Advanced Studies in Contemporary Mathematics
32 (2022), No. 2, pp. 185 - 197
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2022.32.2.185

QUASI BI-SLANT SUBMANIFOLDS OF KENMOTSU MANIFOLDS

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Abstract. The fundamental motivation behind the current paper is to define and study the notion of quasi bi-slant submanifolds of Kenmotsu manifolds as a generalization of slant, semi-slant, hemi-slant, bi-slant and quasi hemi-slant submanifolds. First and foremost, we obtain the necessary and sufficient condition for the integrability of distributions of quasi bi-slant submanifolds of Kenmotsu manifolds and afterwards, we investigate the conditions for quasi bi-slant submanifolds of Kenmotsu manifolds to be totally geodesic. At long last, we additionally provide some examples of such submanifolds.

1. Introduction

In 1969, Tanno [18] characterized connected almost contact metric manifold on the basis of constant sectional curvature of the plane section containing ξ. He classified them as: (i) if $K(X,\xi) > 0$, then the manifold is called a homogeneous Sasakian manifold, (ii) if $K(X,\xi) = 0$, then the manifold is global Riemannian product of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature; and (iii) if $K(X,\xi) < 0$, then the manifold is a warped product space $\mathbb{R} \times C^n$. In 1971, Kenmotsu [10] obtained some tensorial equations to characterize the manifold when $K(X,\xi) < 0$, which nowadays called as Kenmotsu manifold. It may be noticed that a Kenmotsu manifold is not a Sasakian manifold. Also, it is not compact because $\text{div}\xi = 2n$.

Investigation of submanifolds hypothesis has shown an expanding advancement in image processing, economic modelling, computer design along with mathematical physics and mechanics. As such Chen [8, 9] initiated the notion of slant submanifolds as a generalization of both invariant and anti-invariant submanifolds of an almost Hermitian manifold. After this, Papaghiuc [12] introduced a submanifold named as semi-slant submanifold which is a generalization of CR-submanifold, slant submanifold, invariant submanifold and anti-invariant submanifold. The idea of bi-slant submanifolds was given by Carriazo [5, 7] and he called them anti-slant submanifolds. Inspite of the fact that these submanifolds are proposed as hemi-slant submanifolds by Sahin in [16](see also [14, 15, 19]). Hemi-slant submanifolds are one of the particular case of bi-slant submanifolds. Moreover, many geometers studied the concept of slant submanifolds (see also [1, 3, 6, 11, 17]).

2000 *Mathematics Subject Classification.* 53C15, 53C40.

Key words and phrases. Quasi bi-slant submanifolds, Kenmotsu manifolds, integrability conditions, totally geodesic.
Submission Date: 10 March, 2021.

Taking into account of the above studies, we are interested to give the notion of quasi bi-slant submanifolds in which the tangent bundle consists of one invariant and two slant distributions and the Reeb vector field. In this paper, as a generalization of slant, semi-slant, hemi-slant, bi-slant and quasi hemi-slant submanifolds, we introduce quasi bi-slant submanifolds and after that we investigate the geometry of distributions of such submanifolds in detail.

This paper consists 5 sections. In section 2, we mention the basic definitions and formulas related to Kenmotsu manifold and their submanifolds. In section 3, we define quasi bi-slant submanifolds and obtain some lemmas for next section. In section 4, we give some necessary and sufficient conditions for the geometry of distributions. Finally in the last section, we construct some examples of such submanifolds.

2. **Preliminaries**

An odd dimensional C^∞ manifold $\mathcal{N}$ is said to be an almost contact metric manifold if it admits a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η with a Riemannian metric g which satisfy the following relations [4]:

$$\phi^2U = -U + \eta(U)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi U) = 0, \tag{2.1}$$

$$g(U, \xi) = \eta(U), \quad g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \tag{2.2}$$

where U, V are vector fields on $\mathcal{N}$.

Now if an almost contact metric manifold $\mathcal{N}(\phi, \xi, \eta, g)$ holds:

$$(\overline{\nabla}_U\phi)(V) = g(\phi U, V)\xi - \eta(V)\phi U, \quad \overline{\nabla}_U\xi = U - \eta(U)\xi \tag{2.3}$$

for any U, V tangent to $\mathcal{N}$, where $\overline{\nabla}$ is the Levi-civita connection, then $(\mathcal{N}, \phi, \xi, \eta, g)$ is called as Kenmotsu manifold [10, 13].

Also, note that the covariant derivative of ϕ is defined as

$$(\overline{\nabla}_U\phi)V = \overline{\nabla}_U\phi V - \phi\overline{\nabla}_U V. \tag{2.4}$$

Suppose, $\mathcal{M}$ be a Riemannian manifold isometrically immersed in $\mathcal{N}$ associated with the induced Riemannian metric g on $\mathcal{M}$. Let us assume throughout the paper that A represents the shape operator and h represents the second fundamental form of immersion of $\mathcal{M}$ into $\mathcal{N}$. Now the Gauss and Weingarten formulas of $\mathcal{M}$ into $\mathcal{N}$ are given respectively by

$$\overline{\nabla}_U V = \nabla_U V + h(U, V), \tag{2.5}$$

and

$$\overline{\nabla}_U Z = -A_Z U + \nabla^\perp_U Z \tag{2.6}$$

for any vector fields $U, V \in \Gamma(\mathcal{TM})$ and $Z \in \Gamma(\mathcal{T}^\perp\mathcal{M})$; where ∇ denotes the induced Riemannian connection on $\mathcal{M}$ and $\nabla^\perp$ denotes the connection defined on the normal bundle of $\mathcal{M}$.

Moreover, A_Z and h are related to each other by

$$g(h(U, V), Z) = g(A_Z U, V) \tag{2.7}$$

for any vector fields $U, V \in \Gamma(\mathcal{TM})$ and $Z \in \Gamma(\mathcal{T}^\perp\mathcal{M})$.

The mean curvature vector is denoted and defined by the following equation

$$\mathcal{H} = \frac{1}{m} trace(h) = \frac{1}{m} \sum_{r=1}^{m} h(e_r, e_r), \tag{2.8}$$

where m denotes the dimension of submanifold $\mathcal{M}$ and $\{e_r\}_{r=1}^{m}$ is the local orthonormal basis of tangent space at each point of $\mathcal{M}$.

