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ON TYPE 2 DEGENERATE CHANGHEE POLYNOMIALS

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ABSTRACT. The Changhee polynomials which are defined by Kim are one of the important special functions, and various properties of those polynomials have been studied by many researchers. In this paper, we fine some interesting identities related to the type 2 degenerate Changhee polynomials and some important special polynomials and numbers.

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1. INTRODUCTION

In many fields such as algebraic number theory, combinatorics, applied mathematics and mathematical physics, the importance of special functions is well known and widely used. As one of the important special functions, the *Changhee polynomials* are given by

$$(1)\qquad \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!} = \frac{2}{2+t}(1+t)^x, \text{ (see [2, 8, 9, 16, 23, 25, 26, 27]).}$$

In the special case of $x = 0$, $Ch_n = Ch_n(0)$ are called the *Changhee numbers*.

The Changhee polynomials and numbers has been generalized in various ways and its properties have been found by many researchers. In [28], Simsek, Kim and Pyung defined define Changee q-analogue zeta functions and Changee-Barnes q-Euler polynomials by using the p-adic q-measure and found the interpolation formulas of those functions. In [18], Kim and Rim defined new Changhee q-Euler polynomials and numbers and constructed a multivariate Hurwitz type zeta function which interpolates the multivariate q-Euler numbers or polynomials at negative integers. In [2], authors considered the Witt-type formula for the extension of Changchee and Daehee numbers and polynomials, and derived some identities and properties of those numbers and polynomials. In [27], authors considered Witt-type formula for the Changhee numbers and polynomials on the locally constant spaces and derived some interesting identities and properties of those polynomials and numbers. In [23], Lim and Qi considered a λ-analogue of the Changhee polynomials by virtue of the p-adic fermionic integral on $\mathbb{Z}_p$. Kim and Kim derived some new and explicit identities of Changhee and Euler numbers from those nonlinear differential equations in [8]. In [26], Qi, Jang and Kwon studied the Appell-type degenerate q-Changhee polynomials. In [25], authors defined the Changhee-Genocchi polynomials and derived some differential equations arising from those numbers. Nahid, Saif and Araci introduced the new type of Appell-type Changhee-Euler polynomials by combining the Appell-type Changhee polynomials and Euler polynomials and derived the differential equations arising from the generating function of the Appell-type Changhee-Euler polynomials (see [24]).

In [9], authors defined the *type* 2 *Changhee polynomials* by the generating function to be

$$\frac{2}{(1+t)+(1+t)^{-1}}(1+t)^x = \sum_{n=0}^{\infty} c_n(x)\frac{t^n}{n!}. \tag{2}$$

When $x = 0$, $c_n = c_n(0)$ are called the *type* 2 *Changhee numbers.*

For $n \geq 0$, the *Stirling numbers of the first kind* $S_1(n,k)$ and the *Stirling numbers of the second kind* $S_2(n,k)$, respectively, are given by

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \text{ (see [1, 3, 4, 6, 12, 13, 15, 17, 19, 20])}, \tag{3}$$

where $(x)_0 = 1$ and $(x)_n = x(x-1)(x-2)\cdots(x-(n-1))$, $n \geq 1$. By (3), we see that the generating functions of the first and the second Stirling numbers are

$$\frac{1}{k!}\left(\log(1+t)\right)^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \text{ and } \frac{1}{k!}\left(e^t-1\right)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \tag{4}$$

(see [1, 3, 4, 6, 12, 13, 15, 17, 19, 20]).

For any nonzero real number λ, the *degenerate exponential function* is defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \; e_\lambda(t) = (1+\lambda t)^{\frac{1}{\lambda}}, \tag{5}$$

(see [1, 5, 6, 10, 15, 17, 20]).

On the other hand, Kim and Kim defined the *degenerate logarithm function* $\log_\lambda(t)$ as the compositional inverse function of $e_\lambda(t)$ satisfying $\log_\lambda(e_\lambda(t)) = t$. Then we have

$$\log_\lambda(1+t) = \sum_{n=1}^{\infty} \lambda^{n-1}(1)_{n,\frac{1}{\lambda}}\frac{t^n}{n!}, \text{ (see [11, 15, 21])}, \tag{6}$$

where $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$.

