \le \left|\frac{1+Cw(z)}{1+Dw(z)}\right|^{\beta} \le \left(\frac{1+Cr}{1+Dr}\right)^{\beta}. \tag{11}$$

Using (11) and Lemma 2.3 in (10), the results (8) and (9) are obvious.
Sharpness follows for the function $f_n(z)$ defined as

$$D_{\delta}^{p} f_n(z) = \begin{cases} \left(\dfrac{1+C\delta_1 z}{1+D\delta_1 z}\right)^{\beta} (1+B\delta_2 z)^{\frac{(A-B)}{B}}, B \ne 0 \\ \left(\dfrac{1+C\delta_1 z}{1+D\delta_1 z}\right)^{\beta} e^{A\delta_2 z}, B = 0 \end{cases}, |\delta_1| = 1, |\delta_2| = 1. \tag{12}$$

On substituting $\beta = 1$, Theorem 3.2 gives the following result:

Corollary 3.2.1 If $f(z) \in C(\delta; p; A, B; C, D)$, then

$$r(1-Br)^{\frac{A-B}{B}}\left(\frac{1-Cr}{1-Dr}\right) \le \left|D_{\delta}^{p} f(z)\right| \le r(1+Br)^{\frac{A-B}{B}}\left(\frac{1+Cr}{1+Dr}\right), B \ne 0;$$

$$re^{-Ar}\left(\frac{1-Cr}{1-Dr}\right) \le \left|D_{\delta}^{p} f(z)\right| \le re^{-Ar}\left(\frac{1+Cr}{1+Dr}\right), B = 0.$$

For $\delta = 1, p = 0, \beta = 1, A = 1, B = -1$, Theorem 3.2 gives the following result due to Mehrok et al.[10]:

Corollary 3.2.2. Let $f(z) \in CS^*(C, D)$, then

$$\frac{r(1-Cr)}{(1+r)^2(1-Dr)} \le |f(z)| \le \frac{r(1+Cr)}{(1-r)^2(1+Dr)}.$$

For $\delta = 1, p = 0, \beta = 1, A = 1, B = -1, C = 1, D = -1$, Theorem 3.2 gives the result below due to Goel and Sohi [3]:

Corollary 3.2.3. Let $f(z)\in CS^{*}$, then

$$\frac{r(1-r)}{(1+r)^3}\le |f(z)|\le \frac{r(1+r)}{(1-r)^3}.$$

On putting $\delta=1, p=1, \beta=1, A=1, B=-1$ in Theorem 3.2, we get the following result due to Mehrok [7]:

Corollary 3.2.4. Let $f(z)\in C(C,D)$, then

$$\frac{(1-Cr)}{(1+r)^2(1-Dr)}\le |f'(z)|\le \frac{(1+Cr)}{(1-r)^2(1+Dr)}$$

and

$$\int_0^r \frac{(1-Ct)}{(1+t)^2(1-Dt)}dt\le |f(z)|\le \int_0^r \frac{(1+Ct)}{(1-t)^2(1+Dt)}dt.$$

On putting $\delta=1, p=1, \beta=1, A=1, B=-1, C=1, D=-1$ in Theorem 3.2, we get the following result:

Corollary 3.2.5. Let $f(z)\in C$, then

$$\frac{(1-r)}{(1+r)^3}\le |f'(z)|\le \frac{(1+r)}{(1-r)^3}$$

and

$$\int_0^r \frac{(1-t)}{(1+t)^3}dt\le |f(z)|\le \int_0^r \frac{(1+t)}{(1-t)^3}dt.$$

Theorem 3.3. If $f(z)\in C(\delta;p;\beta;A,B;C,D)$, then

$$\left|\arg\frac{D_\delta^p f(z)}{z}\right|\le \beta\sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right)+\frac{(A-B)}{B}\sin^{-1}(Br), B\neq 0; \tag{13}$$

$$\left|\arg\frac{D_\delta^p f(z)}{z}\right|\le \beta\sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right)+Ar, B=0. \tag{14}$$

The results are sharp.