Also, for any vector field $U, V \in \Gamma(\mathcal{TM})$ if $h(U, V) = 0$, then $\mathcal{M}$ is said to be totally geodesic and if $\mathcal{H} = 0$, then $\mathcal{M}$ is said to be a minimal submanifold. From the definition, it is clear that any totally geodesic submanifold is obviously a minimal submanifold.

Furthermore, let $\mathcal{M}$ be a submanifold of a Kenmotsu manifold $\mathcal{N}$ whose structure tensor ξ is tangent to the submanifold $\mathcal{M}$. Since ξ is tangent to $\mathcal{M}$, then from (2.5) we have

$$\overline{\nabla}_U \xi = \nabla_U \xi + h(U, \xi)$$

which due to equation (2.3) yields

$$\nabla_U \xi = U - \eta(U)\xi \tag{2.9}$$

and

$$h(U, \xi) = 0 \quad \text{(or equivalently} \quad A_Z \xi = 0). \tag{2.10}$$

Hence, we also have

$$\overline{\nabla}_U \xi = \nabla_U \xi. \tag{2.11}$$

3. **Quasi bi-slant submanifolds of Kenmotsu manifolds**

In the current part of the paper, we present the quasi bi-slant submanifolds of Kenmotsu manifolds and we acquire the necessary and sufficient conditions for the distributions associated with the definition of such submanifolds to be integrable.

Definition 3.1. *A submanifold $\mathcal{M}$ of a Kenmotsu manifold $(\mathcal{N}, \phi, \xi, \eta)$ is defined as quasi bi-slant submanifold if there exist four orthogonal distributions D, D_1 and D_2 of $\mathcal{M}$, at the point $p \in M$ such that*
(1) $\mathcal{TM}$ possess the orthogonal direct decomposition as

$$\mathcal{TM} = D \oplus D_1 \oplus D_2 \oplus < \xi >,$$

where $< \xi >$ denotes the distribution spanned by ξ.
(2) The distribution D is invariant under ϕ, i.e., $\phi D = D$.
(3) $\phi D_1 \perp D_2$ and $\phi D_2 \perp D_1$.
(4) The distributions D_1 and D_2 are slant with slant angle θ_1, θ_2, respectively.
Taking the dimension of distributions D, D_1 and D_2 as m, m_1 and m_2, respectively. One can easily observe the following conditions:

- *If $m \neq 0$ and $m_1 = m_2 = 0$, then $\mathcal{M}$ is an invariant submanifold.*
- *If $m = m_1 = 0$ and $m_2 \neq 0$, $\theta_2 = \frac{\pi}{2}$, then $\mathcal{M}$ is anti-invariant submanifold.*
- *If $m \neq 0, m_1 \neq 0, \theta_1 = \frac{\pi}{2}$ and $m_2 = 0$, then $\mathcal{M}$ is semi-invariant submanifold.*
- *If $m = 0, m_1 = 0$ and $m_2 \neq 0, 0 < \theta_2 < \frac{\pi}{2}$, then $\mathcal{M}$ is slant submanifold with slant angle θ_2.*
- *If $m = 0, m_1 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ and $m_2 = 0$, then $\mathcal{M}$ is slant submanifold with slant angle θ_1.*
- *If $m = 0, m_1 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ and $m_2 \neq 0$ with $\theta_2 = \frac{\pi}{2}$, then $\mathcal{M}$ is hemi-slant submanifold.*

• *If* $m = 0, m_1 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ *and* $m_2 \neq 0, 0 < \theta_2 < \frac{\pi}{2}$*, then* $\mathcal{M}$ *is bi-slant submanifold.*
• *If* $m \neq 0, m_1 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ *and* $m_2 \neq 0$ *with* $\theta_2 = \frac{\pi}{2}$*, then we may call* $\mathcal{M}$ *as quasi hemi-slant submanifold.*
• *If* $m \neq 0, m_1 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ *and* $m_2 \neq 0, 0 < \theta_2 < \frac{\pi}{2}$*, then* $\mathcal{M}$ *is called proper quasi bi-slant submanifold.*

Remark 1: Above definition can be generalised by taking $\mathcal{TM} = D \oplus D_{\theta_1} \oplus D_{\theta_2} \oplus ... \oplus D_{\theta_k}$. Hence we can define multi-slant submanifolds, quasi multi-slant submanifolds, etc.

Let $\mathcal{M}$ be a quasi bi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$, then for any $X \in \Gamma(\mathcal{TM})$, we have

$$X = PX + P_1X + P_2X + \eta(X)\xi, \tag{3.1}$$

where P, P_1 and P_2 are the projections on the distributions D, D_1 and D_2, respectively. For any $X \in \Gamma(\mathcal{TM})$, we can write

$$\phi X = \nu X + \omega X, \tag{3.2}$$

where νX and ωX are tangential and normal components of ϕX on $\mathcal{M}$, respectively. Similarly, for any $Z \in \mathcal{T}^{\perp}\mathcal{M}$, we have

$$\phi Z = BZ + CZ, \tag{3.3}$$

where $BZ \in \Gamma(\mathcal{TM})$ and $CZ \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$. Using (3.1) and (3.2), we obtain

$$\phi X = \nu PX + \omega PX + \nu P_1X + \omega P_1X + \nu P_2X + \omega P_2X.$$

Since $\phi D = D$, we have $\omega PX = 0$. Therefore, we get

$$\phi X = \nu PX + \nu P_1X + \omega P_1X + \nu P_2X + \omega P_2X. \tag{3.4}$$

Thus we have the following consequences:

$$\phi(\mathcal{TM}) = D \oplus \nu D_1 \oplus \nu D_2, \tag{3.5}$$

and

$$\mathcal{T}^{\perp}\mathcal{M} = \omega D_1 \oplus \omega D_2 \oplus \mu, \tag{3.6}$$

where μ is orthogonal complement of $\omega D_1 \oplus \omega D_2$ in $\mathcal{T}^{\perp}\mathcal{M}$.
The covariant derivative of projection morphisms in (3.2) and (3.3) are defined as [15]

$$(\overline{\nabla}_U \nu)V = \nabla_U \nu V - \nu \nabla_U V, \tag{3.7}$$

$$(\overline{\nabla}_U \omega)V = \nabla^{\perp}_U \omega V - \omega \nabla_U V, \tag{3.8}$$

$$(\overline{\nabla}_U B)Z = \nabla_U BZ - B\nabla^{\perp}_U Z, \tag{3.9}$$

$$(\overline{\nabla}_U C)Z = \nabla^{\perp}_U CZ - C\nabla^{\perp}_U Z \tag{3.10}$$

for any $U, V \in \Gamma(\mathcal{TM})$ and $Z \in \Gamma(\mathcal{T}^{\perp}\mathcal{M})$. Taking into account of the condition (3) in Definition 3.1, (3.2) and (3.3), we have the followings observations [2]:

$$(a)\ \nu D_i \subset D_i, \qquad (b) B\omega D_i = D_i \quad for\ any\ i = 1, 2.$$

