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UNIQUENESS OF $P(f)$ AND $[P(f)]^{(k)}$ CONCERNING WEAKLY WEIGHTED SHARING

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Abstract. In this paper, with the help of the idea of weakly weighted sharing introduced by Lin and Lin [Kodai Math. J. 29(2006), 269-280], we study the uniqueness of polynomial expression $P(f)$ and its derivatives $[P(f)]^{(k)}$ of meromorphic functions f sharing a small function. The main results in the paper significantly improved the result of Liu and Gu [Kodai Math. J. 27(3)(2004), 272-279]. This research work finds certain condition under which the polynomial $P(f)$ is reduced to a non-zero monomial and consequently, the class of the function f is characterized. Examples have been exhibited to show that some conditions in the main results simply can not be removed and also some inequalities are sharp.

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1. Introduction and Results

Let $\mathbb{C}$ be the complex plane, and let f be a non-constant meromorphic function defined on $\mathbb{C}$. We assume that the reader is familiar with the standard definitions and notations used in the Nevanlinna value distribution theory, such as $T(r,f)$, $m(r,f)$, $N(r,f)$ etc (see [9, 18, 20]). By $S(r,f)$ we denote any quantity the condition $S(r,f) - o\,(T(r,f))$ as $r \to \infty$ possibly outside of an exceptional set E of finite linear measure. A meromorphic function $a \equiv a(z)$ is called a small function with respect to f if either $a \equiv \infty$ or $T(r,a) = S(r,f)$. throughout the paper, we denote $S(f)$, the set of all small functions with respect to f. One can easily verify that $\mathbb{C} \cup \{\infty\} \subset S(f)$ and $S(f)$ forms a field over the field of complex numbers.

For $a \in \mathbb{C} \cup \{\infty\}$, the quantities $\delta(a,f)$ and $\Theta(a,f)$ are defined by

$$\begin{aligned}\delta(a,f) &= 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)},\\ \Theta(a,f) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f)}{T(r,f)}\end{aligned}$$

are respectively called the deficiency and ramification index of a for the function f.

For any two non-constant meromorphic functions f and g, and $a \in S(f)$, we say that f and g share a IM (CM) provided that $f - a$ and $g - a$ have the

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same set of zeros ignoring (counting) multiplicities. If $1/f$ and $1/g$ share 0 IM (CM), we say that f and g share ∞ IM (CM).

Definition 1.1. [11] Let $N_E(r,a)$ be the counting function of all the common zeros of $f-a$ and $g-a$ with the same multiplicities, and $N_0(r,a)$ be the counting function of all common zeros with ignoring multiplicities. We denote by $\overline{N}_E(r,a)$ and $\overline{N}_0(r,a)$ the reduced counting function of f and g corresponding to the counting functions $N_E(r,a)$ and $N_0(r,a)$, respectively. If

$$\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{g-a}\right)-2\overline{N}_E(r,a)=S(r,f)+S(r,g),$$

then we say that f and g share a CM. On the other way, if

$$\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{g-a}\right)-2\overline{N}_0(r,a)=S(r,f)+S(r,g),$$

then we say that f and g share a IM.

Definition 1.2. [11] Let k be a positive integer, and let f be a non-constant meromorphic function and $a\in S(f)$.

(i) $\overline{N}_{k)}\left(r,1/(f-a)\right)$ denotes the counting function of those a-points of f whose multiplicities are not greater than k, where each a-point is counted only once.

(ii) $\overline{N}_{(k}\left(r,1/(f-a)\right)$ denotes the counting function of those a-points of f whose multiplicities are not less than k, where each a-point is counted only once.

(iii) $\overline{N}_{k}\left(r,1/(f-a)\right)$ denotes the counting function of those a-points of f, where an a-point of f with multiplicity m counted m times if $m\leqslant k$ and k times if $m>k$.

Definition 1.3. For $a\in\mathbb{C}\cup\{\infty\}$ and $p\in\mathbb{N}$, and for a meromorphic function f, we denote $N_p\left(r,1/(f-a)\right)$ by

$$N_p\left(r,\frac{1}{f-a}\right)=\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-a}\right)+\cdots+\overline{N}_{(p}\left(r,\frac{1}{f-a}\right).$$

It is easy to see that $N_1\left(r,1/(f-a)\right)=\overline{N}\left(r,1/(f-a)\right)$.

Definition 1.4. We denote the quantity $\delta_k(a,f)$ by

$$\delta_k(a,f)=1-\limsup_{r\to\infty}\frac{N_k\left(r,a;f\right)}{T(r,f)},$$

where k is a positive integer. It is easy to see that $\delta_k(a,f)\geqslant\delta(a,f)$.

From the last few decades, the uniqueness theory of entire or meromorphic functions has become a prominent branch of the value distribution theory (see [20]). In 1977, Rubel and Yang [14] first established the result when an entire function f and its derivative f' share two complex values a and b CM, then they are identical *i.e.,* $f\equiv f'$. An easy computation shows that the general solution f of $f\equiv f'$ is $f(z)=ce^z$, where c is a non-zero constant. In 1979, improving the result in [10], analogous result corresponding to IM sharing was obtained by Mues and Steinmetz [13].

In course of time, many researchers such as Brück [6], Ahamed [1], Banerjee and Ahamed [2, 3, 4, 5], Gundersen [8], Yang [17] became more involved to find out the relation between an entire or meromorphic function with its higher order derivatives or with its (linear) differential polynomials, sharing one value or sets of values. Regarding finding the class of the functions, Yang and Zhang [19] (see also [22]) first considered the uniqueness of a power of a meromorphic (entire) function $F = f^n$ and its derivative F' when they share certain value.

In their paper, Yang and Zhang [19] explores the class of the functions satisfying a differential equations of some special forms. Now we are invoking the following results which elaborates the gradual developments to this setting of meromorphic functions. In 2009, Zhang [22] established a theorem improving the results obtained by Yang and Zhang [19].

In 2003, Yu [21] have considered the uniqueness problem of entire and meromorphic function when it shares one small functions with its derivative and proved the following interesting results.

Theorem A. [21] *Let k be a positive integer, and f be a non-constant entire function and $a \in S(f)$ and $a \not\equiv 0, \infty$. If f and $f^{(k)}$ share a CM and $\delta(0; f) > 3/4$, then $f \equiv f^{(k)}$.*

Theorem B. [21] *Let k be a positive positive integer, and f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$, f and a do not have any common pole. If f and $f^{(k)}$ share a CM and*

$$4\delta(0, f) + 2(k + 8)\Theta(\infty, f) > 2k + 19,$$

then $f \equiv f^{(k)}$

In the same paper, Yu [21] posed the following list of open questions.

(i) Can a CM shared value be replaced by an IM shared value in Theorem A?

(ii) Is the condition $\delta(0, f) > 3/4$ sharp in Theorem A?

(iii) Is the condition $4\delta(0, f) + 2(k + 8)\Theta(\infty, f) > 2k + 19$ sharp in Theorem B?

(iv) Can we remove the assumption "f and a do not have any common pole" in Theorem B?

In 2004, applying a different method of proof, Liu and Gu [12] established the following interesting results.

Theorem C. [12] *Let f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0$, ∞. If f and $f^{(k)}$ share the value a CM, and f and a do not have any common pole of same multiplicity and $2\delta(0, f) + 4\Theta(\infty, f) > 5$, then $f \equiv f^{(k)}$.*

Theorem D. [12] *Let f be a non-constant entire function, $a \in S(f)$ and $a \not\equiv 0$, ∞. If f and $f^{(k)}$ share the value a CM, and $\delta(0, f) > 1/2$, then $f \equiv f^{(k)}$.*

In 2006, Lin and Lin [11] introduced the notion of weakly weighted sharing. Let f and g be two non-constant meromorphic functions sharing a IM. For $a \in S(f) \cap S(g)$ and a positive integer k or ∞,

(i) $\overline{N}^{E}_{k)}(r,a)$ denotes the counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, both of their multiplicities are not greater than k, where each a-point is counted only once.