Proof. From (3), we have

$$\left|\arg\frac{D_\delta^p f(z)}{z}\right|\le \beta\left|\arg\left(\frac{1+Cw(z)}{1+Dw(z)}\right)\right|+\left|\arg\frac{g(z)}{z}\right|. \tag{15}$$

It is well known that

$$\left|\arg\left(\frac{1+Cw(z)}{1+Dw(z)}\right)\right|\le \sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right). \tag{16}$$

Using (16) and Lemma 2.4 in (15), the results (13) and (14) can be easily obtained.

Sharpness follows for the function $f_n(z)$ defined in (12), where

$$\delta_1=\frac{z}{r}\left[\frac{-(C+D)r+i\left((1-C^2r^2)(1-D^2r^2)\right)^{\frac{1}{2}}}{1+CDr^2}\right] \text{ and } \delta_2=\frac{z}{r}\left[-Br+i\left(1-B^2r^2\right)^{\frac{1}{2}}\right].$$

On putting $\beta=1$, Theorem 3.3 gives the following result:

Corollary 3.3.1 If $f(z)\in C(\delta;p;A,B;C,D)$, then

$$\left|\arg\frac{D_\delta^p f(z)}{z}\right| \le \sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right)+\frac{(A-B)}{B}\sin^{-1}(Br), B\neq 0;$$
$$\left|\arg\frac{D_\delta^p f(z)}{z}\right| \le \sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right)+Ar, B=0.$$

For $\delta=1, p=0, \beta=1, A=1, B=-1$, Theorem 3.3 gives the following result due to Mehrok et al. [10]:

Corollary 3.3.2. Let $f(z)\in CS^*(C,D)$, then

$$\left|\arg\frac{f(z)}{z}\right| \le \sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right)+2\sin^{-1} r.$$

For $\delta=1, p=0, \beta=1, A=1, B=-1, C=1, D=-1$, Theorem 3.3 gives the result below due to Goel and Sohi [3]:

Corollary 3.3.3. Let $f(z)\in CS^*$, then

$$\left|\arg\frac{f(z)}{z}\right| \le \sin^{-1}\left(\frac{2r}{1+r^2}\right)+2\sin^{-1} r.$$

On putting $\delta=1, p=1, \beta=1, A=1, B=-1$ in Theorem 3.3, we get the following result due to Mehrok [7]:

Corollary 3.3.4. Let $f(z)\in C(C,D)$, then

$$|\arg f'(z)| \le \sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right)+2\sin^{-1} r.$$

On putting $\delta=1, p=1, \beta=1, A=1, B=-1, C=1, D=-1$ in Theorem 3.3, we get the following result:

Corollary 3.3.5. Let $f(z)\in C$, then

$$|\arg f'(z)| \le \sin^{-1}\left(\frac{2r}{1+r^2}\right)+2\sin^{-1} r.$$

Theorem 3.4. Let $f(z)\in C(\delta;p;\beta;A,B;C,D)$, then $f\in S(p,\delta)$ for $|z|<r_1$, where r_1 is the smallest positive root in (0,1) of

$$-ACDr^3+(CD-AC-AD+\beta BC-\beta BD)r^2+(C+D-A-\beta C+\beta D)r+1=0. \tag{17}$$

Proof. As $f(z)\in C(\delta;p;\beta;A,B;C,D)$, then using principle of subordination, we have

$$\frac{D_\delta^p f(z)}{g(z)}=\left(\frac{1+Cw(z)}{1+Dw(z)}\right)^\beta=[P(z)]^\beta,$$

or

$$D_\delta^p f(z)=[P(z)]^\beta g(z). \tag{18}$$

After differentiating (18) logarithmically, it yields

$$\frac{z\left(D_\delta^p f(z)\right)'}{D_\delta^p f(z)}=\beta\frac{zP'(z)}{P(z)}+\frac{zg'(z)}{g(z)}. \tag{19}$$