Lemma 3.2. *Let $\mathcal{M}$ be a quasi bi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$. Then we have the following identities:*
(i) $\nu^2 X_1 + B\omega X_1 = -X_1$,
(ii) $\omega\nu X_1 + C\omega X_1 = 0$,
(iii) $\nu^2 X_2 + B\omega X_2 = -X_2$,
(iv) $\omega\nu X_2 + C\omega X_2 = 0$
for any $X_1 \in D_1$ *and* $X_2 \in D_2$.

Proof. Using (2.1), (3.2) and (3.3) and then comparing tangential and normal components, one can easily get these assertions. □

Lemma 3.3. *Let $\mathcal{M}$ be a quasi bi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$, then we have the following conditions:*
(i) $\nu^2 X_1 = -(cos^2\theta_1)X_1$,
(ii) $g(\nu X_1, \nu Y_1) = (cos^2\theta_1)g(X_1, Y_1)$,
(iii) $g(\omega X_1, \omega Y_1) = (sin^2\theta_1)g(X_1, Y_1)$
for any $X_1, Y_1 \in \Gamma(D_1)$

Proof. (i) For any $X_1, Y_1 \in \Gamma(D_1)$, we have
$cos\theta_1 = \frac{g(\phi X_1, \nu X_1)}{||\phi X_1||.||\nu X_1||} = \frac{-g(X_1, \nu^2 X_1)}{||X_1||.||\nu X_1||}$ and also, $cos\theta_1 = \frac{||\nu X_1||}{||\phi X_1||}$.
Thus we have, $(cos^2\theta_1)||X_1||^2 = -g(X_1, \nu^2 X_1)$, which implies that $g(X_1, X_1)cos^2\theta_1 = -g(X_1, \nu^2 X_1)$. Hence, $\nu^2 X_1 = -(cos^2\theta_1)X_1$.
(ii) For any $X_1, Y_1 \in \Gamma(D_1)$, using (3.2) and Lemma 2(i), we have

$$\begin{aligned} g(\nu X_1, \nu Y_1) &= g(\phi X_1 - \omega X_1, \nu Y_1) \\ &= -g(X_1, \nu^2 Y_1) \\ &= (cos^2\theta_1)g(X_1, Y_1). \end{aligned}$$

(iii) Using (3.2), Lemma 3.3(i) and Lemma 3.3(ii), we have $g(\omega X_1, \omega Y_1) = (sin^2\theta_1)g(X_1, Y_1)$. □

In a similar way as above, one can obtain the following lemma:

Lemma 3.4. *Let $\mathcal{M}$ be a quasi bi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$, then we have the following conditions:*
(i) $\nu^2 X_2 = -(cos^2\theta_2)X_2$,
(ii) $g(\nu X_2, \nu Y_2) = (cos^2\theta_2)g(X_2, Y_2)$,
(iii) $g(\omega X_2, \omega Y_2) = (sin^2\theta_2)g(X_2, Y_2)$
for any $X_2, Y_2 \in \Gamma(D_2)$.

Further, with the help of (2.3), (2.5), (2.6), (3.2) and (3.3), one can easily obtain the following lemma:

Lemma 3.5. *Let $\mathcal{M}$ be a quasi bi-slant submanifold of Kenmotsu manifold $\mathcal{N}$, then for any $U, V \in \Gamma(\mathcal{TM})$, we have*
(i) $\nabla_U \nu V - A_{\omega V}U - \nu\nabla_U V - Bh(U, V) = g(\nu U, V)\xi - \eta(V)\nu U$,
(ii) $h(U, \nu V) + \nabla^{\perp}_U \omega V - \omega\nabla_U V - Ch(U, V) = -\eta(V)\omega U$.

Proof. Since $\mathcal{N}$ is a Kenmotsu manifold, so we have

$$(\overline{\nabla}_U\phi)(V) = g(\phi U, V)\xi - \eta(V)\phi U.$$

By using (2.4) and (3.2), we have

$$\overline{\nabla}_U(\phi V) - \phi(\overline{\nabla}_U V) = g(\nu U + \omega U, V)\xi - \eta(V)(\nu U + \omega U).$$

Now by using (2.5) and (3.2) the above equation becomes

$$\overline{\nabla}_U \nu V + \overline{\nabla}_U \omega V - \phi(\nabla_U V) - \phi(h(U,V)) = g(\nu U, V)\xi - \eta(V)\nu U - \eta(V)\omega U.$$

Again with the help of (2.5), (2.6), (3.2) and (3.3), the last equation leads to

$$\nabla_U \nu V + h(U,\nu V) - A_{\omega V}U + \nabla^{\perp}_U \omega V - \nu\nabla_U V - \omega\nabla_U V - Bh(U,V) - Ch(U,V)$$
$$= g(\nu U, V)\xi - \eta(V)\nu U - \eta(V)\omega U.$$

On comparing the tangential and normal parts in the last equation we get the required assertion. □

Using equations (3.7) and (3.8) in Lemma 3.5, we have the following :

Lemma 3.6. *Let $\mathcal{M}$ be a quasi bi-slant submanifold of Kenmotsu manifold $\mathcal{N}$. Then, we have $(\overline{\nabla}_U \nu)V = A_{\omega V}U + Bh(U,V) + g(\nu U, V)\xi - \eta(V)\nu U$,*
$(\overline{\nabla}_U \omega)V = Ch(U,V) - h(U,\nu V) - \eta(V)\omega U$
for any $U, V \in \Gamma(\mathcal{TM})$.

4. **Integrability of distributions and totally geodesic foliations**

This part comprises the necessary and sufficient condition for integrability of the distributions D, D_1 and D_2.