As degenerate version of the Stirling numbers of the first and second kind in (3), the *degenerate Stirling numbers of the first kind* $S_{1,\lambda}(n,k)$ and the *degenerate Stirling numbers of the second kind* $S_{2,\lambda}(n,k)$ are respectively introduced by Kim-Kim (see [1, 4, 6, 7, 10, 11, 15, 17, 19, 20, 21]) as follows:

$$\frac{1}{k!}\left(\log_\lambda(1+t)\right)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} \text{ and } \frac{1}{k!}\left(e_\lambda(t)-1\right)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}. \tag{7}$$

The degenerate of some special numbers and polynomials was initiated by Carlitz, and he found interesting relationships between Bernoulli polynomials or Eulerian polynomials and some important numbers in combinatorics (see [1]). The study of degenerate versions of various special polynomials or numbers has been studied by many researchers (see [1, 4, 6, 7, 10, 11, 14, 15, 17, 19, 20, 21]).

In this paper, we fine some interesting identities related to the type 2 degenerate Chaghee polynomials, the first and the second kind Stirling numbers, the higher-order type 2 degenerate Euler polynomials and the degenerate central fractional polynomials of the second kind.

2. Type 2 degenerate Changhee polynomials

In this section, we find some relationships between type 2 degeneate Changhee polynomials and some special polynomials.

The *type* 2 *degenerate Changhee polynomials* are given by

$$\frac{2}{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}}+(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}(1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}c^*_{n,\lambda,2}(x)\frac{t^n}{n!}. \tag{8}$$

When $x=0$, $c^*_{n,\lambda,2}=c^*_{n,\lambda,2}(0)$ are called the *type* 2 *degenerate Changhee polynomials* (see [4]).

Note that

$$(1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}S_{1,\lambda}(n,m)(x)_{m,\lambda}\right)\frac{t^n}{n!}. \tag{9}$$

By the definition of type 2 degenerate Changhee polynomials and (9), we get

$$\begin{aligned}&\frac{2}{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}}+(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}(1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}}\\&=\left(\sum_{n=0}^{\infty}c^*_{n,\lambda,2}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}S_{1,\lambda}(n,m)(x)_{m,\lambda}\frac{t^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}S_{1,\lambda}(m,k)c^*_{n-m,\lambda,2}(x)_{k,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{10}$$

By the (10), we obtain the following theorem.

Theorem 2.1. *For each nonnegative integer n, we have*

$$c^*_{n,\lambda,2}(x)=\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}S_{1,\lambda}(m,k)c^*_{n-m,\lambda,2}(x)_{k,\lambda}.$$

In [4], Jang-Kim defined the type 2 degenerate Euler polynomials of order α to be

$$\left(\frac{2}{(1+\lambda t)^{\frac{1}{2\lambda}}+(1+\lambda t)^{-\frac{1}{2\lambda}}}\right)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}E^{(\alpha)}_{n,\lambda}(x)\frac{t^n}{n!}, \tag{11}$$

where $\alpha\in\mathbb{N}$. In particular, if $\alpha=1$, then $E_{n,\lambda}(x)=E^{(1)}_{n,\lambda}(x)$ are called the *type* 2 *degenerate Euler polynomials.* In the special case of $x=0$, $E^{(\alpha)}_{n,\lambda}=E^{(\alpha)}_{n,\lambda}(0)$ are called the *type* 2 *degenerate Euler numbers.*

By replacing t by $\log_\lambda(1+t)$ in (11), we have

$$\begin{aligned}&\sum_{n=0}^{\infty} E_{n,\lambda}(x)\frac{1}{n!}\left(\log_\lambda(1+t)\right)^n \\ &=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} S_{1,\lambda}(n,m)E_{m,\lambda}(x)\right)\frac{t^n}{n!},\end{aligned} \tag{12}$$

and

$$\begin{aligned}&\frac{2}{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}}+(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}(1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}} \\ &=\sum_{n=0}^{\infty} c^*_{n,\lambda,2}(x)\frac{t^n}{n!}.\end{aligned} \tag{13}$$

By (12) and (13), we obtain the following theorem.