(ii) $\overline{N}^{0}_{(k}(r,a)$ denotes the reduced counting function of those a-points of f which are a-points og g, both of their multiplicities are not less than k, where each a-point is counted only once.

Definition 1.5. [11] For $a \in S(f) \cap S(g)$, if k be a positive integer or ∞, and

$$\overline{N}_{k)}(r,a;f) + \overline{N}_{k)}(r,a;g) - 2\overline{N}^{E}_{k)}(r,a) = S(r,f) + S(r,g)$$
$$\overline{N}_{(k+1}(r,a;f) + \overline{N}_{(+1}(r,a;g) - 2\overline{N}^{0}_{(k+1}(r,a) = S(r,f) + S(r,g)$$

or, if $k=0$ and

$$\overline{N}(r,a;f) + \overline{N}(r,a;g) - 2\overline{N}_0(r,a) = S(r,f) + S(r,g),$$

then we say that f and g weakly share a with weight k. Here, we write, f and g share (a,k) to mean that f and g share the value a weakly with weight k.

Obviously, if f and g share (a,k), then f and g share (a,p) for any p $(0 \leqslant p \leqslant k)$. Also, we note that f and g share a IM or CM if and only if f and g share $(a,0)$ or (a,∞), respectively.

Suppose $\mathcal{F}$ and $\mathcal{G}$ share 1 IM. By $N_L(r,1;\mathcal{F})$ we denotes the counting function of the 1-points of $\mathcal{F}$ whose multiplicities are greater than 1-points of $\mathcal{G}$, $N_L(r,1;\mathcal{G})$ is defined similarly.

With the help of the notion of weakly weighted sharing, in 2006, Lin and Lin [11] proved the next result investigating on the uniqueness problem between meromorphic functions f and their kth order derivative sharing a small function.

Theorem E. [11] *Let $k \geqslant 1$ and $2 \leqslant m \leqslant \infty$. Let f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If $f-a$ and $f^{(k)}-a$ share $(0,m)$ and*

$$2\delta_{k+2}(0,f) + 4\Theta(\infty,f) > 5,$$

then $f \equiv f^{(k)}$.

Theorem F. [11] *Let $k \geqslant 1$, and f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If $f-a$ and $f^{(k)}-a$ share $(0,1)$ and*

$$5\delta_{k+2}(0,f) + (k+9)\Theta(\infty,f) > k+12,$$

then $f \equiv f^{(k)}$.

Theorem G. [11] *Let $k \geqslant 1$, and f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share $(0,0)$ and*

$$5\delta_{k+2}(0,f) + (2k+7)\Theta(\infty,f) > 2k+11,$$

then $f \equiv f^{(k)}$.

Remark 1.1. To obtain the relation $f \equiv f^{(k)}$ in Theorems E, F, G, the respective conditions $2\delta_{k+2}(0,f) + 4\Theta(\infty,f) > 5$, $5\delta_{k+2}(0,f) + (k+9)\Theta(\infty,f) > k+12$ and $5\delta_{k+2}(0,f) + (2k+7)\Theta(\infty,f) > 2k+11$, cannot be removed.

Example 1.1. Let $f(z) = ze^z/(e^z+1)$. It is easy to see that

$$f(z) - 1 = \frac{z - e^{-z} - 1}{e^{-z}+1} \quad \text{and} \quad f'(z) - 1 = -\frac{e^{-z}\left(z - e^{-z} - 1\right)}{\left(e^{-z}+1\right)^2}.$$

Clearly, f and f' share the value 1 CM, and $\Theta(\infty,f) = 0$, $\delta_p(0,f) = 1$ for $p \geqslant 2$. A simple computation shows that

(i) $2\delta_{k+2}(0,f) + 4\Theta(\infty,f) = 2 \not> 5$.

(ii) $5\delta_{k+2}(0,f) + (k+9)\Theta(\infty,f) = 5 \not> k+12$.

(iii) $5\delta_{k+2}(0,f) + (2k+7)\Theta(\infty,f) = 5 \not> 2k+11$,

and $f \not\equiv f'$.

Later, in 2011, Xu and Hu [15] generalized Theorems E, F and G by considering $L(f)$ where $L(f)$ is defined by

$$L(f) = a_k f^{(k)} + a_{k-1} f^{(k-1)} + \cdots + a_0 f$$

and proved the results.

Theorem H. [15] *Let $k \geqslant 1$ and $2 \leqslant m \leqslant \infty$. Let f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If $f - a$ and $L(f) - a$ share $(0,m)$ and*

$$2\delta_{k+2}(0,f) + 4\Theta(\infty,f) > 5,$$

then $f \equiv L(f)$.

Theorem I. [15] *Let $k \geqslant 1$, and f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If $f - a$ and $L(f) - a$ share $(0,1)$ and*

$$\delta_{k+2}(0,f) + \frac{3}{2}\delta_2(0,f) + \left(\frac{7}{2} + k\right)\Theta(\infty,f) > k+5,$$

then $f \equiv L(f)$.

Theorem J. [15] *Let $k \geqslant 1$, and f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If $f - a$ and $L(f) - a$ share $(0,0)$ and*

$$2\delta_{k+2}(0,f) + \delta_2(0,f) + 2\Theta(0,f) + (2k+6)\Theta(\infty,f) > 2k+10,$$

then $f \equiv L(f)$.

Remark 1.2. To get the relation $f \equiv L(f)$ in *Theorems H, I, J*, the respective conditions $2\delta_{k+2}(0,f) + 4\Theta(\infty,f) > 5$, $\delta_{k+2}(0,f) + 3/2\delta_2(0,f) + (7/2+k)\,\Theta(\infty,f) > k+5$ and $2\delta_{k+2}(0,f) + \delta_2(0,f) + 2\Theta(0,f) + (2k+6)\Theta(\infty,f) > 2k+10$ cannot be removed.

Example 1.2. Let $f(z)=e^z/\big(e^{e^z}-1\big)$ and

$$L(f)=\frac{2e^z-1}{e^{2z}}f'(z)+\frac{2e^{2z}-2e^z+1}{e^{2z}}f(z).$$

Evidently,

$$f(z)-e^z=\frac{2e^z-e^{e^z}}{e^{e^z}-1}\ \text{ and }\ L(f)-e^z=\frac{2e^z-e^{e^z}}{\left(e^{e^z}-1\right)^2}.$$

Then f and $L(f)$ share the value $a(z)=e^z$ CM, and $\Theta(\infty,f)=0$, $\delta_p(0,f)=1$ for $p\geqslant 2$. A simple computation shows that

(i) $2\delta_{k+2}(0,f)+4\Theta(\infty,f)=2\not> 5$.

(ii) $\delta_{k+2}(0,f)+3/2\delta_2(0,f)+(7/2+k)\,\Theta(\infty,f)=1+3/2\not> k+5$.

(iii) $2\delta_{k+2}(0,f)+\delta_2(0,f)+2\Theta(0,f)+(2k+6)\Theta(\infty,f)=5\not> 2k+10$.

It is easy to see that $f\not\equiv L(f)$.

Remark 1.3. On finding the class of the meromorphic functions which satisfies $f\equiv f^{(k)}$ or $f\equiv L(f)$, we have the following observations.

(i) A non-constant meromorphic function f satisfies $f\equiv f^{(k)}$ or $f\equiv L(f)$, cannot have poles but may have zeros. Therefore, in this case, the solution function f must be entire.