Theorem 4.1. *Let $\mathcal{M}$ be a quasi bi-slant submanifold of Kenmotsu manifold $\mathcal{N}$. The invariant distribution D is integrable if and only if*

$$g(\nabla_U \nu V - \nabla_V \nu U, \nu P_1 X + \nu P_2 X) = g(h(V,\nu U) - h(U,\nu V), \omega P_1 X + \omega P_2 X) \tag{4.1}$$

for any $U, V \in \Gamma(D)$ and $X = P_1 X + P_2 X \in \Gamma(D_1 \oplus D_2)$.

Proof. The invariant distribution D is integrable on $\mathcal{M}$ iff $g([U,V],\xi) = 0$ and $g([U,V],X) = 0$ for any $U, V \in \Gamma(D), X \in \Gamma(D_1 \oplus D_2)$ and $\xi \in \Gamma(\mathcal{TM})$.
Since $\mathcal{M}$ is a quasi bi-slant submanifold of Kenmotsu manifold $\mathcal{N}$. So, we immediately have

$$\begin{aligned} g([U,V],\xi) &= g(\overline{\nabla}_U V,\xi) - g(\overline{\nabla}_V U,\xi) \\ &= Ug(V,\xi) - g(V,\overline{\nabla}_U \xi) - Vg(U,\xi) + g(U,\overline{\nabla}_V \xi) \\ &= g(U, V - \eta(V)\xi) - g(V, U - \eta(U)\xi) = 0. \end{aligned}$$

Thus, invariant distribution D is integrable iff $g([U,V],X) = 0$.
Now, for any $U, V \in \Gamma(D)$ and $X = P_1 X + P_2 X \in \Gamma(D_1 \oplus D_2)$, with the help of (2.2), (2.4), we have

$$\begin{aligned} g([U,V],X) &= g(\phi([U,V]),\phi X) + \eta([U,V])\eta(X) \\ &= g(\phi(\overline{\nabla}_U V - \overline{\nabla}_V U),\phi X) + g([U,V],\xi)g(X,\xi) \\ &= g(\phi(\overline{\nabla}_U V),\phi X) - g(\phi(\overline{\nabla}_V U),\phi X) \\ &= g(\overline{\nabla}_U \phi V - (\overline{\nabla}_U \phi)V,\phi X) - g(\overline{\nabla}_V \phi U - (\overline{\nabla}_V \phi)U,\phi X) \\ &= g(\overline{\nabla}_U \phi V,\phi X) - g((\overline{\nabla}_U \phi)V,\phi X) - g(\overline{\nabla}_V \phi U,\phi X) + g((\overline{\nabla}_V \phi)U,\phi X). \end{aligned}$$

Now using the fact that $\omega U = \omega V = 0$ for any $U, V \in \Gamma(D)$ and (2.2),(2.3) and (3.2), we obtain

$$\begin{aligned} g([U,V],X) &= g(\overline{\nabla}_U(\nu V + \omega V), \phi X) - g(\overline{\nabla}_V(\nu U + \omega U), \phi X) \\ &\quad - \{g(\phi U, V)g(\xi, \phi X) - \eta(V)g(\phi U, \phi X)\} \\ &\quad + \{g(\phi V, U)g(\xi, \phi X) - \eta(U)g(\phi V, \phi X)\} \\ &= g(\overline{\nabla}_U(\nu V), \phi X) - g(\overline{\nabla}_V(\nu U), \phi X). \end{aligned}$$

Using equation (2.5), we have

$$g([U,V],X) = g(\nabla_U(\nu V) - \nabla_V(\nu U), \phi X) + g(h(U,\nu V) - h(V,\nu U), \phi X).$$

Again using equation (3.2) for any $X = P_1X + P_2X \in \Gamma(D_1 \oplus D_2)$, we get

$$\begin{aligned} g([U,V],X) &= g(\nabla_U \nu V - \nabla_V \nu U, \nu P_1 X + \nu P_2 X) \\ &\quad + g(h(U,\nu V) - h(V,\nu U), \omega P_1 X + \omega P_2 X). \end{aligned}$$

This proves the assertion.

□

Theorem 4.2. *Let $\mathcal{M}$ be a quasi bi-slant submanifold of a Kenmotsu $\mathcal{N}$. The slant distribution D_1 is integrable if and only if*

$$\begin{aligned} g(A_{\omega V_1}U_1 - A_{\omega U_1}V_1, \nu X) &= g(A_{\omega\nu V_1}U_1 - A_{\omega\nu U_1}V_1, X) \\ &\quad + g(\nabla^{\perp}_{U_1}\omega V_1 - \nabla^{\perp}_{V_1}\omega U_1, \omega P_2 X) \end{aligned} \tag{4.2}$$

for any $U_1, V_1 \in \Gamma(D_1)$ and $X \in \Gamma(D \oplus D_2)$.

Proof. For any $U_1, V_1 \in \Gamma(D_1)$ and $X = PX + P_2X \in \Gamma(D \oplus D_2)$, the distribution D_1 is integrable on $\mathcal{M}$ if and only if $g([U_1,V_1],\xi) = 0$ and $g([U_1,V_1],X) = 0$, where $\xi \in \Gamma(\mathcal{TM})$. Now, the first case is trivial as in above Theorem 4.1. So, the slant distribution D_1 is integrable if and only if $g([U_1,V_1],X) = 0$.

Now, for any $U_1, V_1 \in \Gamma(D_1)$ and $X = PX + P_2X \in \Gamma(D \oplus D_2)$, by using (2.2) we get

$$\begin{aligned} g([U_1,V_1],X) &= g(\phi[U_1,V_1], \phi X) + \eta([U_1,V_1])\eta(X) \\ &= g(\phi(\overline{\nabla}_{U_1}V_1), \phi X) - g(\phi(\overline{\nabla}_{V_1}U_1), \phi X). \end{aligned}$$

Now using (2.3), (2.4) and (3.3), we get

$$\begin{aligned} g([U_1,V_1],X) &= g(\overline{\nabla}_{U_1}(\nu V_1 + \omega V_1), \phi X) - g(\overline{\nabla}_{V_1}(\nu U_1 + \omega U_1), \phi X) \\ &\quad - \{g(\phi U_1, V_1)g(\xi, \phi X) - \eta(V_1)g(\phi U_1, \phi X)\} \\ &\quad + \{g(\phi V_1, U_1)g(\xi, \phi X) - \eta(U_1)g(\phi V_1, \phi X)\} \\ &= g(\overline{\nabla}_{U_1}(\nu V_1), \phi X) + g(\overline{\nabla}_{U_1}(\omega V_1), \phi X) - g(\overline{\nabla}_{V_1}(\nu U_1), \phi X) \\ &\quad - g(\overline{\nabla}_{V_1}(\omega U_1), \phi X). \end{aligned}$$