Theorem 2.2. *For each $n \in \mathbb{N}\cup\{0\}$, we have*

$$c^*_{n,\lambda,2}(x)=\sum_{m=0}^{n} S_{1,\lambda}(n,m)E_{m,\lambda}(x).$$

In the definition of type 2 degenerate Changhee polynomials, by replacing t by $e_\lambda(t)-1$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty} c^*_{n,\lambda,2}(x)\frac{1}{n!}\left(e_\lambda(t)-1\right)^n \\ &=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} S_{2,\lambda}(n,m)c^*_{m,\lambda,2}(x)\right)\frac{t^n}{n!},\end{aligned} \tag{14}$$

and

$$\frac{2}{(1+\lambda t)^{\frac{1}{2\lambda}}+(1+\lambda t)^{-\frac{1}{2\lambda}}}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!}. \tag{15}$$

By (14) and (15), we obtain the following theorem.

Theorem 2.3. *For each $n \in \mathbb{N}\cup\{0\}$, we have*

$$E_{n,\lambda}(x)=\sum_{m=0}^{n} S_{2,\lambda}(n,m)c^*_{m,\lambda,2}(x).$$

By (8), we see that

$$\begin{aligned}&(1+\lambda\log(1+t))^{\frac{x}{\lambda}} \\ &=\left(\frac{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}}+(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}{2}\right)\sum_{n=0}^{\infty}c^*_{n,\lambda,2}(x)\frac{t^n}{n!}.\end{aligned} \tag{16}$$

Note that

$$
\begin{aligned}
(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}} &= \sum_{n=0}^{\infty}\binom{\frac{1}{2\lambda}}{n}(\lambda\log_\lambda(1+t))^n \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)_{n,\lambda}\sum_{l=m}^{\infty}S_{1,\lambda}(l,n)\frac{t^l}{l!} \qquad (17)\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left(\frac{1}{2}\right)_{m,\lambda}S_{1,\lambda}(n,m)\right)\frac{t^n}{n!},
\end{aligned}
$$

and, by the similar way, we see that

$$
(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(-1)^m\left\langle\frac{1}{2}\right\rangle_{m,\lambda}S_{1,\lambda}(n,m)\right)\frac{t^n}{n!}, \qquad (18)
$$

where $< x >_{0,\lambda}= 1$, $< x >_{n,\lambda}= x(x+\lambda)(x+2\lambda)\cdots(x+(n-1)\lambda)$, $n \in \mathbb{N}$ are the rising factorial of x.

By (17) and (18), we see that

(19)

$$
\begin{aligned}
&\left(\frac{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}}+(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}{2}\right)\sum_{n=0}^{\infty}c^*_{n,\lambda,2}(x)\frac{t^n}{n!}\\
=&\frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\sum_{m=0}^{r}\binom{n}{r}\left(\left(\frac{1}{2}\right)_{m,\lambda}+(-1)^m\left\langle\frac{1}{2}\right\rangle_{m,\lambda}\right)S_{1,\lambda}(r,m)c^*_{n-r,\lambda,2}(x)\right)\frac{t^n}{n!}.
\end{aligned}
$$

By (20) and (19), we obtain the following theorem.

