Advanced Studies in Contemporary Mathematics
32 (2022), No. 2, pp. 157 - 171

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2022.32.2.157

Solvability of some non-linear functional integral equations via measure of noncompactness

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Abstract

In this study, we establish some results related to the existence of solutions for nonlinear functional integral equations, by Darbo's fixed point theorem in Banach algebra, which contains several functional integral equations that arise in mathematical analysis. As an application, we also provide an example of functional integral equations.

Keywords. Measure of non-compactness(MNC), Banach algebra, Fixed point theorem, Functional integral equation(FIE). MSC: 47H10, 90C39.

1 Introduction

Integral equation is an important branch of mathematical analysis and equations of such type are applicable in many physical problems such as in the integro-differential, optimal control, control theory and mathematical physics(see [9, 17]). Recently the theory of such functional integral equations is developed effectively and emerge in the field of analysis, engineering, applied mathematics and nonlinear functional analysis (see [1, 2, 7, 10, 11, 12, 13, 14, 24] and references therein). Here, we try to prove the solvability of generalized FIE:

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$$
\begin{aligned}
x(s) \;=\; & \left(f(s, x(s)) + F\left(s, \int_0^s r(s, z, x(z))dz, \int_0^s g(s, z, x(z))dz, x(s)\right)\right) \\
& \times \left(h(s, x(s)) + G\Big(s, \int_0^b p(s, z, x(z))dz, \int_0^b q(s, z, x(z))dz, x(s)\Big)\right), \quad (1)
\end{aligned}
$$

for $s \in [0, b]$.
The FIE (1) consists many special type of functional integral equations. The goal of this paper is to investigate the method to prove the existence of solutions of (1) with the help of the MNC in [0, b].

1.1 Applications and comparison with some previous well known results

Our proposed integral equation contains several integral equations, considered by several authors as a special case.

- If $F(s, x_1, x_2, x_3) = F(s, x_2, x_3)$ and $G(s, x_1, x_2, x_3) = G(s, x_2, x_3)$, then equation reduces in the FIE, which studied in [16]

$$
\begin{aligned}
x(s) = \left(f(s, x(s)) + F\left(s, \int_0^s g(s, z, x(z))dz, x(s)\right)\right) \times \\
G\left(s, \int_0^b q(s, z, x(z))dz, x(s)\right),
\end{aligned} \quad (2)
$$

  In this article, authors consider solvability of a certain functional-integral equation which contains as particular cases a lot of integral and functional-integral equations, which are applicable in several real world problems of engineering, mechanics, physics, economics and so on. The main tool used in our result is a fixed point theorem which satisfies the Darbo condition with respect to a measure of noncompactness in the Banach algebra of continuous functions in the interval $[0, a]$.

- Taking $F(s, x_1, x_2, x_3) = F(s, x_2, x_3)$, and $G(s, x_1, x_2, x_3) = 1$, equation (1) convert into the following form which has been studied in [22].

$$
x(s) = f(s, x(s)) + F\left(s, \int_0^s g(s, z, x(z))dz, x(s)\right). \quad (3)
$$

  In this artcle, authors established an existence of solutions for some nonlinear functional- integral equations which include many key integral and functional equations that appear in nonlinear analysis and its applications. By using the techniques of noncompactness measures, they applied the basic fixed point theorems such as Darbos theorem to obtain the mentioned aims in Banach algebra.

- On putting $f(s,x_1)=0, F(s,x_1,x_2,x_3)=F(s,x_1,x_2)$, and $G(s,x_1,x_2,x_3)=G(s,x_2,x_3)$ we obtain the following FIE studied in [6, 21].

$$x(s)=F\left(s,\int_0^s g(s,z,x(z))dz,x(s)\right)\times G\left(s,\int_0^b q(s,z,x(z))dz,x(s)\right). \tag{4}$$

  In this article the method of measure of noncompactness with Darbo's fixed point theorem is used to obtain the existence results.

- If $F(s,x_1,x_2,x_3)=x_2$ and $G(s,x_1,x_2,x_3)=1$, then we get the following FIE studied in [5].

$$x(s)=f(s,x(s))+\int_0^s g(s,z,x(z))dz. \tag{5}$$

  In this article authors prove an existence theorem for a nonlinear integral equation being a Volterra counterpart of an integral equation arising in the traffic theory. The method used in the proof allows us to obtain additional characterization in terms of asymptotic stability of solutions of an equation in question. The methods used by authors are measure of noncompactness and Darbo's fixed point theorem.