In account of (2.3), (2.4) and (2.6), we have

$$\begin{aligned} g([U_1,V_1],X) &= -g(\overline{\nabla}_{U_1}\phi(\nu V_1) - (\overline{\nabla}_{U_1}\phi)\nu V_1, X) + g(\overline{\nabla}_{V_1}\phi(\nu U_1) - (\overline{\nabla}_{V_1}\phi)\nu U_1, X) \\ &\quad + g(-A_{\omega V_1}U_1 + \nabla^{\perp}_{U_1}\omega V_1, \phi X) - g(-A_{\omega U_1}V_1 + \nabla^{\perp}_{V_1}\omega U_1, \phi X) \\ &= -g(\overline{\nabla}_{U_1}\phi(\nu V_1), X) + g(\overline{\nabla}_{V_1}\phi(\nu U_1), X) + g(-A_{\omega V_1}U_1 + \nabla^{\perp}_{U_1}\omega V_1, \phi X) \\ &\quad - g(-A_{\omega U_1}V_1 + \nabla^{\perp}_{V_1}\omega U_1, \phi X). \end{aligned}$$

Now for any $X = PX + P_2X \in \Gamma(D \oplus D_1)$, from (3.2) we obtain

$$\begin{aligned} g([U_1,V_1],X) &= -g(\overline{\nabla}_{U_1}\nu^2 V_1, X) - g(\overline{\nabla}_{U_1}\omega\nu V_1, X) + g(\overline{\nabla}_{V_1}\nu^2 U_1, X) + g(\overline{\nabla}_{V_1}\omega\nu U_1, X) \\ &\quad - g(A_{\omega V_1}U_1 - A_{\omega U_1}V_1, \nu X + \omega X) \\ &\quad + g(\nabla^{\perp}_{U_1}\omega V_1 - \nabla^{\perp}_{V_1}\omega U_1, \phi PX + \phi P_2X) \end{aligned}$$

which leads to

$$\begin{aligned} g([U_1,V_1],X) &= cos^2\theta_1 g(\overline{\nabla}_{U_1}V_1 - \overline{\nabla}_{V_1}U_1, X) - g(\overline{\nabla}_{U_1}\omega\nu V_1 - \overline{\nabla}_{V_1}\omega\nu U_1, X) \\ &\quad - g(A_{\omega V_1}U_1 - A_{\omega U_1}V_1, \nu X) + g(\nabla^{\perp}_{U_1}\omega V_1 - \nabla^{\perp}_{V_1}\omega U_1, \omega P_2X) \\ &= cos^2\theta_1 g([U_1,V_1],X) \\ &\quad - g(-A_{\omega\nu V_1}U_1 + \nabla^{\perp}_{U_1}\omega\nu V_1 + A_{\omega\nu U_1}V_1 - \nabla^{\perp}_{V_1}\omega\nu U_1, X) \\ &\quad - g(A_{\omega V_1}U_1 - A_{\omega U_1}V_1, \nu X) + g(\nabla^{\perp}_{U_1}\omega V_1 - \nabla^{\perp}_{V_1}\omega U_1, \omega P_2X) \\ &= cos^2\theta_1 g([U_1,V_1],X) - g(-A_{\omega\nu V_1}U_1 + A_{\omega\nu U_1}V_1, X) \\ &\quad - g(A_{\omega V_1}U_1 - A_{\omega U_1}V_1, \nu X) + g(\nabla^{\perp}_{U_1}\omega V_1 - \nabla^{\perp}_{V_1}\omega U_1, \omega P_2X) \end{aligned}$$

which in view of Lemma 3.3 (i) and using the fact that $\omega PX = 0$ the above equation leads to

$$\begin{aligned} sin^2\theta_1 g([U_1,V_1],X) &= g(A_{\omega\nu V_1}U_1 - A_{\omega\nu U_1}V_1, X) - g(A_{\omega V_1}U_1 - A_{\omega U_1}V_1, \nu X) \\ &\quad + g(\nabla^{\perp}_{U_1}\omega V_1 - \nabla^{\perp}_{V_1}\omega U_1, \omega P_2X). \end{aligned}$$

Thus the proof follows. □

Likewise the above hypothesis, we have the following sufficient conditions for the slant distribution D_1 to be integrable:

Theorem 4.3. *Let $\mathcal{M}$ be a quasi bi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$. If*

$$\begin{aligned} &A_{\omega V_1}U_1 - A_{\omega U_1}V_1 \in D_1, \\ &A_{\omega\nu V_1}U_1 - A_{\omega\nu U_1}V_1 \in D_1, \\ &\nabla^{\perp}_{U_1}\omega V_1 - \nabla^{\perp}_{V_1}\omega U_1 \in \omega D_1 \oplus \mu \end{aligned} \tag{4.3}$$

for any $U_1, V_1 \in \Gamma(D_1)$, then the slant distribution D_1 is integrable.

Proof. From the above theorem, for any $U_1, V_1 \in \Gamma(D_1)$ and $X \in \Gamma(D \oplus D_2)$, we obtain

$$\begin{aligned} sin^2\theta_1 g([U_1,V_1],X) &= g(A_{\omega\nu V_1}U_1 - A_{\omega\nu U_1}V_1, X) - g(A_{\omega V_1}U_1 - A_{\omega U_1}V_1, \nu X) \\ &\quad + g(\nabla^{\perp}_{U_1}\omega V_1 - \nabla^{\perp}_{V_1}\omega U_1, \omega P_2X) \end{aligned}$$

which shows that if $A_{\omega V_1}U_1 - A_{\omega U_1}V_1 \in D_1$, $A_{\omega\nu V_1}U_1 - A_{\omega\nu U_1}V_1 \in D_1$ and $\nabla^{\perp}_{U_1}\omega V_1 - \nabla^{\perp}_{V_1}\omega U_1 \in \omega D_1 \oplus \mu$, then $g([U_1,V_1],X) = 0$ and hence D_1 is integrable. □

In a similar way to the above theorems, we can also obtain the following assertions:

Theorem 4.4. *Let $\mathcal{M}$ be a quasi bi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$. Then the distribution D_2 is integrable if and only if*

$$g(A_{\omega V_2}U_2 - A_{\omega U_2}V_2, \nu X) = g(A_{\omega\nu V_2}U_2 - A_{\omega\nu U_2}V_2, X) + g(\nabla^{\perp}_{U_2}\omega V_2 - \nabla^{\perp}_{V_2}\omega U_2, \omega P_1 X) \tag{4.4}$$

for any $U_2, V_2 \in \Gamma(D_2)$ and $X \in \Gamma(D \oplus D_1)$.