Now for $g\in S^*(A,B)$, we have

$$\mathrm{Re}\left(\frac{zg'(z)}{g(z)}\right)\ge\frac{1-Ar}{1-Br}.$$

Also from (11), we have

$$\left|\frac{1+Cw(z)}{1+Dw(z)}\right| = |P(z)| \leq \frac{1+Cr}{1+Dr}.$$

So

$$\left|\frac{zP'(z)}{P(z)}\right| \leq \frac{r(C-D)}{(1+Cr)(1+Dr)}. \tag{20}$$

So using (20), (19) yields,

$$\operatorname{Re}\left(\frac{z\left(D_\delta^p f(z)\right)'}{D_\delta^p f(z)}\right) \geq \operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) - \beta\left|\frac{zP'(z)}{P(z)}\right|$$

$$\geq \frac{1-Ar}{1-Br} - \beta\frac{r(C-D)}{(1+Cr)(1+Dr)}$$

$$= \frac{-ACDr^3 + (CD - AC - AD + \beta BC - \beta BD)r^2 + (C + D - A - \beta C + \beta D)r + 1}{(1-Br)(1+Cr)(1+Dr)}.$$

Hence $f(z) \in S(p,\delta)$ in $|z| < r_1$, where r_1 is the smallest positive root in (0,1) of

$$-ACDr^3 + (CD - AC - AD + \beta BC - \beta BD)r^2 + (C + D - A - \beta C + \beta D)r + 1 = 0.$$

Sharpness follows if we take $f_n(z)$ to be same as in (7).

On substituting $\beta = 1$, Theorem 3.4 gives the following result:

Corollary 3.4.1. Let $f(z) \in C(\delta; p; A,B;C,D)$, then $f \in S(\delta; p)$ for $|z| < r_2$, where r_2 is the smallest positive root in (0,1) of

$$-ACDr^3 + (CD - AC - AD + BC - BD)r^2 + (2D - A)r + 1 = 0.$$

4. The class $C_1(\delta; p; \beta; A,B;C,D)$

Theorem 4.1. If $f(z) \in C_1(\delta; p; \beta; A,B;C,D)$, then

$$|a_n| \leq \frac{1}{[1+(n-1)\delta]^p}\left\{\frac{1}{n!}\prod_{j=2}^{n}(A-(j-1)B) + \beta(C-D)\left[1 + \sum_{k=2}^{n-1}\frac{1}{k!}\prod_{j=2}^{k}(A-(j-1)B)\right]\right\}, \quad n \geq 2. \tag{21}$$

The bounds are sharp.

Proof. In Definition 1.3, using Lemma 2.1 and Lemma 2.5 and following the procedure of Theorem 3.1, the result (21) is obvious.

For $n = 2$, equality sign in (21) hold for the function $f_n(z)$ defined by

$$D_\delta^p f_n(z) = \frac{1}{Az}\left(\frac{1+C\delta_1 z}{1+D\delta_1 z}\right)^\beta\left[(1+B\delta_2 z)^{\frac{A}{B}} - 1\right], |\delta_1| = 1, |\delta_2| = 1. \tag{22}$$

On putting $\beta = 1$, Theorem 4.1 gives the following result:

Corollary 4.1.1. If $f(z) \in C_1(\delta; p; A,B;C,D)$, then

$$|a_n| \leq \frac{1}{[1+(n-1)\delta]^p}\left\{\frac{1}{n!}\prod_{j=2}^{n}(A-(j-1)B) + (C-D)\left[1 + \sum_{k=2}^{n-1}\frac{1}{k!}\prod_{j=2}^{k}(A-(j-1)B)\right]\right\}, \quad n \geq 2.$$

For $\delta = 1, p = 0, \beta = 1, A = 1, B = -1$, Theorem 4.1 gives the following result due to Mehrok et al.[11]:

Corollary 4.1.2. Let $f(z)\in CS_1^*(C,D)$, then

$$|a_n|\le 1+(n-1)(C-D).$$

On putting $\delta=1,p=0,\beta=1,A=1,B=-1,C=1,D=-1$ in Theorem 4.1, we get the following result:

Corollary 4.1.3. Let $f(z)\in CS_1^*$, then

$$|a_n|\le 2n-1.$$

For $\delta=1,p=1,\beta=1,A=1,B=-1$, Theorem 4.1 yields the following result due to Mehrok and Singh [8]:

Corollary 4.1.4. Let $f(z)\in C_1(C,D)$, then

$$|a_n|\le \frac{1}{n}+\frac{(n-1)(C-D)}{n}.$$

For $\delta=1,p=1,\beta=1,A=1,B=-1,C=1,D=-1$, Theorem 4.1 gives the following result due to Abdel Gawad and Thomas [1]:

Corollary 4.1.5. Let $f(z)\in C_1$, then

$$|a_n|\le 2-\frac{1}{n}.$$

Theorem 4.2. If $f(z)\in C_1(\delta;p;\beta;A,B;C,D)$, then for $|z|=r$, $0<r<1$, we have

$$\frac{1}{A}\left[1-(1-Br)^{\frac{A}{B}}\right]\left(\frac{1-Cr}{1-Dr}\right)^{\beta}\le\left|D_\delta^p f(z)\right|\le\frac{1}{A}\left[(1+Br)^{\frac{A}{B}}-1\right]\left(\frac{1+Cr}{1+Dr}\right)^{\beta}, B\neq 0; \tag{23}$$

$$\frac{1}{A}\left[1-e^{-Ar}\right]\left(\frac{1-Cr}{1-Dr}\right)^{\beta}\le\left|D_\delta^p f(z)\right|\le\frac{1}{A}\left[e^{Ar}-1\right]\left(\frac{1+Cr}{1+Dr}\right)^{\beta}, B=0. \tag{24}$$

Estimates are sharp.

Proof. Using Lemma 2.6 and following the procedure of Theorem 3.2, the results (23) and (24) are obvious.

Sharpness follows for the function $f(z)$ defined as

$$D_\delta^p f_n(z)=\begin{cases}\dfrac{1}{Az}\left(\dfrac{1+C\delta_1 z}{1+D\delta_1 z}\right)^{\beta}\left[(1+B\delta_2 z)^{\frac{A}{B}}-1\right], B\neq 0\\[2ex] \dfrac{1}{Az}\left(\dfrac{1+C\delta_1 z}{1+D\delta_1 z}\right)^{\beta}e^{A\delta_2 z}, B=0\end{cases}, |\delta_1|=1, |\delta_2|=1. \tag{25}$$

On putting $\beta=1$ in Theorem 4.2, we obtain the results:

Corollary 4.2.1. If $f(z)\in C_1(\delta;p;A,B;C,D)$, then for $|z|=r$, $0<r<1$, we have

$$\frac{1}{A}\left[1-(1-Br)^{\frac{A}{B}}\right]\left(\frac{1-Cr}{1-Dr}\right)\le\left|D_\delta^p f(z)\right|\le\frac{1}{A}\left[(1+Br)^{\frac{A}{B}}-1\right]\left(\frac{1+Cr}{1+Dr}\right), B\neq 0;$$

$$\frac{1}{A}\left[1-e^{-Ar}\right]\left(\frac{1-Cr}{1-Dr}\right)\le\left|D_\delta^p f(z)\right|\le\frac{1}{A}\left[e^{Ar}-1\right]\left(\frac{1+Cr}{1+Dr}\right), B=0.$$

For $\delta=1,p=0,\beta=1,A=1,B=-1$, Theorem 4.2 yields the following result due to Mehrok et al.[11]:

Corollary 4.2.2. Let $f(z)\in CS_1^*(C,D)$, then

$$\frac{r(1-Cr)}{(1+r)(1-Dr)}\le|f(z)|\le\frac{r(1+Cr)}{(1-r)(1+Dr)}.$$

On putting $\delta=1,p=0,\beta=1,A=1,B=-1,C=1,D=-1$ in Theorem 4.2, we get the following result:

Corollary 4.2.3. Let $f(z)\in CS_1^*$, then

$$\frac{r(1-r)}{(1+r)^2}\le|f(z)|\le\frac{r(1+r)}{(1-r)^2}.$$

For $\delta=1,p=1,\beta=1,A=1,B=-1$, Theorem 4.2 gives the following result due to Mehrok and Singh [8]:

Corollary 4.2.4. Let $f(z)\in C_1(C,D)$, then

$$\frac{(1-Cr)}{(1+r)(1-Dr)}\le|f'(z)|\le\frac{(1+Cr)}{(1-r)(1+Dr)}$$

and

$$\int_0^r\frac{(1-Ct)}{(1+t)(1-Dt)}dt\le|f(z)|\le\int_0^r\frac{(1+Ct)}{(1-t)(1+Dt)}dt.$$