Theorem 2.4. *For each $n \in \mathbb{N}\cup\{0\}$, we have*

$$
\begin{aligned}
&2\sum_{m=0}^{n}S_{1,\lambda}(n,m)(x)_{m,\lambda}\\
=&\sum_{r=0}^{n}\sum_{m=0}^{r}\binom{n}{r}\left(\left(\frac{1}{2}\right)_{m,\lambda}+(-1)^m\left\langle\frac{1}{2}\right\rangle_{m,\lambda}\right)S_{1,\lambda}(r,m)c^*_{n-r,\lambda,2}(x)
\end{aligned}
$$

From now on, we consider the *type* 2 *degenerate Changhee polynomials of order* α which are defined by the generating function to be

(20)

$$
\begin{aligned}
&\left(\frac{2}{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}}+(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}\right)^{\alpha}(1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}}\\
=&\sum_{n=0}^{\infty}c^{(\alpha,*)}_{n,\lambda,2}(x)\frac{t^n}{n!},
\end{aligned}
$$

where $\alpha \in \mathbb{R}$. When $x = 0$, $c^{(\alpha,*)}_{n,\lambda,2} = c^{(\alpha,*)}_{n,\lambda,2}(0)$ are called the *type* 2 *degenerate Changhee numbers of order* α.

Note that, by (7) and (20), we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty} c_{n,\lambda,2}^{(\alpha,*)}(x+y)\frac{t^n}{n!}\\
&=\left(\frac{2}{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}}+(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}\right)^{\alpha}(1+\lambda\log_\lambda(1+t))^{\frac{x+y}{\lambda}}\\
&=\left(\sum_{n=0}^{\infty} c_{n,\lambda,2}^{(\alpha,*)}(x)\frac{t^n}{n!}\right)(1+\lambda\log_\lambda(1+t))^{\frac{y}{\lambda}}\\
&=\left(\sum_{n=0}^{\infty} c_{n,\lambda,2}^{(\alpha,*)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(y)_{n,\lambda}\frac{1}{n!}\left(\log_\lambda(1+t)\right)^n\right)\\
&=\left(\sum_{n=0}^{\infty} c_{n,\lambda,2}^{(\alpha,*)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n} S_{1,\lambda}(n,m)(y)_{m,\lambda}\frac{t^n}{n!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{r=0}^{m}\binom{n}{m}S_{1,\lambda}(m,r)c_{n-m,\lambda,2}^{(\alpha,*)}(x)(y)_{r,\lambda}\right)\frac{t^n}{n!}.
\end{aligned}
\tag{21}
$$

By (21), we obtain the following theorem.

Theorem 2.5. *For each nonnegative integer, we have*

$$c_{n,\lambda,2}^{(\alpha,*)}(x+y)=\sum_{m=0}^{n}\sum_{r=0}^{m}\binom{n}{m}S_{1,\lambda}(m,r)c_{n-m,\lambda,2}^{(\alpha,*)}(x)(y)_{r,\lambda}.$$

By replacing t by $e_\lambda(t)-1$ in (20), we get

$$\left(\frac{2}{(1+\lambda t)^{\frac{1}{2\lambda}}+(1+\lambda t)^{-\frac{1}{2\lambda}}}\right)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}E_{n,\lambda}^{(\alpha)}(x)\frac{t^n}{n!},\tag{22}$$

and

$$
\begin{aligned}
\sum_{n=0}^{\infty} c_{n,\lambda,2}^{(\alpha,*)}(x)\frac{1}{n!}\left(e_\lambda(t)-1\right)^n&=\sum_{n=0}^{\infty} c_{n,\lambda,2}^{(\alpha,*)}(x)\sum_{l=n}^{\infty}S_{2,\lambda}(l,n)\frac{t^l}{l!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}S_{2,\lambda}(n,m)c_{m,\lambda,2}^{(\alpha,*)}(x)\right)\frac{t^n}{n!}.
\end{aligned}
\tag{23}
$$

By (22) and (23), we obtain the following theorem.