(ii) The general solution of $f\equiv f^{(k)}$ is

$$f(z)=c_1e^z+c_2e^{\theta^2 z}+\cdots+c_2e^{\theta^{k-1}z},$$

where c_i $(i=1,2,\ldots,k)$ are complex constants, and $\theta=\cos\left(2\pi/k\right)+i\sin\left(2\pi/k\right)$.

(iii) The general solution of $f=L(f)$ is

$$f(z)=b_1e^{\beta_1 z}+b_2e^{\beta_2 z}+\ldots+b_ke^{\beta_k z},$$

where d_i $(i=1,2,\ldots,k)$ are complex constants, and β_j $(j=1,2,\ldots,k)$ are the roots of the equation

$$w^k+a_{k-1}w^{k-1}+\cdots+a_1w+a_0=0.$$

(iv) Let μ be a complex constant satisfying $\mu^k=1$. For positive integer n, if we choose $f(z)=ce^{\mu z/n}$, then it is not hard to verify that $f^n(z)-a(z)$ and $\left(f^n(z)\right)^{(k)}-a(z)$ share the value 0 CM, where $a(z)$ is a small function of f with $f^n(z)=\left(f^n(z)\right)^{(k)}$.

Therefore, it is natural to ask the question : *what happen if we consider some power of a meromorphic function f so that $f^n-a(z)$ and $(f^n)^{(k)}-a(z)$ share 0 CM?*

Zhang [22], Zhang and Yang [23] answered the above question and obtained a uniqueness result between f^n and $(f^n)^{(k)}$. They have shown that the function f actually assume the form

$$f(z)=ce^{\lambda z/n},$$

for non-zero constants c and λ with $\lambda^k=1$.

Our aim is to investigate on the question: what could be the possible relationship between a polynomial expression $P(f)$ of a meromorphic function f and its k-order derivatives $[P(f)]^{(k)}$ and also the solution class in Theorem E to Theorem J? Since no attempts till now made by any researcher, so investigating and exploring the above situation is the main motivation of writing this paper. Henceforth, throughout this paper, for a meromorphic function f, we consider a polynomial expression $P(f)$, which is a more general setting of power of f, defined by

$$P(f) = a_n f^n + \cdots + a_1 f + a_0.$$

In connection with the above discussions, it is therefore reasonable to raise some questions as below.

Question 1.1. (i) What happens if $P(f) - a$ and $[P(f)]^{(k)} - a$ share $(0, m)$ in all the above mentioned results?

(ii) Can we get an uniqueness relation between $P(f)$ and $[P(f)]^{(k)}$?

(iii) Can we also get a specific form of the function f if the answer of the questions is true?

Taking the above questions into background, we investigate to find the possible answers of them. To make our investigation easier, we will take some help of transformations (see [3]). Hence, we factorize the expression $P(f)$ as

$$P(f) = a_n (z - d_{p_1})^{p_1} (z - d_{p_2})^{p_2} \cdots (z - d_{p_s})^{p_s},$$

where a_j $(j = 0, 1, 2, \ldots, n-1)$, $a_n (\neq 0)$ and d_{p_i} $(i = 1, 2, \ldots, s)$ are distinct finite complex numbers, and $p_1, p_2, \ldots, p_s$, n and k all are positive integers with $\sum_{j=1}^{s} p_j = n$. Let $p = \max\{p_1, p_2, \ldots, p_s\}$, and we consider an arbitrary polynomial

$$\begin{aligned} Q(f_*) &= a_n \prod_{j=1, p_j \neq p}^{s} \left(f_* + d_p - d_{p_j}\right)^{p_j} \\ &= c_m f_*^m + \cdots + c_1 f_* + c_0, \end{aligned}$$

where $a_n = c_m$, $f_* = f - d_p$ and $m = n - p$. Obviously, we have

$$P(f) = f_*^p Q(f_*).$$

In particular, when $d_p = 0$, it is easy to see that $f_* = f$ and $P(f) = f^p Q(f)$.

We now state the main results of this paper.

Theorem 1.1. *Let $k \geqslant 1$ and $2 \leqslant m \leqslant \infty$, be two integers. Let f be a non-constant meromorphic function, and $a \in S(f)$ with $a \not\equiv 0, \infty$. If $P(f) - a$ and $[P(f)]^{(k)} - a$ share $(0, m)$ and $2n\delta_{k+2}(0, f) + 4\Theta(\infty, f) > 5$, then $P(f) \equiv [P(f)]^{(k)}$ i.e., $f_*^p Q(f_*) = [f_*^p Q(f_*)]^{(k)}$. Furthermore, if $p > k + 1$ then*

(i) *$Q(f_*)$ reduces to a non-zero monomial $c_j f_*^j (\not\equiv 0)$ for some $j \in \{0, 1, 2, \ldots, m\}$.*

(ii) *$f(z)$ takes the form*

$$f(z) = c e^{\lambda z/(p+j)} + d_p, \quad \text{where } c, \lambda \in \mathbb{C} \setminus \{0\} \text{ with } \lambda^k = 1.$$

Theorem 1.2. *Let $k(\geqslant 1)$ be an integer, f be a non-constant meromorphic function, and $a \in S(f)$ where $a \not\equiv 0, \infty$. If $P(f) - a$ and $[P(f)]^{(k)} - a$ share $(0, 1)$ and*

$$5n\delta_{k+2}(0, P(f)) + (k+9)\Theta(\infty, f) > 3n + k + 9,$$

then the conclusion of Theorem 1.1 holds.

Remark 1.4. The conclusion of *Theorem 1.2* ceases to be hold if we remove the condition

$$5n\delta_{k+2}(0, P(f)) + (k+9)\Theta(\infty, f) > 3n + k + 9.$$

Example 1.3. Let

$$f(z) = -\frac{a_1}{2a_2} + \frac{1}{18a_2}\left(9e^{3z} + 6z + 2\right)^{1/2},$$

where a_1, a_2 are two non-zero constants and $P(f) = a_2 f^2(z) + a_1 f(z)$. Clearly, we have $P(f) = \frac{1}{9}(9e^{3z} + 6z + 2)$ and

$$3(P(f) - z) = [P(f)]' - z.$$

Therefore, $P(f) - z$ and $[P(f)]' - z$ share $(0, \infty)$ with $\delta_3(0, P(f)) = 0$ and $\Theta(\infty, f) = 1$. Evidently,

$$5n\delta_{k+2}(0, P(f)) + (k+9)\Theta(\infty, f) = 10 \not> 17 = 3n + k + 9.$$

Hence $P(f) \not\equiv [P(f)]^{(k)}$.

Regarding IM sharing, we proved the following result.

Theorem 1.3. *Let $k(\geqslant 1)$ be an integer, f be a non-constant meromorphic function, and $a \in S(f)$ where $a \not\equiv 0, \infty$. If $P(f) - a$ and $[P(f)]^{(k)} - a$ share 0 IM and*

$$(k-1)\Theta(\infty, f) + \delta_2(\infty, f) + n[\delta_k(0, P(f)) + \delta_{k+1}(0, P(f))] > k + n,$$

then the conclusion of Theorem 1.1 holds.

Remark 1.5. The condition

$$(k-1)\Theta(\infty, f) + \delta_2(\infty, f) + n[\delta_k(0, P(f)) + \delta_{k+1}(0, P(f))] > k + n$$

in Theorem 1.3 is sharp which can be seen from the next example.

Example 1.4. Suppose that, $P(f) = f$, and $k = 1$, where

$$f(z) = \frac{2e^{2z}}{e^{2z} - 1}.$$

We see that

$$N(r, f) \sim T(r, f) \text{ and so we have } \delta_2(\infty, f) = 0.$$

It is easy to see that $\Theta(0, P(f)) = \delta_2(0, P(f)) = 1$. Clearly, $P(f)$ and $[P(f)]^{(k)}$ share the value 1 IM and

$$(k-1)\Theta(\infty, f) + \delta_2(\infty, f) + n[\delta_k(0, P(f)) + \delta_{k+1}(0, P(f))] = k + n$$

but $P(f) \not\equiv [P(f)]^{(k)}$.