- Taking $G(s,x_1,x_2,x_3)=1$ and $F(s,x_1,x_2,x_3)=\hat{f}(s,x)x_2$, then equation (1) has the following form studied in [23].

$$x(s)=\hat{f}(s,x(s))\int_0^s g(s,z,x(z))dz. \tag{6}$$

  Again, the method of measure of noncompactnes and Darbo's fixed point theorem is used by authors to establish the main results in this article.

- On putting $G(s,x_1,x_2,x_3)=1$ and $F(s,x_1,x_2,x_3)=a(s)+x_2$, then we get following non-linear Volterra integral equation

$$x(s)=a(s)+\int_0^s g(s,z,x(z))dz. \tag{7}$$

- Taking $F(s,x_1,x_2,x_3)=1$ and $G(s,x_1,x_2,x_3)=n(s)+x_2$, then we obtain Urysohn integral equation

$$x(s)=n(s)+\int_0^b q(s,z,x(z))dz. \tag{8}$$

The method of measure of noncompactness together with Darbo's fixed point is used in almost all the research article cited above. In present article we also apply the same method but advantage of our work over above cited work is that our equation contains all the equations that are studied in above articles. So our work is a generalization of above cited work.

2 Preliminaries

Assume that X is a real Banach space with the norm $||.||$. Denote by $B(x, \hat{r})$ the closed ball centered at z_0 with radius $\hat{r}$. For P a nonempty subset of X, denote by $\overline{P}$ and Conv$P$ the closure and convex closure of P respectively. Moreover, M_X denote the family of bounded subsets of X and N_X denote its subfamily contains of all relatively compact sets.

Definition 2.1. [3] Assume $P \in M_X$ and

$$\mu(P) = \inf \left\{ \varepsilon > 0 : P = \bigcup_{i=1}^{n} P_j \text{ with } \mathrm{diam} P_j \leq \varepsilon,\ j = 1, 2, ..., n \right\}.$$

where,

$$\mathrm{diam}\ P = \sup\{\|\alpha - \beta\| : \alpha, \beta \in P\}.$$

Hence, $0 \leq \mu(P) < \infty$. $\mu(P)$ is called the Kuratowski MNC.

Theorem 2.1. *Assume $P, P^* \in M_X$ and $\lambda \in \mathbb{R}$. Then*

(i) $\mu(P) = 0$ *if and only if* $P \in N_X$*;*

(ii) $P \subseteq P^* \implies \mu(P) \leq \mu(P^*)$*;*

(iii) $\mu(\overline{P}) = \mu(ConvP) = \mu(P)$*;*

(iv) $\mu(P \cup P^*) = \max\{\mu(P), \mu(P^*)\}$*;*

(v) $\mu(\lambda P) = |\lambda| \mu(P)$*, where* $\lambda P = \{\lambda x : x \in P\}$*;*

(vi) $\mu(P + P^*) \leq \mu(P) + \mu(P^*)$*, where* $P + P^* = \{x + x^* : x \in P, x^* \in P^*\}$*;*

(vii) $|\mu(P) - \mu(P^*)| \leq 2d_{\hat{h}}(P, P^*)$*, where* $d_{\hat{h}}(P, P^*)$ *denotes the Hausdorff metric of* P *and* P^**, i.e.*

$$d_{\hat{h}}(P, P^*) = \max \left\{ \sup_{x^* \in P^*} d(x^*, P), \sup_{x \in P} d(x, P^*) \right\}.$$

where $d(.,..)$ *is the distance from an element* X *to a set of* X*.*

Further, facts concerning MNC may be found in [3].
Suppose that Q is a nonempty subset of a Banach space X and $H : Q \to X$ is a continuous operator which transforming bounded subsets of Q into bounded ones. Moreover, let μ be a regular measure of non-compactness in X.

Definition 2.2. [3] Let Q be a nonempty, convex, bounded and closed subset of X and let $H : Q \to Q$ be continuous mapping such that $\exists$ a constant $k \in [0, 1)$, with

$$\mu(HP) \leq k\mu(P)$$

for any subset of P of Q. Then H has a fixed point in Q.

Now, we discuss on $C[0,b]$ which contains set of all real continuous functions defined on the interval $[0,b]$ with the standard norm

$$||x|| = \sup\{|x(s)| : s \in [0,b]\}.$$

Clearly, the space $C[0,b]$ has also the structure of Banach algebra.
Now, we will focus on a regular MNC defined in [4] (cf also [3]).

Now, we fix a set $P \in X_{C[0,b]}$. For $x \in P$ and given $\epsilon > 0$ denote by $\omega(x,\epsilon)$ the modulus of continuity of$x$, i.e.,

$$\omega(x,\epsilon) = \sup\{|x(s) - x(\hat{s})| : s, \hat{s} \in [0,b], |s - \hat{s}| \leq \epsilon\}.$$

Further,

$$\begin{aligned} \omega(P,\epsilon) &= \sup\{\omega(x,\epsilon) : x \in P\}, \\ \omega_0(P) &= \lim_{\epsilon \to 0} \omega(P,\epsilon). \end{aligned}$$

Thus $\omega_0(P)$ is a regular measure of non-compactness in $C[0,b]$.