Theorem 2.6. *For each nonnegative integer n, we have*

$$E_{n,\lambda}^{(\alpha)}(x)=\sum_{m=0}^{n}S_{2,\lambda}(n,m)c_{m,\lambda,2}^{(\alpha,*)}(x).$$

By replacing t by $\log_\lambda(1+t)$ in (11), we get

(24)
$$\left(\frac{2}{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}}+(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}\right)^{\alpha}(1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}}$$
$$=\sum_{n=0}^{\infty}c_{n,\lambda,2}^{(\alpha,*)}(x)\frac{t^n}{n!},$$

and

(25)
$$\begin{aligned}\sum_{n=0}^{\infty}E_{n,\lambda}^{(\alpha)}(x)\frac{1}{n!}\left(\log_\lambda(1+t)\right)^n&=\sum_{n=0}^{\infty}E_{n,\lambda}^{(\alpha)}(x)\sum_{l=n}^{\infty}S_{1,\lambda}(l,n)\frac{t^l}{l!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}S_{1,\lambda}(n,m)E_{m,\lambda}^{(\alpha)}(x)\right)\frac{t^n}{n!}.\end{aligned}$$

By (24) and (25), we obtain the following theorem.

Theorem 2.7. *For each nonnegative integer n, we have*
$$c_{n,\lambda,2}^{(\alpha,*)}(x)=\sum_{m=0}^{n}S_{1,\lambda}(n,m)E_{m,\lambda}^{(\alpha)}(x).$$

Let $r\in\mathbb{N}$ with $\alpha>r$. By (20),

(26)
$$\left(\frac{2}{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}}+(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}\right)^{\alpha}(1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}}$$
$$=\left(\frac{2}{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}}+(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}\right)^{\alpha-r}$$
$$\times\left(\frac{2}{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}}+(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}\right)^{r}(1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}}$$
$$=\left(\sum_{n=0}^{\infty}c_{n,\lambda,2}^{(\alpha-r,*)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}c_{n,\lambda,2}^{(r,*)}(x)\frac{t^n}{n!}\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}c_{n-m,\lambda,2}^{(\alpha-r,*)}c_{m,\lambda,2}^{(r,*)}(x)\right)\frac{t^n}{n!},$$

and by (20) and (26), we obtain the following theorem.

Theorem 2.8. *Let $\alpha\in\mathbb{R}$ and let r be an positive integer with $\alpha>r$. For each nonnegative integer n and α,r with $\alpha>r$, we have*
$$c_{n,\lambda,2}^{(\alpha,*)}(x)=\sum_{m=0}^{n}\binom{n}{m}c_{n-m,\lambda,2}^{(\alpha-r,*)}c_{m,\lambda,2}^{(r,*)}(x).$$

In [10], Kim-Kim defined the degenerate central factorial polynomials of the second kind as follows:

(27)
$$\frac{1}{k!}\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)^k e_\lambda^x(t)=\sum_{n=k}^{\infty}T_{2,\lambda}(n,k|x)\frac{t^n}{n}.$$

When $x = 0$, $T_{2,\lambda}(n,k) = T_{2,\lambda}(n,k|0)$ are called the degenerate central fractional numbers.

If we take $\alpha = -r$ and t replacing by $e_\lambda(t) - 1$ in (20), then, by (24), we get

(28)
$$\begin{aligned}
&\left(\frac{2}{(1+\lambda t)^{\frac{1}{2\lambda}} + (1+\lambda t)^{-\frac{1}{2\lambda}}}\right)^{-r} = 2^{-r}\left((1+\lambda t)^{\frac{1}{2\lambda}} - (1+\lambda t)^{-\frac{1}{2\lambda}} + 2(1+\lambda t)^{-\frac{1}{2\lambda}}\right)^{r}\\
=&2^{-r}\sum_{l=0}^{\infty}\binom{r}{l}\left((1+\lambda t)^{\frac{1}{2\lambda}} - (1+\lambda t)^{-\frac{1}{2\lambda}}\right)^{l}\left(2\left(1+\lambda t\right)^{-\frac{1}{2\lambda}}\right)^{r-l}\\
=&\sum_{l=0}^{\infty}\frac{(r)_l}{2^l}\frac{1}{l!}\left((1+\lambda t)^{\frac{1}{2\lambda}} - (1+\lambda t)^{-\frac{1}{2\lambda}}\right)^{l}(1+\lambda t)^{\frac{l-r}{2\lambda}}\\
=&\sum_{l=0}^{\infty}\frac{(r)_l}{2^l}\sum_{m=l}^{\infty}T_{2,\lambda}(m,l)\frac{t^m}{m!}\sum_{s=0}^{\infty}\binom{\frac{l-r}{2\lambda}}{s}(\lambda t)^s\\
=&\sum_{n=0}^{\infty}\left(\sum_{a=0}^{n}\sum_{l=0}^{a}\binom{n}{a}\frac{(r)_l}{2^l}T_{2,\lambda}(a,l)\left(\frac{l-r}{2}\right)_{n-a,\lambda}\right)\frac{t^n}{n!},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty}c_{n,\lambda,2}^{(-r,*)}(x)\frac{1}{n!}\left(e_\lambda(t)-1\right)^n =&\sum_{n=0}^{\infty}c_{n,\lambda,2}^{(-r,*)}(x)\sum_{l=n}^{\infty}S_{2,\lambda}(l,n)\frac{t^l}{l!}\\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}c_{m,\lambda,2}^{(-r,*)}(x)S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{29}$$