2. LEMMAS

In this section, we present some necessary lemmas which will play key role to prove the main results. We define the functions $\mathfrak{F}$, $\mathfrak{G}$ and $\mathfrak{H}$ by

$$\mathfrak{F}=\frac{P(f)}{a}, \quad \mathfrak{G}=\frac{[P(f)]^{(k)}}{a}. \tag{2.1}$$

$$\mathfrak{H}=\left(\frac{\mathfrak{F}''}{\mathfrak{F}'}-2\frac{\mathfrak{F}'}{\mathfrak{F}-1}\right)-\left(\frac{\mathfrak{G}''}{\mathfrak{G}'}-2\frac{\mathfrak{G}'}{\mathfrak{G}-1}\right). \tag{2.2}$$

Lemma 2.1. [16] *Let f be a non-constant meromorphic function and let*

$$P(f)=a_nf^n+a_{n-1}f^{n-1}+\ldots+a_1f+a_0,$$

where $a_i\in S(f)$ for $i=0,1,2\ldots,n$; $a_n\neq 0$, be a polynomial in f of degree n. Then

$$T\left(r,P(f)\right)=n\,T(r,f)+S(r,f).$$

Lemma 2.2. [10] *If $N\left(r,1/g^{(k)}\middle|\, g\neq 0\right)$ denotes the counting function of those zeros of $g^{(k)}$ which are not the zeros of g, where a zero of $g^{(k)}$ is counted according to its multiplicity, then*

$$N\left(r,\frac{1}{g^{(k)}}\middle| g\neq 0\right)\leqslant k\overline{N}(r,g)+N_{k)}\left(r,\frac{1}{g}\right)+k\overline{N}_{(k}\left(r,\frac{1}{g}\right)+S(r,g)$$

Lemma 2.3. [20] *Let g be a non-constant meromorphic function, and let k be a positive integer. Then*

(i) $N\left(r,\frac{1}{g^{(k)}}\right)\leqslant N\left(r,\frac{1}{g}\right)+k\overline{N}(r,g)+S(r,g).$

(ii) $N\left(r,\frac{1}{g^{(k)}}\right)\leqslant T\left(r,g^{(k)}\right)-T(r,g)+N\left(r,\frac{1}{g}\right)+S(r,g).$

Lemma 2.4. *Let f be a non-constant meromorphic function, and let k be a positive integer. Then*

(i) $N_2\left(r,\frac{1}{[P(f)]^{(k)}}\right)\leqslant N\left(r,\frac{1}{P(f)}\right)+k\overline{N}(r,f)+S(r,f).$

(ii) $N_2\left(r,\frac{1}{[P(f)]^{(k)}}\right)\leqslant T\left(r,P[f]^{(k)}\right)-nT(r,f)+N_{k+2}\left(r,\frac{1}{P(f)}\right)+S(r,f).$

Proof. (i) By (i) of Lemma 2.3, replacing g by $P(f)$, it is easy to see that

$$\begin{aligned}&N_2\left(r,\frac{1}{[P(f)]^{(k)}}\right)+\sum_{j=3}^{\infty}\overline{N}\left(r,\frac{1}{[P(f)]^{(k)}}\middle|\geqslant j\right)\\ \leqslant\ &N_{k+2}\left(r,\frac{1}{P(f)}\right)+\sum_{j=k+3}^{\infty}\overline{N}\left(r,\frac{1}{P(f)}\middle|\geqslant j\right)+k\overline{N}(r,P(f))+S(r,f).\end{aligned}$$

i.e.,

$$\begin{aligned}
& N_2\left(r,\frac{1}{[P(f)]^{(k)}}\right)\\
\leqslant\;& N_{k+2}\left(r,\frac{1}{P(f)}\right)+\sum_{j=k+3}^{\infty}\overline{N}\left(r,\frac{1}{P(f)}\bigg|\geqslant j\right)-\sum_{j=3}^{\infty}\overline{N}\left(r,\frac{1}{[P(f)]^{(k)}}\bigg|\geqslant j\right)\\
& +k\overline{N}(r,P(f))+S(r,f)\\
\leqslant\;& N_{k+2}\left(r,\frac{1}{P(f)}\right)+k\overline{N}(r,f)+S(r,f)
\end{aligned}$$

(ii) We see that

$$\begin{aligned}
& N_2\left(r,\frac{1}{[P(f)]^{(k)}}\right)\\
\leqslant\;& N\left(r,\frac{1}{[P(f)]^{(k)}}\right)-\sum_{p=3}^{\infty}N\left(r,\frac{1}{[P(f)]^{(k)}}\bigg|\geqslant p\right)\\
=\;& T\left(r,[P(f)]^{(k)}\right)-m\left(r,\frac{1}{[P(f)]^{(k)}}\right)-\sum_{p=3}^{\infty}N\left(r,\frac{1}{[P(f)]^{(k)}}\bigg|\geqslant p\right)+O(1)\\
\leqslant\;& T\left(r,[P(f)]^{(k)}\right)-m\left(r,\frac{1}{P(f)}\right)-m\left(r,\frac{[P(f)]^{(k)}}{P(f)}\right)-\sum_{p=3}^{\infty}N\left(r,\frac{1}{[P(f)]^{(k)}}\bigg|\geqslant p\right)\\
& +S(r,f)\\
\leqslant\;& T\left(r,[P(f)]^{(k)}\right)-nT\left(r,f\right)+N\left(r,\frac{1}{P(f)}\right)\\
& -\sum_{p=3}^{\infty}N\left(r,\frac{1}{[P(f)]^{(k)}}\bigg|\geqslant p\right)+S(r,f)\\
\leqslant\;& T\left(r,[P(f)]^{(k)}\right)-nT\left(r,f\right)+N_{k+2}\left(r,\frac{1}{P(f)}\right)+\sum_{p=k+3}^{\infty}\overline{N}\left(r,\frac{1}{P(f)}\bigg|\geqslant p\right)\\
& -\sum_{p=3}^{\infty}N\left(r,\frac{1}{[P(f)]^{(k)}}\bigg|\geqslant p\right)+S(r,f)\\
\leqslant\;& T\left(r,[P(f)]^{(k)}\right)-nT\left(r,f\right)+N_{k+2}\left(r,\frac{1}{P(f)}\right)+S(r,f).
\end{aligned}$$

□

Lemma 2.5. [15] *Let $\mathfrak{F}$ and $\mathfrak{G}$ be two non-constant meromorphic functions such that they share (1,0), then*

$$\overline{N}_L\left(r,\frac{1}{\mathfrak{F}-1}\right)\leqslant\frac{1}{2}\overline{N}\left(r,\frac{1}{\mathfrak{F}}\right)+\frac{1}{2}\overline{N}(r,\mathfrak{F})+S(r,\mathfrak{F}).$$

Lemma 2.6. [15] *Let $\mathfrak{F}$ and $\mathfrak{G}$ be two non-constant meromorphic functions such that they share (0,0), then*

$$\overline{N}_L\left(r,\frac{1}{\mathfrak{F}-1}\right)\leqslant\overline{N}\left(r,\frac{1}{\mathfrak{F}}\right)+\overline{N}(r,\mathfrak{F})+S(r,\mathfrak{F}).$$

Lemma 2.7. [11] *Let m be a non-negative integer or ∞. Let $\mathfrak{F}$ and $\mathfrak{G}$ be two non-constant meromorphic functions sharing $(1,m)$ and $\mathfrak{H}$ be given by (2.2). If $\mathfrak{H}\not\equiv 0$, then*