Theorem 2.2. *[4] Suppose that Q is a closed, convex and bounded subset of $C[0,b]$ and H_1 and H_2 be the operators which transform continuously the set Q into $C[0,b]$ in this way that $H_1(Q)$ and $H_2(Q)$ are bounded. Again, suppose that operator $H = H_1.H_2$ transform Q into itself. If the operators H_1 and H_2 satisfies the Darbo's condition on the set Q with the constant K_1 and K_2, respectively, then the operator H satisfies the Darbo's condition on Q with the constant*

$$||H_1(Q)||K_2 + ||H_2(Q)||K_1.$$

If $||H_1(Q)||K_2 + ||H_2(Q)||K_1 < 1$, then H will be called contraction with respect to the measure ω_0 and has a fixed point in the set Q.

3 Main Results

In this article, we study about the solvability of the FIE (1) for $x \in C[0,b]$ under the following estimate.

(1) $f, h : [0,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $F, H : [0,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and $\exists$ the constants C_1 and $C_2 \geq 0$ such that

$$\begin{aligned} |f(s,0)| &\leq C_1, \\ |h(s,0)| &\leq C_1, \\ |F(s,0,0,0)| &\leq C_2, \\ |G(s,0,0,0)| &\leq C_2. \end{aligned}$$

(2) There exists the continuous functions $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8 : [0, b] \to [0, b]$ such that

$$\begin{aligned} |f(s,x_1) - f(s,y_1)| &\leq b_1(s)|x_1 - y_1|, \\ |h(s,x_1) - h(s,y_1)| &\leq b_2(s)|x_1 - y_1|, \\ |F(s,x_1,x_2,x_3) - F(s,y_1,y_2,y_3)| &\leq b_3(s)|x_1 - y_1| + b_4(s)|x_2 - y_2| + b_5(s)|x_3 - y_3| \\ |G(s,x_1,x_2,x_3) - G(s,y_1,y_2,y_3)| &\leq b_6(s)|x_1 - y_1| + b_7(s)|x_2 - y_2| + b_8(s)|x_3 - y_3|, \end{aligned}$$

for all $s \in [0, b]$ and $y_1, y_2, y_3, y_4, z_1, z_2 \in \mathbb{R}$.

(3) $r = r(s, z, x(z)), g = g(s, z, x(z)), p = p(s, z, x(z))$, and $q = q(s, z, x(z)) : [0, b] \times [0, b] \times \mathbb{R} \to \mathbb{R}$ transform continuously the interval $[0, b]$ into itself.

(4) $\exists$ a non-negative constant K such that

$$\max\{b_1(s), b_2(s), b_3(s), b_4(s), b_5(s), b_6(s), b_7(s), b_8(s)\} \leq K, \text{ for } s \in [0, b].$$

(5) (Sub-linearity condition) $\exists$ constant ζ and η such that

$$\begin{aligned} |p(s,z,x(z))| &\leq \zeta + \eta|x|, \\ |q(s,z,x(z))| &\leq \zeta + \eta|x|, \\ |r(s,z,x(z))| &\leq \zeta + \eta|x|, \\ |g(s,z,x(z))| &\leq \zeta + \eta|x|. \end{aligned}$$

for all $s,\ z \in [0, b]$ and $x \in \mathbb{R}$.

(6) $4\gamma\sigma < 1$ for, $\gamma = 2K + 2Kb\eta$ and $\sigma = C_1 + 2Kb\zeta + C_2$.

Theorem 3.1. *Under the assumptions* (1)$-$(6) *equation (1) has at least one solution in* $C[0, b]$.

Proof. Taking operators H_1 and H_2 defined on $C[0, b]$ by the formula

$$\begin{aligned} (H_1x)(s) &= f(s,x(s)) + F\left(s, \int_0^s r(s,z,x(z))dz, \int_0^s g(s,z,x(z))dz, x(z)\right), \\ (H_2x)(s) &= g(s,x(s)) + G\left(s, \int_0^b p(s,z,x(z))dz, \int_0^b q(s,z,x(z))dz, x(s)\right), \end{aligned}$$

for $s \in [0, b]$.
From assumptions (1) and (3), we see that H_1 and H_2 transform $C[0, b]$ into itself. Now, we put

