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ABSTRACT. The aim of this paper is to express several identities involving 'degenerate formal power series' as those including degenerate Stirling numbers of the second kind, degenerate Bell polynomials, degenerate Fubini polynomials and degenerate poly-Bernoulli polynomials.

1. INTRODUCTION

In recent years, studying degenerate versions of some special polynomials and numbers regained interests of some mathematicians, which include the degenerate Bernoulli numbers of the second kind, the degenerate Stirling numbers of both kinds, the degenerate Cauchy numbers, the degenerate Bell numbers and polynomials, the degenerate complete Bell polynomials and numbers, and so on (see [7,9,10,13,14,16,18]) and the references therein). It is remarkable that this study of degenerate versions is not only limited to polynomials and numbers but also extended to transcendental functions like the gamma functions (see [11,12]). They have been studied by various means like combinatorial methods, generating functions, differential equations, umbral calculus, λ-umbral calculus, p-adic analysis, and probability theory.

The aim of this paper is to express several identities involving 'degenerate formal power series' as those including degenerate Stirling numbers of the second kind, degenerate Bell polynomials, degenerate Fubini polynomials and degenerate poly-Bernoulli polynomials (see Theorems 2,5,6,9-12). Along the way, we also obtain some related identities which involve the λ-falling factorials, the degenerate Stirling numbers of both kinds, the degenerate Bernoulli numbers, the degenerate Fubini polynomials and the degenerate Euler polynomials. For the rest of this section, we recall the facts that will be used throughout this paper.

For any $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by

$$(1) \qquad e_\lambda^x(t) = \sum_{n=0}^{\infty} \frac{(x)_{n,\lambda}}{n!} t^n, \quad (\text{see } [4,7,9,10]),$$

where the λ-falling factorials are given by $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$, $(n \geq 1)$.

When $x = 1$, we write $e_\lambda(t) = e_\lambda^1(t)$. Note that $\lim_{\lambda \to 0} e_\lambda^x(t) = e^{xt}$.

Let us consider the 'degenerate formal power series' which is given by

$$(2) \qquad f_\lambda(t) = \sum_{k=0}^{\infty} a_k(t)_{k,\lambda} = a_0 + a_1(t)_{1,\lambda} + a_2(t)_{2,\lambda} + \cdots \in \mathbb{C}[[t]],$$

where the $a_i's$ are constant complex numbers. Then we let

$$(3) \qquad f(t) = \lim_{\lambda \to 0} f_\lambda(t) = \sum_{k=0}^{\infty} a_k t^n \in \mathbb{C}[[t]], \text{ (see [3])}.$$

The Carlitz degenerate Bernoulli polynomials are defined by

$$(4) \qquad \frac{t}{e_\lambda(t) - 1} e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}, \quad \text{(see [6])}.$$

Note that $\lim_{n \to \infty} \beta_{n,\lambda} = B_n(x)$, where $B_n(x)$ are Bernoulli polynomials given by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \text{(see [1, 14 − 18])}.$$

Kim-Kim considered the degenerate Stirling numbers of the first kind defined by

$$(5) \qquad (x)_n = \sum_{k=0}^{n} S_{1,\lambda}(n,k)(x)_{k,\lambda}, \quad (n \geq 0), \quad \text{(see [5,7])},$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$.

As the inversion formula of (5), the degenerate Stirling numbers of the second kind are given by

$$(6) \qquad (x)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k)(x)_k, \quad (n \geq 0), \quad \text{(see [7,9,10])}.$$

In [13], the degenerate Bell polynomials are defined by

$$(7) \qquad e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} \phi_{n,\lambda}(x) \frac{t^n}{n!}.$$

Note that $\phi_{n,\lambda}(x) = \sum_{k=0}^{n} S_{2,\lambda}(n,k)x^k$. When $x = 1$, $\phi_{n,\lambda} = \phi_{n,\lambda}(1)$ are called the degenerate Bell numbers.

For any $\lambda \in \mathbb{R}$, the degenerate Fubini polynomials are defined by Kim-Kim as

$$(8) \qquad \frac{1}{1 - x(e_\lambda(t) - 1)} = \sum_{n=0}^{\infty} F_{n,\lambda}(x) \frac{t^n}{n!}, \quad \text{(see [14])}.$$

It is well known that the polylogarithmic function is defined by

$$(9) \qquad \text{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^k}{n^k}, \quad (k \in \mathbb{Z}), \quad \text{(see [2])}.$$

Note that $\text{Li}_1(x) = -\log(1-x)$.