Proof. For any $U_2, V_2 \in \Gamma(D_2)$ and $X = PX + P_1X \in \Gamma(D \oplus D_1)$, the distribution D_2 is integrable on $\mathcal{M}$ iff $g([U_2, V_2], \xi) = 0$ and $g([U_2, V_2], X) = 0$, where $\xi \in \Gamma(\mathcal{TM})$. Now, $g([U_2, V_2], \xi) = 0$ is obvious. So the slant distribution D_2 is integrable if and only if $g([U_2, V_2], X) = 0$.

Now, for any $U_2, V_2 \in \Gamma(D_2)$ and $X = PX + P_1X \in \Gamma(D \oplus D_1)$, with the help of (2.2) we get

$$\begin{aligned} g([U_2, V_2], X) &= g(\phi[U_2, V_2], \phi X) + \eta([U_2, V_2])\eta(X) \\ &= g(\phi(\overline{\nabla}_{U_2}V_2), \phi X) - g(\phi(\overline{\nabla}_{V_2}U_2), \phi X). \end{aligned}$$

Now using (2.3), (2.4) and (3.3), we get

$$\begin{aligned} g([U_2, V_2], X) &= g(\overline{\nabla}_{U_2}(\nu V_2 + \omega V_2), \phi X) - g(\overline{\nabla}_{V_2}(\nu U_2 + \omega U_2), \phi X) \\ &\quad - \{g(\phi U_2, V_2)g(\xi, \phi X) - \eta(V_2)g(\phi U_2, \phi X)\} \\ &\quad + \{g(\phi V_2, U_2)g(\xi, \phi X) - \eta(U_2)g(\phi V_2, \phi X)\} \\ &= g(\overline{\nabla}_{U_2}(\nu V_2), \phi X) + g(\overline{\nabla}_{U_2}(\omega V_2), \phi X) - g(\overline{\nabla}_{V_2}(\nu U_2), \phi X) \\ &\quad - g(\overline{\nabla}_{V_2}(\omega U_2), \phi X). \end{aligned}$$

In account of (2.3), (2.4) and (2.6), we have

$$\begin{aligned} g([U_2, V_2], X) &= -g(\overline{\nabla}_{U_2}\phi(\nu V_2) - (\overline{\nabla}_{U_2}\phi)\nu V_2, X) + g(\overline{\nabla}_{V_2}\phi(\nu U_2) - (\overline{\nabla}_{V_2}\phi)\nu U_2, X) \\ &\quad + g(-A_{\omega V_2}U_2 + \nabla^{\perp}_{U_2}\omega V_2, \phi X) - g(-A_{\omega U_2}V_2 + \nabla^{\perp}_{V_2}\omega U_2, \phi X) \\ &= -g(\overline{\nabla}_{U_2}\phi(\nu V_2), X) + g(\overline{\nabla}_{V_2}\phi(\nu U_2), X) + g(-A_{\omega V_2}U_2 + \nabla^{\perp}_{U_2}\omega V_2, \phi X) \\ &\quad - g(-A_{\omega U_2}V_2 + \nabla^{\perp}_{V_2}\omega U_2, \phi X). \end{aligned}$$

Now for any $X = PX + P_1X \in \Gamma(D \oplus D_1)$, from (3.2) we obtain

$$\begin{aligned} g([U_2, V_2], X) &= -g(\overline{\nabla}_{U_2}\nu^2 V_2, X) - g(\overline{\nabla}_{U_2}\omega\nu V_2, X) + g(\overline{\nabla}_{V_2}\nu^2 U_2, X) + g(\overline{\nabla}_{V_2}\omega\nu U_2, X) \\ &\quad - g(A_{\omega V_2}U_2 - A_{\omega U_2}V_2, \nu X + \omega X) \\ &\quad + g(\nabla^{\perp}_{U_2}\omega V_2 - \nabla^{\perp}_{V_2}\omega U_2, \phi PX + \phi P_1 X) \end{aligned}$$

which leads to

$$\begin{aligned}
g([U_2,V_2],X) &= cos^2\theta_2 g(\overline{\nabla}_{U_2}V_2 - \overline{\nabla}_{V_2}U_2, X) - g(\overline{\nabla}_{U_2}\omega\nu V_2 - \overline{\nabla}_{V_2}\omega\nu U_2, X)\\
&- g(A_{\omega V_2}U_2 - A_{\omega U_2}V_2, \nu X) + g(\nabla^{\perp}_{U_2}\omega V_2 - \nabla^{\perp}_{V_2}\omega U_2, \omega P_1 X)\\
&= cos^2\theta_2 g([U_2,V_2],X)\\
&- g(-A_{\omega\nu V_2}U_2 + \nabla^{\perp}_{U_2}\omega\nu V_2 + A_{\omega\nu U_2}V_2 - \nabla^{\perp}_{V_2}\omega\nu U_2, X)\\
&- g(A_{\omega V_2}U_2 - A_{\omega U_2}V_2, \nu X) + g(\nabla^{\perp}_{U_2}\omega V_2 - \nabla^{\perp}_{V_2}\omega U_2, \omega P_1 X)\\
&= cos^2\theta_2 g([U_2,V_2],X) - g(-A_{\omega\nu V_2}U_2 + A_{\omega\nu U_2}V_2, X)\\
&- g(A_{\omega V_2}U_2 - A_{\omega U_2}V_2, \nu X) + g(\nabla^{\perp}_{U_2}\omega V_2 - \nabla^{\perp}_{V_2}\omega U_2, \omega P_1 X)
\end{aligned}$$

which in view of Lemma 3.4(i) and the fact that $\omega PX = 0$ turns to

$$\begin{aligned}
sin^2\theta_2 g([U_2,V_2],X) &= g(A_{\omega\nu V_2}U_2 - A_{\omega\nu U_2}V_2, X) - g(A_{\omega V_2}U_2 - A_{\omega U_2}V_2, \nu X)\\
&+ g(\nabla^{\perp}_{U_2}\omega V_2 - \nabla^{\perp}_{V_2}\omega U_2, \omega P_1 X).
\end{aligned}$$