$$Hx = (H_1x)(H_2x).$$

Clearly, H transform $C[0, b]$ into itself.
Now, fix $x \in C[0, b]$. Then,

$$|(Hx)(s)| = |(H_1x)(s)|.|(H_2x)(s)|$$

$$
\begin{aligned}
&= \left(\left| f(s,x(s)) + F\left(s, \int_0^s r(s,z,x(z))dz, \int_0^s g(s,z,x(z))dz, x(s)\right) \right| \right) \\
&\quad \times \left| g(s,x(s)) + G\left(s, \int_0^b p(s,z,x(z))dz, \int_0^b q(s,z,x(z))dz, x(s)\right) \right|, \\
&\leq \Bigg(|f(s,x(s)) - f(s,0)| + |f(s,0)| \\
&\quad + \left| F\Big(s, \int_0^s r(s,z,x(z))dz, \int_0^s g(s,z,x(z)))dz, x(s)\Big) - F(s,0,0,0) \right| + |F(s,0,0,0)| \Bigg) \\
&\quad \times \Bigg(|g(s,x(s)) - g(s,0)| + |g(s,0)| \\
&\quad + \left| G\Big(s, \int_0^b p(s,z,x(z))dz, \int_0^b q(s,z,x(z))dz, x(s)\Big) - G(s,0,0,0) \right| \\
&\quad + |G(s,0,0,0)| \Bigg), \\
&\leq \Bigg(b_1(s)|x(s)| + C_1 + b_3(s) \int_0^s |r(s,z,x(z)))|dz \\
&\quad + b_4(s) \int_0^s |g(s,z,x(z))|dz + b_5|(x(s))| + C_2 \Bigg) \\
&\quad \times \Bigg(b_2(s)|x(s)| + C_1 + b_6(s) \int_0^b |p(s,z,x(z))|dz \\
&\quad + b_7(s) \int_0^b |q(s,z,x(z))|dz + b_8(s)|x(s)| + C_2 \Bigg) \\
&\leq \Big(2K||x|| + C_1 + 2Kb(\zeta + \eta||x||) + C_2 \Big) \\
&\quad \times \Big(2K||x|| + C_1 + 2Kb(\zeta + \eta||x||) + C_2 \Big) \\
&\leq \Big((2K + 2Kb\eta)||x|| + C_1 + 2Kb\zeta + C_2 \Big)^2
\end{aligned}
$$

Taking $\gamma = 2K + 2Kb\eta$ and $\sigma = C_1 + 2Kb\zeta + C_2$ then we have

$$
\begin{aligned}
||H_1 x|| &\leq \gamma||x|| + \sigma, && (9) \\
||H_2 x|| &\leq \gamma||x|| + \sigma, && (10) \\
||Hx|| &\leq (\gamma||x|| + \sigma)^2. && (11)
\end{aligned}
$$

for $x \in C[0,b]$.

From (11), we reduce the operator H maps the ball $B_r \subset C[0,b]$ into itself for

$r_1 \le r \le r_2$, where

$$r_1 = \frac{(1-2\gamma\sigma) - \sqrt{1-4\gamma\sigma}}{2\gamma^2}.$$

$$r_2 = \frac{(1-2\gamma\sigma) + \sqrt{1-4\gamma\sigma}}{2\gamma^2}.$$

Also, from estimate (9) and (10),

$$\begin{aligned} ||H_1 B_r|| &\le \gamma\, r + \sigma, &(12)\\ ||H_2 B_r|| &\le \gamma\, r + \sigma. &(13)\end{aligned}$$

Next, we prove that the operator H is continuous on the ball B_r. To do this, fix $\epsilon > 0$ and take arbitrary $x, y \in B_r$ such that $||x-y|| \le \epsilon$. Then for $s \in [0, b]$, we get

$$\begin{aligned} |(H_1x)(s) - (H_1y)(s)| &= \Bigg| f(s,x(s)) + F\left(s, \int_0^s r(s,z,x(z))dz, \int_0^s g(s,z,x(z))dz, x(s)\right) \\ &\quad -f(s,y(s)) + F\left(s, \int_0^s r(s,z,y(z))dz, \int_0^s g(s,z,y(z))dz, y(s)\right)\Bigg| \\ &\le b_1(s)|x(s)-y(s)| \\ &\quad +\Bigg|F\left(s, \int_0^s r(s,z,x(z))dz, \int_0^s g(s,z,x(z))dz, x(s)\right) \\ &\quad -F\left(s, \int_0^s r(s,z,y(s))dz, \int_0^s g(s,z,x(z))dz, x(s)\right) + \\ &\quad F\left(s, \int_0^s r(s,z,y(s))dz, \int_0^s g(s,z,x(z))dz, x(s)\right) \\ &\quad -F\left(s, \int_0^s r(s,z,x(s))dz, \int_0^s g(s,z,x(z))dz, x(s)\right)\Bigg| \\ &\le b_1(s)|x(s)-y(s)| \\ &\quad +b_4(s)\int_0^s g(s,z,x(z)) - g(s,z,y(s))|dz + b_5(s)|x(s)) - y(s))| \\ &\quad +b_3(s)\left|\int_0^s r(s,z,x(z))dz - \int_0^s r(s,z,y(z))dz\right| \\ &\le 2K||x-y|| + Kb\omega(g,\epsilon) + Kb\omega(r,\epsilon), \\ &\le 2K\epsilon + Kb(\omega(g,\epsilon) + \omega(r,\epsilon)), \end{aligned}$$