(i) *for $2\leqslant m\leqslant\infty$*

$$\begin{aligned}&T(r,\mathfrak{F})\\ \leqslant\ & N_2(r,\mathfrak{F})+N_2\left(r,\frac{1}{\mathfrak{F}}\right)+N_2(r,\mathfrak{G})+N_2\left(r,\frac{1}{\mathfrak{G}}\right)+S(r,\mathfrak{F})+S(r,\mathfrak{G}).\end{aligned}$$

(ii) *for $m=1$*

$$\begin{aligned}&T(r,\mathfrak{F})\\ \leqslant\ & N_2(r,\mathfrak{F})+N_2\left(r,\frac{1}{\mathfrak{F}}\right)+N_2(r,\mathfrak{G})+N_2\left(r,\frac{1}{\mathfrak{G}}\right)+\overline{N}^L\left(r,\frac{1}{\mathfrak{F}-1}\right)\\ &+S(r,\mathfrak{F})+S(r,\mathfrak{G}).\end{aligned}$$

(iii) *for $m=0$*

$$\begin{aligned}&T(r,\mathfrak{F})\\ \leqslant\ & N_2(r,\mathfrak{F})+N_2\left(r,\frac{1}{\mathfrak{F}}\right)+N_2(r,\mathfrak{G})+N_2\left(r,\frac{1}{\mathfrak{G}}\right)+2\overline{N}^L\left(r,\frac{1}{\mathfrak{F}-1}\right)\\ &+\overline{N}^L\left(r,\frac{1}{\mathfrak{G}-1}\right)+S(r,\mathfrak{F})+S(r,\mathfrak{G}).\end{aligned}$$

The same inequality holds for $T(r,\mathfrak{G})$ also.

Lemma 2.8. *Let f be a transcendental meromorphic function, and $\alpha(\not\equiv 0,\infty)$ be a meromorphic function such that $T(r,\alpha)=S(r,f)$. Let b,c are any two finite non-zero distinct complex number. If*

$$\Psi(f)=\alpha P(f)[P(f)]^{(k)}, \tag{2.3}$$

where $k\geqslant 1$ is an integer, then

$$\begin{aligned}&2nT\left(r,f\right)\\ \leqslant\ & 2N\left(r,\frac{1}{P(f)}\right)+N\left(\frac{1}{\Psi(f)-b}\right)+N\left(r,\frac{1}{\Psi(f)-c}\right)-N(r,\Psi(f))\\ &-N\left(r,\frac{1}{\Psi'(f)}\right)+S(r,f).\end{aligned}$$

Proof. Since f is a non-constant meromorphic function, hence it is easy to see that $\Psi(f)$ is also. From (2.3), we obtain

$$\frac{1}{\alpha[P(f)]^2}=\frac{1}{\Psi(f)}\frac{[P(f)]^{(k)}}{P(f)}.$$

It is also easy to see that

$$m\left(r,\frac{1}{\alpha[P(f)]^2}\right)\leqslant m\left(r,\frac{1}{\Psi(f)}\right)+m\left(r,\frac{[P(f)]^{(k)}}{P(f)}\right)+O(1),$$

$$m\left(r,\frac{1}{\alpha[P(f)]^2}\right)=T\left(r,\alpha[P(f)]^2\right)-N\left(r,\frac{1}{\alpha[P(f)]^2}\right)+O(1)$$

and

$$m\left(r,\frac{1}{\Psi(f)}\right)=T\left(r,\frac{1}{\Psi(f)}\right)-N\left(r,\frac{1}{\Psi(f)}\right)+O(1).$$

Combining all the above relations, we can obtain

$$\begin{aligned}(2.4)\quad & T\left(r,\alpha[P(f)]^2\right)\\ \leqslant\ & N\left(r,\frac{1}{\alpha[P(f)]^2}\right)+T(r,\Psi(f))-N\left(r,\frac{1}{\Psi(f)}\right)+m\left(r,\frac{[P(f)]^{(k)}}{P(f)}\right)\\ & +O(1)\\ \leqslant\ & N\left(r,\frac{1}{\alpha[P(f)]^2}\right)+T(r,\Psi(f))-N\left(r,\frac{1}{\Psi(f)}\right)+O(1).\end{aligned}$$

Applying *Second Main Theorem*, we obtain

$$\begin{aligned}T(r,\Psi(f))\ \leqslant\ & N\left(r,\frac{1}{\Psi(f)}\right)+N\left(\frac{1}{\Psi(f)-b}\right)+N\left(r,\frac{1}{\Psi(f)-c}\right)\\ (2.5)\qquad\qquad & -N_1(r,\Psi(f))+S(r,f),\end{aligned}$$

where $N_1(r,\Psi(f))=2N(r,\Psi(f))-N(r,\Psi'(f))+N\left(r,\frac{1}{\Psi'(f)}\right)$.

Let z_0 be a pole of f of multiplicity $m(\geqslant 1)$. Then z_0 will be a pole of both $\Psi(f)$ and $\Psi'(f)$, of respective multiplicities $2mn+k+q$ and $2mn+k+q+1$, where $q=0$. If z_0 is neither a pole nor a zero of α, $q=t$, if z_0 is a pole of α of multiplicity t, and $q=-t$, if z_0 is a zero of α of multiplicity t, where t is a positive integer.

Therefore, we have

$$\begin{aligned}2(2mn+k+q)-(2mn+k+q+1)\ &=\ 2mn+k+q-1\\ &=\ m+n+2mn+k+q-m-1\\ &\geqslant\ m+n,\end{aligned}$$

because $2mn+k+q-m-1\geqslant k-1\geqslant 0$.

Since $T(r,\alpha)=S(r,f)$, hence it follows that

$$(2.6)\qquad N_1(r,\Psi(f))\geqslant N(r,\Psi(f))+N\left(r,\frac{1}{\Psi'(f)}\right)+S(r,f).$$

Therefore, we obtained from (2.4), (2.5) and (2.6) that

$$\begin{aligned}& T\left(r,\alpha[P(f)]^2\right)\\ \leqslant\ & N\left(r,\frac{1}{\alpha[P(f)]^2}\right)+N\left(\frac{1}{\Psi(f)-b}\right)+N\left(r,\frac{1}{\Psi(f)-c}\right)-N(r,\Psi(f))\\ & -N\left(r,\frac{1}{\Psi'(f)}\right)+S(r,f).\end{aligned}$$

A simple computation shows that

$$\begin{aligned}2nT(r,f)\ \leqslant\ & 2N\left(r,\frac{1}{P(f)}\right)+N\left(\frac{1}{\Psi(f)-b}\right)+N\left(r,\frac{1}{\Psi(f)-c}\right)\\ & -N(r,\Psi(f))-N\left(r,\frac{1}{\Psi'(f)}\right)+S(r,f).\end{aligned}$$

□

Lemma 2.9. *Let $\mathfrak{F}$ and $\mathfrak{G}$ be two non-constant meromorphic functions defined as in (2.1) be such that $\mathfrak{F} \equiv \mathfrak{G}$. If $p > k+1$ then*

(i) *$Q(f_*)$ reduces to a non-zero monomial $c_j f_*^j$ for some $j \in \{0, 1, 2, \ldots, m\}$.*
(ii) *$f(z)$ takes the form*

$$f(z) = ce^{\lambda z/(p+j)} + d_p, \quad \textit{where } c, \lambda \in \mathbb{C} \setminus \{0\} \textit{ with } \lambda^k = 1.$$

Proof. Since $\mathfrak{F} \equiv \mathfrak{G}$ *i.e.,* $P(f) \equiv [P(f)]^{(k)}$, hence we see that

$$f_*^p Q(f_*) = [f_*^p Q(f_*)]^{(k)}. \tag{2.7}$$

(i) Our aim is to show that $Q(f_8)$ reduces to a non-zero monomial $c_j f_*^j$ for some $j \in \{0, 1, 2, \ldots, m\}$.