Hence, the proof follows. □

Also, we have the following sufficient conditions for the slant distribution D_2 to be integrable:

Theorem 4.5. *Let $\mathcal{M}$ be a quasi bi-slant submanifold of a Kenmotsu manifold $\mathcal{N}$. If*

$$\begin{aligned}
&A_{\omega V_2}U_2 - A_{\omega U_2}V_2 \in D_2\\
&A_{\omega\nu V_2}U_2 - A_{\omega\nu U_2}V_2 \in D_2,\\
&\nabla^{\perp}_{U_2}\omega V_2 - \nabla^{\perp}_{V_2}\omega U_2 \in \omega D_2 \oplus \mu
\end{aligned} \tag{4.5}$$

for any $U_2, V_2 \in \Gamma(D_2)$, then the slant distribution D_2 is integrable.

Proof. From the above theorem, for any $U_2, V_2 \in \Gamma(D_2)$ and $X \in \Gamma(D \oplus D_1)$, we have the following result

$$\begin{aligned}
sin^2\theta_2 g([U_2,V_2],X) &= g(A_{\omega\nu V_2}U_2 - A_{\omega\nu U_2}V_2, X) - g(A_{\omega V_2}U_2 - A_{\omega U_2}V_2, \nu X)\\
&+ g(\nabla^{\perp}_{U_2}\omega V_2 - \nabla^{\perp}_{V_2}\omega U_2, \omega P_1 X)
\end{aligned}$$

which implies that if $A_{\omega V_2}U_2 - A_{\omega U_2}V_2 \in D_2$, $A_{\omega\nu V_2}U_2 - A_{\omega\nu U_2}V_2 \in D_2$ and $\nabla^{\perp}_{U_2}\omega V_2 - \nabla^{\perp}_{V_2}\omega U_2 \in \omega D_2 \oplus \mu$, then $g([U_2,V_2],X) = 0$ and hence D_2 is integrable. □

Now, we investigate the geometry of leaves of an invariant distribution D, slant distributions D_1 and D_2.

Theorem 4.6. *Let $\mathcal{M}$ be a quasi bi-slant submanifold of $\mathcal{N}$. Then the invariant distribution D does not define totally geodesic foliation on $\mathcal{M}$.*

Proof. The invariant distribution D defines a totally geodesic foliation on $\mathcal{M}$ iff $g(\overline{\nabla}_U V, \xi) = 0, g(\overline{\nabla}_U V, Z) = 0$ and $g(\overline{\nabla}_U V, W) = 0$, for any $U, V \in \Gamma(D)$,
$Z = P_1 Z + P_2 Z \in \Gamma(D_1 \oplus D_2)$ and $W \in \Gamma(T^{\perp}M)$.
Now, since we know that

$$\begin{aligned}
g(\overline{\nabla}_U V, \xi) &= \overline{\nabla}_U\{g(V,\xi)\} - g(V, \overline{\nabla}_U \xi)\\
&= Ug(V,\xi) - g(V, \overline{\nabla}_U \xi)\\
&= -g(V, \overline{\nabla}_U \xi) \qquad \{\because g(V,\xi) = 0\}.
\end{aligned}$$

Using equation (2.3), we get

$$\begin{aligned} g(\overline{\nabla}_U V, \xi) &= -g(V, U - \eta(U)\xi) \\ &= -g(V, U) + \eta(U)g(V, \xi) \\ &= -g(V, U) \\ &\neq 0 \quad for\ some\ U, V \in \Gamma(D). \end{aligned}$$

As $g(\overline{\nabla}_U V, \xi) \neq 0$, therefore the invariant distribution D does not define totally geodesic foliation on $\mathcal{M}$. □

Similarly as above we have the following theorems for the slant distribiutions D_1 and D_2:

Theorem 4.7. *Let $\mathcal{M}$ be a quasi bi-slant submanifold of $\mathcal{N}$. Then the slant distribution D_1 with slant angle θ_1 does not define totally geodesic foliation on $\mathcal{M}$.*

Theorem 4.8. *Let $\mathcal{M}$ be a quasi bi-slant submanifold of $\mathcal{N}$. Then the slant distribution D_2 with slant angle θ_2 does not define totally geodesic foliation on $\mathcal{M}$.*

Example 1. Let us consider an 11-dimensional manifold
$\overline{\mathcal{M}} = \{(x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5, z) \in \mathbb{R}^{11} : z \neq 0\}$, where, $(x_i, y_i, z), i = 1, 2, 3, 4, 5$ are standard coordinates in $\mathbb{R}^{11}$. We choose the vector fields
$\epsilon_i = e^{-z}\frac{\partial}{\partial x_i}$, $\epsilon_{5+i} = e^{-z}\frac{\partial}{\partial y_i}$, $\epsilon_{11} = \frac{\partial}{\partial z}$
which are linearly independent at each points of $\overline{\mathcal{M}}$. Let g be the Riemannian metric defined by

$$g = e^{2z}(dx \otimes dx + dy \otimes dy) + \eta \otimes \eta,$$

where η is the 1- form defined by $\eta(X) = g(X, \epsilon_{11})$ for any vector field X on $\overline{\mathcal{M}}$. Hence $\{\epsilon_1, \epsilon_2, ..., \epsilon_{11}\}$ is an orthonormal basis of $\overline{\mathcal{M}}$. We define (1,1) tensor field ϕ as $\phi\{\sum_{i=1}^{5}(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}) + z\frac{\partial}{\partial z}\} = \sum_{i=1}^{5}(x_i\frac{\partial}{\partial y_i} - y_i\frac{\partial}{\partial x_i})$. Thus, we get
$\phi(\epsilon_1) = \epsilon_6$, $\phi(\epsilon_2) = \epsilon_7$, $\phi(\epsilon_3) = \epsilon_8$,
$\phi(\epsilon_4) = \epsilon_9$, $\phi(\epsilon_5) = \epsilon_{10}$, $\phi(\epsilon_6) = -\epsilon_1$,
$\phi(\epsilon_7) = -\epsilon_2$, $\phi(\epsilon_8) = -\epsilon_3$, $\phi(\epsilon_9) = -\epsilon_4$,
$\phi(\epsilon_{10}) = -\epsilon_5$, $\phi(\epsilon_{11}) = 0$.
The linearity property of g and ϕ yields that