□

where
$\omega(g,\epsilon) = sup\{|g(s,z,x) - g(s,z,y)| : s,z \in [0,b]; x,y \in [-r,r]; ||x-y|| \le \epsilon\}$,
$\omega(r,\epsilon) = sup\{|r(s,z,x) - r(s,z,y)| : s,z \in [0,b]; x,y \in [-r,r]; ||x-y|| \le \epsilon\}$,
The function $r = r(s,z,x)$ and $g = g(s,z,y)$ are uniform continuous on the bounded subset $[0,b] \times [0,b] \times [-r,r]$, then $\omega(r,\epsilon)$ and $\omega(g,\epsilon) \to 0$ as $\epsilon \to 0$. Thus, H_1 is continuous on B_r.

Similarly, we have

$$
\begin{aligned}
|(H_2x)(s)-(H_2y)(s)| &= \Big| h(s,x(s)) + G\left(s, \int_0^b p(s,z,x(z))dz, \int_0^b q(s,z,x(z))dz, x(s)\right) \\
&\quad -h(s,y(s)) - G\left(s, \int_0^b p(s,z,y(z))dz, \int_0^b q(s,z,y(z))dz, y(s)\right) \Big| \\
&\leq b_2(s)|x(s)-y(s)| \\
&\quad +\Big| G\left(s, \int_0^b p(s,z,x(z))dz, \int_0^b q(s,z,x(z))dz, x(s)\right) \\
&\quad -G\left(s, \int_0^b p(s,z,y(s))dz, \int_0^b q(s,z,x(z))dz, x(s)\right) + \\
&\quad G\left(s, \int_0^b p(s,z,y(s))dz, \int_0^b q(s,z,x(z))dz, x(s)\right) \\
&\quad -G\left(s, \int_0^b p(s,z,x(s))dz, \int_0^b q(s,z,x(z))dz, x(s)\right) \Big| \\
&\leq b_2(s)|x(s)-y(s)| \\
&\quad +b_7(s)\int_0^b |p(s,z,x(z))-p(s,z,y(s))|\, dz \\
&\quad +b_8(s)|x(s))-y(s))| \\
&\quad +b_6(s)\int_0^b |q(s,z,x(z))-q(s,z,y(z))|\, dz \\
&\leq 2K||x-y|| + Kb\omega(p,\epsilon) + Kb\omega(q,\epsilon), \\
&\leq 2K\epsilon + Kb(\omega(p,\epsilon)+\omega(q,\epsilon)),
\end{aligned}
$$

where
$\omega(p,\epsilon) = sup\{|p(s,z,x)-p(s,z,y)| : s,z \in [0,b]; x,y \in [-r,r]; ||x-y|| \leq \epsilon\}$,
$\omega(q,\epsilon) = sup\{|q(s,z,x)-q(s,z,y)| : s,z \in [0,b]; x,y \in [-r,r]; ||x-y|| \leq \epsilon\}$,
The function $p = p(s,z,x)$ and $q = q(s,z,y)$ are uniform continuous on the bounded subset $[0,b]\times[0,b]\times[-r,r]$, then $\omega(p,\epsilon)$ and $\omega(q,\epsilon) \to 0$ as $\epsilon \to 0$. Thus, H_2 is continuous on B_r. Hence, H is a continuous operator on B_r.
Now, we prove that the H_1 and H_2 satisfy the Darbo's condition with respect to the measure ω_0, defined in section 2, in the ball B_r. Assume that a non empty subset P of B_r and $x \in P$, Let $\epsilon > 0$ be fixed and $s_1, s_2 \in [0,b]$ such that, without loss of generality, then we put $s_1 \leq s_2$ and $s_2 - s_1 \leq \epsilon$, we obtain