On contrary, we suppose that $Q(f_*) = c_m f_*^m + \ldots + c_1 f^* + c_0$, in which at least two terms present. It then follows from (2.7) that f cannot have any poles *i.e.,* in other words f must be an entire function. Again since $p > k+1$, so one can check that 0 is an Picard exceptional value of f_*. Therefore, we can write f as $f_* = e^{h(z)}$, where h is a non-constant entire function.

It is easy to see that

$$c_j \left[f_*^{p+j} - \left(f_*^{p+j}\right)^{(k)} \right] = \phi_j \left(h', \ldots, h^{(k)}\right) e^{(p+j)h}, \tag{2.8}$$

where $\phi_j \equiv \phi_j \left(h', \ldots, h^{(k)}\right)$ $(j = 0, 1, \ldots, m)$ are differential polynomials in $h', \ldots, h^{(k)}$.

From (2.7) and (2.8), we get that

$$\phi_m e^{mh} + \ldots + \phi_1 e^h + \phi_0 \equiv 0. \tag{2.9}$$

Since $T(r, \phi_j) = S(r, f)$ for $(j = 0, 1, \ldots, m)$, therefore by Borel unicity theorem (see [7]), we obtain from (2.9) that $\phi_j \equiv 0$. Since $Q(f_*)$ contains at least two terms, so there must exist s, $t \in \{0, 1, 2\}$ with $s \neq t$, we obtain from (2.8) that

$$f_*^{p+s} \equiv [f_*^{p+s}]^{(k)} \text{ and } f_*^{p+t} \equiv [f_*^{p+t}]^{(k)},$$

which is a contradiction, otherwise, in this case, the function f would have two different forms. Therefore, it is easy to see that $Q(f_*) = c_j f_*^j$ for some $j \in \{0, 1, 2, \ldots, m\}$.

(ii) We note that the form of the function f_* satisfying $f_*^{p+j} = [f_*^{p+j}]^{(k)}$ will be $f_* = ce^{\lambda z/(p+j)}$, where c is a non-zero constant and $\lambda^k = 1$. Hence, we see that

$$f(z) = ce^{\lambda z/(p+j)} + d_p.$$

□

3. Proof of the main results

Proof of Theorem 1.1. Let $\mathfrak{F}$ and $\mathfrak{G}$ be defined as in (2.1) and $\mathfrak{H}$ be as in (2.2). Since $P(f)-a$ and $[P(f)]^{(k)}-a$ share $(0,m)$, so it follows that $\mathfrak{F}$ and $\mathfrak{G}$ share $(1,m)$.

Let $\mathfrak{H}\not\equiv 0$. Therefore, it follows from Lemma 2.4 and (i) of Lemma 2.7, we obtain

$$\begin{aligned}
& T(r,\mathfrak{G}) \\
\leqslant\ & N_2(r,\mathfrak{F})+N_2\left(r,\frac{1}{\mathfrak{F}}\right)+N_2(r,\mathfrak{G})+N_2\left(r,\frac{1}{\mathfrak{G}}\right)+S(r,\mathfrak{F})+S(r,\mathfrak{G}) \\
\leqslant\ & N_2(r,P(f))+N_2\left(r,\frac{1}{P(f)}\right)+N_2(r,[P(f)]^{(k)})+N_2\left(r,\frac{1}{[P(f)]^{(k)}}\right) \\
\leqslant\ & T\left(r,[P(f)]^{(k)}\right)-nT(r,f)+N_2\left(r,\frac{1}{P(f)}\right)+N_{k+2}\left(r,\frac{1}{P(f)}\right) \\
& +4\overline{N}(r,f)+S(r,f).
\end{aligned}$$

By Lemma 2.1, we obtain

$$\begin{aligned}
& T\left(r,[P(f)]^{(k)}\right) \\
\leqslant\ & T\left(r,[P(f)]^{(k)}\right)-nT(r,f)+N_2\left(r,\frac{1}{P(f)}\right)+N_{k+2}\left(r,\frac{1}{P(f)}\right) \\
& +4\overline{N}(r,f)+S(r,f).
\end{aligned}$$

A simple computation shows that

$$nT(r,f)\leqslant 2N_{k+2}\left(r,\frac{1}{P(f)}\right)+4\overline{N}(r,f)+S(r,f).$$

Therefore, for arbitrary $\epsilon>0$, we obtain

$$\left[2\delta_{k+2}\left(0,P(f)\right)+4\Theta(\infty,f)\right]T(r,f)\leqslant(6-n+\epsilon)T(r,f)+S(r,f),$$

which contradicts

$$2\delta_{k+2}\left(0,P(f)\right)+4\Theta(\infty,f)>6-n.$$

Therefore, we have $\mathfrak{H}\equiv 0$. By integration, we obtain

$$\frac{1}{\mathfrak{F}-1}=\frac{\mathfrak{A}}{\mathfrak{G}-1}+\mathfrak{B},$$

where $\mathfrak{A}(\neq 0)$ and $\mathfrak{B}$ are constants.

It is easy to see that

$$\mathfrak{F}=\frac{(\mathfrak{B}+1)\mathfrak{G}+(\mathfrak{A}-\mathfrak{B}-1)}{\mathfrak{B}\mathfrak{G}+(\mathfrak{A}-\mathfrak{B})} \tag{3.1}$$

and hence

$$T(r,\mathfrak{F})=T(r,\mathfrak{G})+S(r,f).$$

We discuss the following three cases.

Case 1. Suppose that $\mathfrak{B}\neq -1,0$.

Subcase 1.1. If $\mathfrak{A}-\mathfrak{B}-1\neq 0$, then from (3.1), we obtain

$$\overline{N}\left(r,\frac{1}{\mathfrak{G}+\dfrac{\mathfrak{A}-\mathfrak{B}-1}{\mathfrak{B}+1}}\right)=\overline{N}\left(r,\frac{1}{\mathfrak{F}}\right).$$

Applying *Second Main Theorem*, we obtain

$$\begin{aligned}
T(r,\mathfrak{G}) &\leqslant \overline{N}(r,\mathfrak{G})+\overline{N}\left(r,\frac{1}{\mathfrak{G}}\right)+\overline{N}\left(r,\frac{1}{\mathfrak{G}+\dfrac{\mathfrak{A}-\mathfrak{B}-1}{\mathfrak{B}+1}}\right)+S(r,\mathfrak{G})\\
&\leqslant \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{[P(f)]^{(k)}}\right)+\overline{N}\left(r,\frac{1}{P(f)}\right)+S(r,f)\\
&\leqslant \overline{N}(r,f)+T\left(r,[P(f)]^{(k)}\right)-nT(r,f)+\overline{N}_{k+2}\left(r,\frac{1}{P(f)}\right)\\
&\quad +\overline{N}\left(r,\frac{1}{P(f)}\right)+S(r,f).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&T\left(r,[P(f)]^{(k)}\right)\\
\leqslant\ &\overline{N}(r,f)+T\left(r,[P(f)]^{(k)}\right)-nT(r,f)+\overline{N}_{k+2}\left(r,\frac{1}{P(f)}\right)+\overline{N}\left(r,\frac{1}{P(f)}\right)\\
&+S(r,f),
\end{aligned}$$

which implies

$$\begin{aligned}
&nT(r,f)\\
\leqslant\ &\overline{N}(r,f)+\overline{N}_{k+2}\left(r,\frac{1}{P(f)}\right)+\overline{N}\left(r,\frac{1}{P(f)}\right)+S(r,f)\\
\leqslant\ &\overline{N}(r,f)+2\overline{N}_{k+2}\left(r,\frac{1}{P(f)}\right)+S(r,f).
\end{aligned}$$

This shows that, for arbitrary $\epsilon>0$,

$$\Big[2\delta_{k+2}\left(0,P(f)\right)+\Theta(\infty,f)\Big]T(r,f)\leqslant(3-n+\epsilon)T(r,f)+S(r,f),$$

which contradicts

$$2\delta_{k+2}\left(0,P(f)\right)+4\Theta(\infty,f)>6-n.$$

Subcase 1.2. Thus we have $\mathfrak{A}-\mathfrak{B}-1=0$. Then, it follows from (3.1) that

$$\overline{N}\left(r,\frac{1}{\mathfrak{G}+\dfrac{1}{\mathfrak{B}}}\right)=\overline{N}(r,\mathfrak{F}).$$

By the same argument as above, we can reached in a contradiction.