By (30) and (29), we obtain the following theorem.

Theorem 2.9. *For $r \in \mathbb{R}$ and each nonnegative integer n, we have*

$$\sum_{m=0}^{n}c_{m,\lambda,2}^{(-r,*)}(x)S_{2,\lambda}(n,m) = \sum_{a=0}^{n}\sum_{l=0}^{a}\binom{n}{a}\frac{(r)_l}{2^l}T_{2,\lambda}(a,l)\left(\frac{l-r}{2}\right)_{n-a,\lambda}.$$

If we take $\alpha = -r$ in (20), then, we get

(30)

$$\begin{aligned}
&\left(\frac{2}{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}}+(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}\right)^{-r}\\
=&\left(\frac{1}{2}\right)^r\left((1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}}-(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}+2\,(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}\right)^r\\
=&\sum_{l=0}^{\infty}(r)_l\frac{1}{l!}\left((1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}}-(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}\right)^l\frac{1}{2^l}\,(1+\lambda\log_\lambda(1+t))^{\frac{l-r}{2\lambda}}\\
=&\sum_{l=0}^{\infty}\frac{(r)_l}{2^l}\sum_{a=l}^{\infty}T_{2,\lambda}(a,l)\frac{1}{a!}\left(\log_\lambda(1+t)\right)^a(1+t)^{\frac{l-r}{2}}\\
=&\sum_{m=0}^{\infty}\sum_{l=0}^{m}\frac{(r)_l}{2^l}T_{2,\lambda}(m,l)\frac{1}{m!}\left(\log_\lambda(1+t)\right)^m(1+t)^{\frac{l-r}{2}}\\
=&\sum_{m=0}^{\infty}\sum_{l=0}^{m}\frac{(r)_l}{2^l}T_{2,\lambda}(m,l)\sum_{s=m}^{\infty}S_{1,\lambda}(s,m)\frac{t^s}{s!}\sum_{b=0}^{\infty}\frac{1}{2^b}(l-r)_{b,2}\frac{t^b}{b!}\\
=&\sum_{n=0}^{\infty}\left(\sum_{c=0}^{n}\sum_{m=0}^{c}\sum_{l=0}^{m}\binom{n}{c}\frac{(r)_l(l-r)_{n-c,2}}{2^{l+n-c}}T_{2,\lambda}(m,l)S_{1,\lambda}(c,m)\right)\frac{t^n}{n!}.
\end{aligned}$$

By the definition of the type 2 degenerate Changhee numbers of order α and (30), we obtain the following theorem.

Theorem 2.10. *For $r \in \mathbb{R}$ and each nonnegative integer n, we have*

$$c_{n,\lambda,2}^{(-r,*)} = \sum_{c=0}^{n}\sum_{m=0}^{c}\sum_{l=0}^{m}\binom{n}{c}\frac{(r)_l(l-r)_{n-c,2}}{2^{l+n-c}}T_{2,\lambda}(m,l)S_{1,\lambda}(c,m).$$

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