$$
\begin{aligned}
|(H_1x)(s_2)-(H_1x)(s_1)| &= \Big| f(s_2,x(s_2)) \\
&\quad +F\left(s_2, \int_0^{s_2} r(s_2,z,x(z))dz, \int_0^{s_2} g(s_2,z,x(z))dz, x(z)\right) \\
&\quad -f(s_1,x(s_1)) \\
&\quad -F\left(s_1, \int_0^{s_1} r(s_1,z,x(z))dz, \int_0^{s_1} g(s_1,z,x(z))dz, x(z)\right) \Big| \\
&\leq |f(s_2,x(s_2))-f(s_2,x(s_1))| + |f(s_2,x(s_1))-f(s_1,x(s_1))|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|F\left(s_2, \int_0^{s_2} r(s_2,z,x(z))dz, \int_0^{s_2} g(s_2,z,x(z))dz, x(s_2)\right)\right.\\
&\left.-F\left(s_2, \int_0^{s_1} r(s_1,z,x(z))dz \int_0^{s_1} g(s_1,z,x(z))dz, x(s_1)\right)\right|\\
&+\left|F\left(s_2, \int_0^{s_1} r(s_1,z,x(z))dz, \int_0^{s_1} g(s_1,z,x(z))dz, x(s_1))\right)\right|\\
\leq\ & b_1(s)|x(s_2)-x(s_1)| + \omega_f(\epsilon)\\
&+b_3(s)|\int_0^{s_2} r(s_2,z,x(z))dz - r(s_1,z,x(z))dz|\\
&+b_4(s)\left|\int_0^{s_2} g(s_2,z,x(z))dz - \int_0^{s_1} g(s_1,z,x(z))dz\right|\\
&+b_5(s)|x(s_2)-x(s_1))| + \omega_F(\epsilon)\\
\leq\ & K\omega(x,\epsilon) + \omega_f(\epsilon) + K\omega(x,\epsilon)) + \omega_F(\epsilon)\\
&+K\{\int_0^{s_2} r(s_2,z,x(z))dz - r(s_1,z,x(z))|dz\\
&+\int_{s_1}^{s_2} |r(s_2,z,x(z))|dz\}\\
&+K\{\int_0^{s_2} |g(s_2,z,x(z))dz - g(s_1,z,x(z))|dz\\
&+\int_{s_1}^{s_2} |g(s_2,z,x(z))|dz\}
\end{aligned}
$$

$$
\begin{aligned}
\omega(H_1x,\epsilon) \leq\ & K\omega(x,\epsilon) + \omega_f(\epsilon) + K\omega(x,\epsilon) + \omega_F(\epsilon)\\
&+ K\{\omega_r(\epsilon)b + K_1\epsilon\} + K\{\omega_g(\epsilon)b + K_1\epsilon\}, \qquad (14)
\end{aligned}
$$

where
$\omega_f(\epsilon, ..) = sup\{|f(s,s_1) - f(\acute{s},s_1)| : s,\acute{s} \in [0,b]; |s-\acute{s}| \leq \epsilon : s_1,s_2 \in [-r,r]\}$
$\omega_r(\epsilon, ..) = sup\{|r(s,z,x) - r(\acute{s},z,x)| : s,\acute{s} \in [0,b]; |s-\acute{s}| \leq \epsilon : x \in [-r,r]\}$
$\omega_g(\epsilon, ..) = sup\{|g(s,z,x) - g(\acute{s},z,x)| : s,\acute{s} \in [0,b]; |s-\acute{s}| \leq \epsilon : x \in [-r,r]\}$
$\omega_F(\epsilon, ..) = sup\{|F(s,x_1,x_2,x_3) - F(\acute{s},y_1,y_2,y_3)| : s,\acute{s} \in [0,b]; |s-\acute{s}| \leq \epsilon$
$: x_3 \in [-r,r]; x_1,x_2 \in [-K_1b, K_1b]\}$
$K_1 = sup\{|r(s,z,x)|, |g(s,z,x)| : s,z \in [0,b]; x \in [-r,r]\}$
Since the function $f = f(s,x_1)$ and $F = F(s,x_1,x_2,x_3)$ are uniform continuous on the set $[0,b]$, $[0,b]\times\mathbb{R}\times\mathbb{R}$ and $[0,b]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, respectively. The function $r = r(s,z,x)$ and $g = g(s,z,x)$ are uniform continuous on the set $[0,b]\times[0,b]\times\mathbb{R}$. Hence we infer that $\omega_f(\epsilon...) \to 0, \omega_r(\epsilon...) \to 0, \omega_g(\epsilon...) \to 0$ and $\omega_F(\epsilon...) \to 0$ as $\epsilon \to 0$, we get