Case 2. Let $\mathfrak{B}=-1$.

Subcase 2.1. Suppose that $\mathfrak{A}+1\neq 0$. Then from (3.1) it is easy to see that

$$\overline{N}\left(r,\frac{1}{\mathfrak{G}_(\mathfrak{A}+1)}\right)=\overline{N}(r,\mathfrak{F}).$$

By the similar argument to the Case 1, we can arrive at a contradiction.

Subcase 2.2. Let $\mathfrak{A}+1=0$. Then from (3.1), we see that $\mathfrak{F}\mathfrak{G}=1$. Therefore, we have

$$P(f)[P(f)]^{(k)}=a^2. \tag{3.2}$$

Subcase 2.2.1. Let f be a rational function. Then, $P(f)$ and $[P(f)]^{(k)}$ are also rationals. Therefore, from (3.2), we see that a is a non-zero constant. So from (2.1), we see that $P(f)$ has no zero and pole. Since f is non-constant, hence we arrive at a contradiction.

Subcase 2.2.2. Let f be a transcendental meromorphic function. Then by Lemma 2.8 in view of (3.2), we obtain

$$\begin{aligned} 2nT(r,f) &\leqslant 2N\left(r,\frac{1}{P(f)}\right)+2T\left(r,P(f)[P(f)]^{(k)}\right)+S(r,f)\\ &\leqslant 2N\left(r,\frac{1}{P(f)}\right)+2T\left(r,a^2\right)+S(r,f)\\ &\leqslant 2N\left(r,\frac{1}{a^2}\right)+S(r,f)\\ &\leqslant S(r,f), \end{aligned}$$

which is a contradiction.

Case 3. Suppose that $\mathfrak{B}\equiv 0$.

Subcase 3.1. Let $\mathfrak{A}-1\neq 0$. Then, from (3.1) it is easy to see that

$$\overline{N}\left(r,\frac{1}{\mathfrak{G}+(\mathfrak{A}-1)}\right)=\overline{N}\left(r,\frac{1}{\mathfrak{F}}\right).$$

Similar to the argument as in Case 1, we can arrive at a contradiction.

Subcase 3.2. Therefore, we have $\mathfrak{A}-1=0$. From (3.1), we deduce that $\mathfrak{F}\equiv\mathfrak{G}$. In view of Lemma 2.9, it is easy to see that $Q(f_*)$ reduces to a non-zero monomial $c_jf_*^p$. Therefore, a simple computation shows that the function f takes the form

$$f(z)=ce^{\lambda z/(p+j)}+d_p,$$

where c is a non-zero constant and $\lambda^k=1$. □

Proof of Theorem 1.2. Let $\mathfrak{F}$ and $\mathfrak{G}$ be defined by (2.1). Then it is clear that $\mathfrak{F}-1=P(f)-a/a$ and $\mathfrak{G}=[P(f)]^{(k)}-a/a$. Since $P(f)-a$ and $[P(f)]^{(k)}-a$ share $(0,1)$, hence it follows that $\mathfrak{F}$ and $\mathfrak{G}$ share $(1,1)$ except the zeros and poles of $a(z)$.

Let $\mathfrak{H}$ be defined by (2.2) and we suppose that $\mathfrak{H} \not\equiv 0$. Then by Lemma 2.7, we obtain

$$\begin{aligned}(3.3)\quad & T(r,\mathfrak{G})\\ \leqslant\ & N_2(r,\mathfrak{F}) + N_2\left(r,\frac{1}{\mathfrak{F}}\right) + N_2(r,\mathfrak{G}) + N_2\left(r,\frac{1}{\mathfrak{G}}\right) + \overline{N}^L\left(r,\frac{1}{\mathfrak{G}-1}\right)\\ & +S(r,\mathfrak{F}) + S(r,\mathfrak{G}).\end{aligned}$$

By a simple computation, it is easy to see that

$$\begin{aligned}\overline{N}^L\left(r,\frac{1}{\mathfrak{G}-1}\right) &\leqslant \frac{1}{2}N\left(r,\frac{\mathfrak{G}}{\mathfrak{G}'}\right)\\ &\leqslant N\left(r,\frac{\mathfrak{G}'}{\mathfrak{G}}\right) + S(r,\mathfrak{G})\\ &\leqslant \frac{1}{2}\overline{N}(r,\mathfrak{G}) + \frac{1}{2}\overline{N}\left(r,\frac{1}{\mathfrak{G}'}\right) + S(r,\mathfrak{G})\\ &\leqslant \frac{1}{2}\overline{N}(r,f) + \frac{1}{2}\overline{N}\left(r,\frac{1}{[P(f)]^{(k)}}\right) + S(r,f).\end{aligned}$$

In view of Lemma 2.4, from (3.3), we obtain

$$\begin{aligned}& T\left(r,[P(f)]^{(k)}\right)\\ \leqslant\ & N_2(r,f) + N_2\left(r,\frac{1}{P(f)}\right) + N_2(r,[P(f)]^{(k)}) + \frac{1}{2}\overline{N}(r,f) + \frac{1}{2}\overline{N}\left(r,\frac{1}{[P(f)]^{(k)}}\right)\\ & +S(r,f)\\ \leqslant\ & N_{k+2}\left(r,\frac{1}{P(f)}\right) + T\left(r,[P(f)]^{(k)}\right) - nT(r,f) + N_{k+2}\left(r,\frac{1}{P(f)}\right)\\ & +\frac{1}{2}N_{k+2}\left(r,\frac{1}{P(f)}\right)\frac{k+9}{2}\overline{N}(r,f) + S(r,f).\end{aligned}$$

Therefore, for arbitrary $\epsilon > 0$, we obtain

$$\begin{aligned}nT(r,f) &\leqslant \frac{5}{2}N_{k+2}\left(r,\frac{1}{P(f)}\right) + \frac{k+9}{2}\overline{N}(r,f) + S(r,f)\\ &\leqslant \left\{\frac{5n+k+9}{2} - \frac{5n}{2}\delta_{k+2}(0,P(f)) - \frac{k+9}{2}\Theta(\infty,f) + \epsilon\right\}T(r,f)\\ &\quad +S(r,f),\end{aligned}$$

which shows that

$$5n\delta_{k+2}(0,P(f)) + (k+9)\Theta(\infty,f) \leqslant 3n+k+9.$$

This contradicts

$$5n\delta_{k+2}(0,P(f)) + (k+9)\Theta(\infty,f) > 3n+k+9.$$

Thus we have $\mathfrak{H} \equiv 0$. We obtain conclusion of Theorem 1.2 following the rest of the proof of Theorem 1.1. □

Proof of Theorem 1.3. Let $\mathfrak{F}$ and $\mathfrak{G}$ be defined by (2.1). Then it is clear that $\mathfrak{F} - 1 = P(f) - a/a$ and $\mathfrak{G} = [P(f)]^{(k)} - a/a$. Since $P(f) - a$ and $[P(f)]^{(k)} - a$ share 0 IM, it is evident that $\mathfrak{F}$ and $\mathfrak{G}$ share 1 except the zeros and poles of $a(z)$.