2. Some identities of degenerate special polynomials

Here we consider the 'degenerate formal power series' given by

$$(10) \qquad \begin{aligned} f_\lambda(t) &= \sum_{k=0}^{\infty} a_k(t)_{k,\lambda} \\ &= a_0 + a_1(t)_{1,\lambda} + a_2(t)_{2,\lambda} + a_3(t)_{3,\lambda} + \cdots \in \mathbb{C}[[t]], \end{aligned}$$

and express several identities involving 'degenerate formal power series' as those including degenerate Stirling numbers of the second kind, degenerate Bell polynomials, degenerate Fubini polynomials and degenerate poly-Bernoulli polynomials (see Theorems 2,5,6,9-12). From (6), we note that

$$(11) \qquad \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!} = \frac{1}{k!}\left(e_\lambda(t)-1\right)^k = \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e_\lambda^l(t)$$

$$= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\sum_{n=0}^{\infty}(l)_{n,\lambda}\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left\{\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(l)_{n,\lambda}\right\}\frac{t^n}{n!}.$$

By comparing the coefficients on both sides of (11), we get

$$(12) \qquad \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(l)_{n,\lambda} = S_{2,\lambda}(n,k),$$

with the understanding that $S_{2,\lambda}(n,k)=0$, for $0 \le n < k$. It is not difficult to show that

$$(13) \qquad (y+zk)_{m,\lambda} = \sum_{l=0}^{m}\binom{m}{l}(zk)_{l,\lambda}(y)_{m-l,\lambda}, \quad (m \ge 0).$$

We observe that

$$(14) \qquad (zk)_{p,\lambda} = (zk)(zk-\lambda)\cdots(zk-(p-1)\lambda)$$

$$= k^p z\left(z-\frac{\lambda}{k}\right)\cdots\left(z-(p-1)\frac{\lambda}{k}\right)$$

$$= k^p(z)_{p,\frac{\lambda}{k}} = z^p(k)_{p,\frac{\lambda}{z}}, \quad (p \ge 0).$$

From (13) and (14), we note that

$$(15) \qquad (y+zk)_{m,\lambda} = \sum_{p=0}^{m}\binom{m}{p}k^p(z)_{p,\frac{\lambda}{k}}(y)_{m-p,\lambda}$$

$$= \sum_{p=0}^{m}\binom{m}{p}z^p(y)_{m-p,\lambda}(k)_{p,\frac{\lambda}{z}}.$$

By (12) and (15), we get

$$(16) \quad \sum_{k=0}^{n}\binom{n}{k}(-1)^k(y+zk)_{m,\lambda} = \sum_{k=0}^{n}\binom{n}{k}(-1)^k\left\{\sum_{p=0}^{m}\binom{m}{p}z^p(y)_{m-p,\lambda}(k)_{p,\frac{\lambda}{z}}\right\}$$

$$= \sum_{p=0}^{m}\binom{m}{p}z^p(y)_{m-p,\lambda}\left\{\sum_{k=0}^{n}\binom{n}{k}(-1)^k(k)_{p,\frac{\lambda}{z}}\right\}$$

$$= (-1)^n n!\sum_{p=0}^{m}\binom{m}{p}z^p(y)_{m-p,\lambda}\left(\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(k)_{p,\frac{\lambda}{z}}\right)$$

$$= (-1)^n n!\sum_{p=0}^{m}\binom{m}{p}z^p(y)_{m-p,\lambda}S_{2,\frac{\lambda}{z}}(p,n).$$

Therefore, by comparing the coefficients on both sides of (16), we obtain the following theorem.