$$\omega_0(H_1P) \leq 2K\omega_0(P). \qquad (15)$$

Similarly, we write

$$
\begin{aligned}
|(H_2x)(s_2) - (H_2x)(s_1)| =\ & \Big|h(s_2, x(s_2))\\
&+G\left(s_2, \int_0^b p(s_2,z,x(z))dz, \int_0^b q(s_2,z,x(z))dz, x(z)\right)
\end{aligned}
$$

$$
\begin{aligned}
& -h(s_1, x(s_1)) \\
& -G\left(s_1, \int_0^b p(s_1, z, x(z))dz, \int_0^b q(s_1, z, x(z))dz, x(z)\right)\Bigg| \\
\leq\ & |h(s_2, x(s_2)) - h(s_2, x(s_1))| + |h(s_2, x(s_1)) - h(s_1, x(s_1))| \\
& +\Bigg|G\left(s_2, \int_0^b p(s_2, z, x(z))dz, \int_0^b q(s_2, z, x(z))dz, x(s_2)\right) \\
& -G\left(s_2, \int_0^b p(s_1, z, x(z))dz \int_0^b q(s_1, z, x(z))dz, x(s_1)\right)\Bigg| \\
& +\Bigg|G\left(s_2, \int_0^b p(s_1, z, x(z))dz, \int_0^b q(s_1, z, x(z))dz, x(s_1))\right)\Bigg| \\
\leq\ & b_2(s)|x(s_2) - x(s_1)| + \omega_h(\epsilon) \\
& +b_6(s)\left|\int_0^b p(s_2, z, x(z))dz - p(s_1, z, x(z))dz\right| \\
& +b_7(s)\left|\int_0^b q(s_2, z, x(z))dz - \int_0^b q(s_1, z, x(z))dz\right| \\
& +b_8(s)|x(s_2) - x(s_1))| + \omega_G(\epsilon) \\
\leq\ & K\omega(x, \epsilon) + \omega_h(\epsilon) + K\omega(x, \epsilon)) + \omega_G(\epsilon) \\
& +K\{\int_0^b p(s_2, z, x(z))dz - p(s_1, z, x(z))|dz \\
& + \int_0^b |p(s_2, z, x(z))|dz\} \\
& +K\{\int_0^b |q(s_2, z, x(z))dz - q(s_1, z, x(z))|dz \\
& + \int_0^b |q(s_2, z, x(z))|dz\}
\end{aligned}
$$

$$
\begin{aligned}
\omega(H_2x, \epsilon) \leq\ & K\omega(x, \epsilon) + \omega_h(\epsilon) + K\omega(x, \epsilon) + \omega_G(\epsilon) \\
& + K\{\omega_p(\epsilon)b + K_1\epsilon\} + K\{\omega_q(\epsilon)b + K_2\epsilon\}, \qquad (16)
\end{aligned}
$$

where
$\omega_h(\epsilon, ..) = sup\{|h(s, s_1) - h(\acute{s}, s_1)| : s, \acute{s} \in [0, b]; |s - \acute{s}| \leq \epsilon : s_1, s_2 \in [-r, r]\}$
$\omega_p(\epsilon, ..) = sup\{|p(s, z, x) - p(\acute{s}, z, x)| : s, \acute{s} \in [0, b]; |s - \acute{s}| \leq \epsilon : x \in [-r, r]\}$
$\omega_q(\epsilon, ..) = sup\{|q(s, z, x) - q(\acute{s}, z, x)| : s, \acute{s} \in [0, b]; |s - \acute{s}| \leq \epsilon : x \in [-r, r]\}$
$\omega_G(\epsilon, ..) = sup\{|G(s, x_1, x_2, x_3) - G(\acute{s}, y_1, y_2, y_3)| : s, \acute{s} \in [0, b]; |s - \acute{s}| \leq \epsilon$
$: x_3 \in [-r, r]; x_1, x_2 \in [-K_2b, K_2b]\}$
$K_2 = sup\{|p(s, z, x)|, |q(s, z, x)| : s, z \in [0, b]; x \in [-r, r]\}$
Since the function $h = h(s, x_1)$ and $G = G(s, x_1, x_2, x_3)$ are uniform continuous on the set $[0, b]$, $[0, b] \times \mathbb{R} \times \mathbb{R}$ and $[0, b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, respectively. The function $p = p(s, z, x)$ and $q = q(s, z, x)$ are uniform continuous on the set $[0, b] \times [0, b] \times \mathbb{R}$. Hence we infer that $\omega_h(\epsilon...) \to 0, \omega_p(\epsilon...) \to 0, \omega_q(\epsilon...) \to 0$ and $\omega_G(\epsilon...) \to 0$ as $\epsilon \to 0$, we get

$$\omega_0(H_2P) \leq 2K\omega_0(P). \qquad (17)$$

Finally, we get H satisfies the Darbo condition on B_r with respect to the measure ω_0 with constant $(\gamma r+\sigma)\ 2K+(\gamma r+\sigma)\ 2K$. Now, we have