We suppose that $\mathfrak{F} \not\equiv \mathfrak{G}$. Set

$$\begin{aligned}\Psi : &= \frac{1}{\mathfrak{F}}\left(\frac{\mathfrak{G}'}{\mathfrak{G}-1} - (k+1)\frac{\mathfrak{F}'}{\mathfrak{F}-1}\right)\\ &= \frac{\mathfrak{G}}{\mathfrak{F}}\left(\frac{\mathfrak{G}'}{\mathfrak{G}-1} - \frac{\mathfrak{G}'}{\mathfrak{G}}\right) - (k+1)\left(\frac{\mathfrak{F}'}{\mathfrak{F}-1} - \frac{\mathfrak{F}'}{\mathfrak{F}}\right).\end{aligned}$$

We first suppose that $\Psi \equiv 0$. Then, we have

$$\mathfrak{G} - 1 = c\,(\mathfrak{F}-1)^{(k+1)}, \tag{3.4}$$

where c is a non-zero constant. Let z_0 be a pole of f with multiplicity $p(\geqslant 1)$ such that $a(z_0) \neq 0, \infty$. Clearly, z_0 is a pole of $\mathfrak{G}-1$ with multiplicity $np+k$, and a pole of $(\mathfrak{G}-1)^{k+1}$ with multiplicity $np(k+1)$. Then it follows from (3.4), we must have $np+k = np(k+1)$. If $np \geqslant 2$, then we arrive at a contradiction. Therefore, we have

$$N_{(2}(r,f) = S(r,f). \tag{3.5}$$

By the assumption

$$(k-1)\Theta(\infty,f) + \delta_2(\infty,f) + n[\delta_k(0,P(f)) + \delta_{k+1}(0,P(f))] > k+n,$$

which in turn shows that

$$(k-1)\Theta(\infty,f) + \delta_2(\infty,f) + n\delta_k(0,P(f)) > n.$$

By Lemma 2.1, we obtain

$$\begin{aligned} n(k+1)T(r,f) &= (k+1)T(r,\mathfrak{F}) + S(r,f)\\ &\leqslant T\left(r,(\mathfrak{F}-1)^{k+1}\right) + S(r,f)\\ &\leqslant T(r,\mathfrak{G}) + S(r,f)\\ &\leqslant T\left(r,\frac{\mathfrak{G}}{\mathfrak{F}}\right) + T(r,\mathfrak{F}) + S(r,f)\\ &\leqslant N\left(r,\frac{[P(f)]^{(k)}}{P(f)}\right) + m\left(r,\frac{[P(f)]^{(k)}}{P(f)}\right) + S(r,f)\\ &\leqslant k\overline{N}(r,f) + N_k(r,0;P(f)) + nT(r,f) + S(r,f)\\ &\leqslant (k-1)\overline{N}(r,f) + N_2(r,f) + N_k(r,0;P(f)) + nT(r,f)\\ &\quad +S(r,f)\\ &\leqslant (k+2n-(k-1)\Theta(\infty,f) - \delta_2(\infty,f) - n\delta_k(0,P(f)+\epsilon))\,T(r,f)\\ &\quad +S(r,f),\end{aligned}$$

which implies $(k-1)\Theta(\infty,f)+\delta_2(\infty,f)+n\delta_k(0,P(f)) \leqslant n+(1-k)n \leqslant n$. This contradicts

$$(k-1)\Theta(\infty,f) + \delta_2(\infty,f) + n\delta_k(0,P(f)) > n.$$

Therefore, we must have $\Psi \not\equiv 0$. By the *Fundamental estimate of logarithmic derivative* it follows that $m(r,\Psi) = S(r,f)$.

Let z_1 be a pole of f with multiplicity $q(\geqslant 1)$ such that $a(z_1) \neq 0,\ \infty$. Then a simple computation shows that

$$\Psi(z) = \begin{cases} O((z-z_1)^n) & q=1\\ O((z-z_1)^{n(q-1)}) & q \geqslant 2\end{cases}$$

In view of the definition of Ψ, we obtain

$$
\begin{aligned}
(3.6)\quad & N(r,\Psi)\\
&\leqslant \overline{N}\left(r,\frac{1}{\mathfrak{F}}\right)+\overline{N}_{k+1}\left(r,\frac{1}{P(f)}\right)+S(r,f)\\
&\leqslant N\left(r,\frac{1}{\frac{\mathfrak{F}-\mathfrak{G}}{\mathfrak{F}}}\right)+\overline{N}_{k+1}\left(r,\frac{1}{P(f)}\right)+S(r,f)\\
&\leqslant T\left(r,\frac{\mathfrak{G}}{\mathfrak{F}}\right)+\overline{N}_{k+1}\left(r,\frac{1}{P(f)}\right)+S(r,f)\\
&\leqslant N\left(r,\frac{[P(f)]^{(k)}}{P(f)}\right)+m\left(r,\frac{[P(f)]^{(k)}}{P(f)}\right)+\overline{N}_{k+1}\left(r,\frac{1}{P(f)}\right)+S(r,f)\\
&\leqslant k\overline{N}(r,f)+N_k\left(r,\frac{1}{P(f)}\right)+\overline{N}_{k+1}\left(r,\frac{1}{P(f)}\right)+S(r,f).
\end{aligned}
$$

Using (3.6), we deduce that

$$
\begin{aligned}
& N(r,f)-\overline{N}_{(2}(r,f)\\
&\leqslant N(r,0;\Psi)\\
&\leqslant T\left(r,\frac{1}{\Psi}\right)-m\left(r,\frac{1}{\Psi}\right)+S(r,f)\\
&\leqslant T(r,\Psi)-m\left(r,\frac{1}{\Psi}\right)+S(r,f)\\
&\leqslant N(r,\Psi)+m(r,\Psi)-m\left(r,\frac{1}{\Psi}\right)+S(r,f)\\
&\leqslant k\overline{N}(r,f)+N_k\left(r,\frac{1}{P(f)}\right)+N_{k+1}\left(r,\frac{1}{P(f)}\right)-m\left(r,\frac{1}{\Psi}\right)+S(r,f).
\end{aligned}
$$

From the definition of Ψ, it is easy to see that

$$
(3.7)\qquad m(r,P(f))\leqslant m\left(r,\frac{1}{\Psi}\right)+S(r,f).
$$

Therefore, it follows from (3.6) and (3.7) that

$$
\begin{aligned}
& nT(r,f)\\
&\leqslant (k-1)\overline{N}(r,f)+N_2(r,f)+N_k\left(r,\frac{1}{P(f)}\right)+N_{k+1}\left(r,\frac{1}{P(f)}\right)+S(r,f)\\
&\leqslant \left\{k+2n(k-1)\Theta(\infty,f)+\delta_2(\infty,f)+n[\delta_k(0,P(f))+\delta_{k+1}(0,P(f))]+\epsilon\right\}T(r,f)\\
&\quad +S(r,f),
\end{aligned}
$$

which contradicts

$$
(k-1)\Theta(\infty,f)+\delta_2(\infty,f)+n[\delta_k(0,P(f))+\delta_{k+1}(0,P(f))]>k+n.
$$

Therefore, it is evident $\mathfrak{F}\equiv\mathfrak{G}$ *i.e.*, $P(f)\equiv[P(f)]^{(k)}$. Since $p>k+1$, therefore, by Lemma 2.9, we obtain

(i) $Q(f_*)$ reduces to a non-zero monomial $c_j f_*^j$ for some $j\in\{0,1,\dots,m\}$.

(ii) $f(z)$ takes the form

$$f(z) = c\, e^{\lambda z/(p+j)} + d_p, \text{ where } c,\ \lambda \in \mathbb{C} \setminus \{0\} \text{ and } \lambda^k = 1.$$

□

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