For $\delta=1,p=1,\beta=1,A=1,B=-1,C=1,D=-1$, Theorem 4.2 gives the following result due to Abdel-Gawad and Thomas [1]:

Corollary 4.2.5. Let $f(z)\in C_1$, then

$$\frac{(1-r)}{(1+r)^2}\le|f'(z)|\le\frac{(1+r)}{(1-r)^2}$$

and

$$-\log(1+r)+\frac{2r}{1+r}\le|f(z)|\le\log(1-r)+\frac{2r}{1-r}.$$

Theorem 4.3. If $f(z)\in C_1(\delta;p;\beta;A,B;C,D)$, then

$$\left|\arg\frac{D_\delta^p f(z)}{z}\right|\le\beta\sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right)+\frac{A}{B}\sin^{-1}(Br), B\neq 0; \tag{26}$$

$$\left|\arg\frac{D_\delta^p f(z)}{z}\right|\le\beta\sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right)+Ar, B=0. \tag{27}$$

The results are sharp.

Proof. Using Lemma 2.7 and following the procedure of Theorem 3.3, the results (26) and (27) are obvious.

Sharpness follows for $f_n(z)$ to be same as in (25) where δ_1 and δ_2 are defined in Theorem 3.3.

On putting $\beta=1$ in Theorem 4.3, it yields the following result:

Corollary 4.3.1. If $f(z)\in C_1(\delta;p;A,B;C,D)$, then

$$\left|\arg\frac{D_\delta^p f(z)}{z}\right|\le\sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right)+\frac{A}{B}\sin^{-1}(Br), B\neq 0;$$

$$\left|\arg\frac{D_\delta^p f(z)}{z}\right| \le \sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right)+Ar, B=0.$$

On putting $\delta=1, p=0, \beta=1, A=1, B=-1$ in Theorem 4.3, we get the following result due to Mehrok et al. [11]:

Corollary 4.3.2. Let $f(z)\in CS_1^*(C,D)$, then

$$\left|\arg\frac{f(z)}{z}\right| \le \sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right)+\sin^{-1} r.$$

On putting $\delta=1, p=0, \beta=1, A=1, B=-1, C=1, D=-1$ in Theorem 4.3, we get the following result:

Corollary 4.3.3. Let $f(z)\in CS_1^*$, then

$$\left|\arg\frac{f(z)}{z}\right| \le \sin^{-1}\left(\frac{2r}{1+r^2}\right)+\sin^{-1} r.$$

For $\delta=1, p=1, \beta=1, A=1, B=-1$, Theorem 4.3 gives the following result due to Mehrok and Singh [8]:

Corollary 4.3.4. Let $f(z)\in C_1(C,D)$, then

$$|\arg f'(z)| \le \sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right)+\sin^{-1} r.$$

On putting $\delta=1, p=1, \beta=1, A=1, B=-1, C=1, D=-1$ in Theorem 4.3, we get the following result due to Abdel-Gawad and Thomas [1]. :

Corollary 4.3.5. Let $f(z)\in C$, then

$$|\arg f'(z)| \le \sin^{-1}\left(\frac{2r}{1+r^2}\right)+\sin^{-1} r.$$

Theorem 4.4. Let $f(z)\in C(\delta;p;\beta;A,B;C,D)$, then $f\in S(\delta,p)$ for $|z|<r_3$, where r_3 is the smallest positive root in (0,1) of

$$(CD+\beta BC-\beta BD)r^2+(C+D-\beta C+\beta D)r+1=0. \tag{28}$$

Results is sharp .

Proof. Following the procedure of Theorem 3.4 and using the result that for $h\in K(A,B)$, $\text{Re}\left(\frac{zh'(z)}{h(z)}\right)\ge\frac{1}{1-Br}$, the result (28) is obvious.

Sharpness follows for the function $f_n(z)$ defined in (22).

For $\beta=1$, Theorem 4.4 gives the following result:

Corollary 4.4.1. Let $f(z)\in C_1(\delta;p;A,B;C,D)$, then $f\in S(\delta;p)$ for $|z|<r_4$, where r_4 is the smallest positive root in (0,1) of

$$(CD+BC-BD)r^2+2Dr+1=0.$$

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