$$\begin{aligned}(\gamma r+\sigma)\ 2K+(\gamma r+\sigma)\ 2K &= 4K(\gamma r+\sigma)\\ &= 4K(\gamma r_1+\sigma)\\ &= 4K\left(\gamma\left(\frac{(1-2\gamma\sigma)-\sqrt{1-4\gamma\sigma}}{2\gamma^2}\right)+\sigma\right)\\ &= 4K\left(\frac{1-\sqrt{1-4\gamma\sigma}}{2\gamma}\right)\\ &< 1.\end{aligned}$$

Hence, H is a contraction on B_r with respect to ω_0. Thus, H has at least one fixed point in the ball B_r, by applying Theorem 2.2. Consequently, the nonlinear FIE (1) has at least one solution in ball B_r.

4 An example

Now, we present an example of a functional-integral equation and consequently, see the existence of its solutions by using Theorem 3.1.

Example 4.1. *Consider the following nonlinear functional integral equation:*

$$\begin{aligned}x(s) = &\left[\frac{s^2}{8(1+s^2)}\sin x(s)+\frac{s}{7}\int_0^s\left\{\left(\frac{s\sin x(\sqrt{z})}{2}+(2+s)\ln(1+|x(\sqrt{z})|)\right)\right.\right.\\ &\left.\left.+\left(\frac{s\cos(x(1-z))}{2}+(2+s)\arctan\left(\frac{|x(1-z)|}{1+|x(1-z)|}\right)\right)\right\}dz\right]\\ &\times\left[\frac{s^2}{12}\arctan|x(s)|+\frac{1}{14}\int_0^1\left\{\left(\frac{\cos(x(1-z))}{2}+3s^2\arctan\left(\frac{|x(1-z)|}{1+|x(1-z)|}\right)\right)\right.\right.\\ &\left.\left.+\left(\frac{t\sin x(\sqrt{z})}{2}+3s^2\ln(1+|x(\sqrt{z})|)\right)\right\}dz\right], \qquad (18)\end{aligned}$$

where $s\in[0,1]$.
Observe that equation (18) *is a particular case of equation* (1). *Let us take* $f,h:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}; F,H:[0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *and* $r,g,p,q:[0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$ *and comparing* (18) *with equation* (1), *we get*

$$\begin{aligned}f(s,x_1) &= \frac{s^2}{8(1+s^2)}\sin x_1,\\ h(s,x_1) &= \frac{s^2}{12}\arctan|x_1|,\\ F(s,x_1,x_2,x_3) &= \frac{s}{7}x_2,\end{aligned}$$

$$
\begin{aligned}
G(s,x_1,x_2,x_3) &= \frac{1}{14}x_2,\\
r(s,z,x) &= \frac{s\sin x}{2} + (2+s)\ln(1+|x|),\\
g(s,z,x) &= \frac{s\cos x}{2} + (2+s)\arctan\left(\frac{|x|}{1+|x|}\right),\\
p(s,z,x) &= \frac{\cos x}{2} + 3s^2\arctan\left(\frac{|x|}{1+|x|}\right),\\
q(s,z,x) &= \frac{s\sin x}{2} + 3s^2\ln(1+|x|),
\end{aligned}
$$

then we can easily check that the assumptions of Theorem 3.1 are satisfied with constants $b_1 = \frac{1}{16}, b_2 = \frac{1}{12}, b_3 = b_5 = b_6 = b_8 = 0, b_4 = \frac{1}{7}, b_7 = \frac{1}{14}$.
In this case, we have

$$K = \max\left\{\frac{1}{16}, \frac{1}{12}, 0, \frac{1}{7}, \frac{1}{14}\right\} = \frac{1}{7}.$$

Further,

$$|p(s,z,x)| \leq \frac{1}{2} + 3|x|,$$

$$|q(s,z,x)| \leq \frac{1}{2} + 3|x|,$$

$$|r(s,z,x)| \leq \frac{1}{2} + 3|x|,$$

$$|g(s,z,x)| \leq \frac{1}{2} + 3|x|,$$

It is observed that $C_1 = C_2 = 0, \zeta = \frac{1}{2}, \eta = 3$ *and* $b = 1$.
Finally, we see that

$$4\gamma\sigma = 4(2K + 2Kb\eta)(C_1 + 2Kb\zeta + C_2) < 1.$$

Hence, all the assumptions from (1) *to* (6) *are satisfied. Now, based on result obtained in Theorem 3.1, we deduce that equation* (18) *has at least one solution in Banach algebra* $C[0,1]$.

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