$$\eta(\epsilon_{11}) = g(\epsilon_{11},\ \epsilon_{11}) = 1,\ \phi^2 X = -X + \eta(X)\epsilon_{11},\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X, Y on $\overline{\mathcal{M}}$. Thus, for $\epsilon_{11} = \xi$, $\overline{\mathcal{M}}(\phi, \xi, \eta, g)$ defines an almost contact metric manifold. We can easily show that for any vector fields X, Y, Z on $\overline{\mathcal{M}}$, $\overline{\mathcal{M}}(\phi, \xi, \eta, g)$ is a Kenmotsu manifold.
Now, let $\mathcal{M}$ be a subset of $\overline{\mathcal{M}}$ and consider the immersion $f : \mathcal{M} \to \overline{\mathcal{M}}$ defined as:
$f(u, v, w, r, s, t, z) = (u, 0, w, 0, s, v cos\theta_1, v sin\theta_1, r cos\theta_2, r sin\theta_2, t, z)$.
If we take
$X_1 = \epsilon_1$, $X_2 = cos\theta_1\epsilon_6 + sin\theta_1\epsilon_7$, $X_3 = \epsilon_3$, $X_4 = cos\theta_2\epsilon_8 + sin\theta_2\epsilon_9$
$X_5 = \epsilon_5$, $X_6 = \epsilon_{10}$, $X_7 = \xi = \epsilon_{11}$,
then the restriction of $X_1, X_2, ..., X_7$ to $\mathcal{M}$ forms an orthonormal frame of the tangent bundle $\mathcal{TM}$. Obviously, we get
$\phi X_1 = \epsilon_6$, $\phi X_2 = -cos\theta_1\epsilon_1 - sin\theta_1\epsilon_2$,
$\phi X_3 = \epsilon_8$, $\phi X_4 = -cos\theta_2\epsilon_3 - sin\theta_2\epsilon_4$,
$\phi X_5 = \epsilon_{10}$, $\phi X_6 = -\epsilon_5$, $\phi X_7 = 0$.

Let us put $D_1 = span\{X_1, X_2\}, D_2 = span\{X_3, X_4\}$ and $D = span\{X_5, X_6\}$. Then obviously D_1, D_2 and D satisfy the definition of quasi bi-slant submanifold of a Kenmotsu manifold. Hence, submanifold $\mathcal{M}$ defined by f is quasi bi-slant submanifold of $\mathbb{R}^{11}$ with bi-slant angles θ_1 and θ_2.

Example 2. Consider, (x_i, y_i, z) be cartesian coordinates on $\mathbb{R}^{2n+1}$ for $i = 1, 2, ..., n$ with an almost contact metric structure (ϕ, ξ, η, g) which is defined as follows: $\phi(\sum_{i=1}^{n}(a_i\frac{\partial}{\partial x_i} + b_i\frac{\partial}{\partial y_i}) + c\frac{\partial}{\partial z}) = -\sum_{i=1}^{n} b_i\frac{\partial}{\partial x_i} + \sum_{i=1}^{n} a_i\frac{\partial}{\partial y_i}$,
where, $\xi = \frac{\partial}{\partial z}$ and a_i, b_i, c are C^∞ real valued functions defined on $\mathbb{R}^{2n+1}$. Let $\eta = dz, g$ is Euclidean metric and $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z}\}, i = 1, 2, ..., n$ is orthonormal base field of vectors on $\mathbb{R}^{2n+1}$. We can easily show that (ϕ, ξ, η, g) is Kenmotsu structure on $\mathbb{R}^{2n+1}$. Hence, it is a Kenmotsu manifold.
Consider, a submanifold $\overline{\mathcal{M}}$ of $\mathbb{R}^{11}$ defined by

$$f(u, v, w, r, s, t, q) = (u, \tfrac{w}{2}, 0, \tfrac{s}{2}, v, \tfrac{\sqrt{3}}{2}r, \tfrac{r}{2}, \tfrac{t}{2}, \tfrac{t}{2}, q).$$

By direct computation it is easy to check that the tangent bundle of $\overline{\mathcal{M}}$ is spanned by the set $\{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}$, where

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial y_1}, \quad X_3 = \frac{1}{2}(\frac{\partial}{\partial x_2} + \sqrt{3}\frac{\partial}{\partial y_3}),$$
$$X_4 = \frac{\partial}{\partial x_3}, \quad X_5 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_5}), \quad X_6 = \frac{\partial}{\partial y_4}, X_7 = \xi = \frac{\partial}{\partial z}.$$

Obviously, we get

$$\phi X_1 = \frac{\partial}{\partial y_1}, \quad \phi X_2 = -\frac{\partial}{\partial x_1}, \quad X_3 = \frac{1}{2}(\frac{\partial}{\partial y_2} - \sqrt{3}\frac{\partial}{\partial x_3}),$$
$$\phi X_4 = \frac{\partial}{\partial y_3}, \quad \phi X_5 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial y_4} - \frac{\partial}{\partial x_5}), \quad \phi X_6 = -\frac{\partial}{\partial x_4}, \quad \phi X_7 = 0.$$

Hence the distributions $D = span\{X_1, X_2\}, D_1 = span\{X_3, X_4\}$ and $D_2 = span\{X_5, X_6\}$ are invariant, slant with slant angle $\frac{\pi}{6}$ and slant with slant angle $\frac{\pi}{4}$, respectively. Also the distributions D, D_1, D_2 satisfy the definition of quasi bi-slant submanifold of Kenmotsu manifold. Hence submanifold $\overline{\mathcal{M}}$ defined by f is quasi bi-slant submanifold of $\mathbb{R}^{11}$ with bi-slant angles $\frac{\pi}{6}$ and $\frac{\pi}{4}$.

Acknowledgement. The authors are thankful to the editor and anonymous referees for their valuable suggestions towards the improvement of the paper.

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