Theorem 1. *For $n \geq 0$, the following identity holds.*

$$\sum_{k=0}^{n} \binom{n}{k}(-1)^k(y+zk)_{m,\lambda} = (-1)^n n! \sum_{p=0}^{m} \binom{m}{p} z^p(y)_{m-p,\lambda} S_{2,\frac{\lambda}{z}}(p,n).$$

Remark 2. *As $S_{2,\lambda}(n,k) = 0$, for $k > n$ and $k < 0$, we see from Theorem 1 that we have*

$$\sum_{k=0}^{n} \binom{n}{k}(-1)^k(y+zk)_{m,\lambda} = 0, \text{ if } n > m.$$

From (10), we note that

$$(17) \qquad\qquad f_\lambda(y+zk) = \sum_{m=0}^{\infty} a_m(y+zk)_{m,\lambda}.$$

By (12), (15), (17) and Theorem 1, we get

$$(18) \qquad \sum_{k=0}^{n} \binom{n}{k}(-1)^k f_\lambda(y+zk) = \sum_{k=0}^{n} \binom{n}{k}(-1)^k \sum_{m=0}^{\infty} a_m(y+zk)_{m,\lambda}$$

$$= \sum_{m=0}^{\infty} a_m(-1)^n n! \sum_{p=0}^{m} \binom{m}{p} S_{2,\frac{\lambda}{z}}(p,m) z^p(y)_{m-p,\lambda}$$

$$= (-1)^n n! \sum_{m=0}^{\infty} a_m \left\{ \sum_{p=0}^{m} \binom{m}{p} S_{2,\frac{\lambda}{z}}(p,n) z^p(y)_{m-p,\lambda} \right\}.$$

Therefore, we obtain the following theorem.

Theorem 3. *For $n \geq 0$, the following identity is valid.*

$$\sum_{k=0}^{n} \binom{n}{k}(-1)^k f_\lambda(y+zk) = (-1)^n n! \sum_{m=0}^{\infty} a_m \left\{ \sum_{p=0}^{m} \binom{m}{p} S_{2,\frac{\lambda}{z}}(p,n) z^p(y)_{m-p,\lambda} \right\}.$$

In particular, for $y = 0$,

$$\sum_{k=0}^{n} \binom{n}{k}(-1)^k f_\lambda(zk) = (-1)^n n! \sum_{m=0}^{\infty} a_m S_{2,\frac{\lambda}{z}}(m,n) z^m.$$

If $n = m$ in Theorem 1, then we have

$$(19) \qquad \sum_{k=0}^{n} \binom{n}{k}(-1)^k(y+zk)_{n,\lambda} = (-1)^n n! \sum_{p=0}^{n} \binom{n}{p} z^p(y)_{n-p,\lambda} S_{2,\frac{\lambda}{z}}(p,n)$$

$$= (-1)^n n! z^n.$$

Therefore, by (19), we obtain the following theorem.

Theorem 4. *For $n \geq 0$, we have the following identity:*

$$\sum_{k=0}^{n} \binom{n}{k}(-1)^k(y+zk)_{n,\lambda} = (-1)^n n! z^n.$$

For $n \in \mathbb{N}$, we have

$$(20) \qquad\qquad (x)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k)(x)_k = \sum_{k=1}^{n} S_{2,\lambda}(n,k)(x)_k.$$

Dividing (20) by x and then taking $x = 0$, we have

$$(21) \qquad\qquad (-1)^{n-1}\lambda^{n-1}(n-1)! = \sum_{k=1}^{n} S_{2,\lambda}(n,k)(-1)^{k-1}(k-1)!.$$

Therefore, by (21), we obtain the following lemma.

Lemma 5. *For $n \in \mathbb{N}$, the following identity holds true.*

$$(-1)^{n-1}\lambda^{n-1}(n-1)! = \sum_{k=1}^{n} S_{2,\lambda}(n,k)(-1)^{k-1}(k-1)!.$$

From Theorem 3, we note that

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}f_{\lambda}(zk) = (-1)^{n}n!\sum_{m=0}^{\infty}a_{m}z^{m}S_{2,\frac{\lambda}{z}}(m,n).$$

By multiplying $\frac{1}{n}$ on both sides, we get

$$\frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}f_{\lambda}(zk) = (-1)^{n}(n-1)!\sum_{m=0}^{\infty}a_{m}z^{m}S_{2,\frac{\lambda}{z}}(m,n),$$

where $n = 1,2,3,\ldots$.

Thus, by using Lemma 5, we have

(22) $$\sum_{n=1}^{\infty}\frac{1}{n}\left\{\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}f_{\lambda}(zk)\right\} = \sum_{n=1}^{\infty}(-1)^{n}(n-1)!\sum_{m=0}^{\infty}a_{m}z^{m}S_{2,\frac{\lambda}{z}}(m,n)$$

$$= \sum_{m=1}^{\infty}a_{m}z^{m}\sum_{n=1}^{\infty}(-1)^{n}(n-1)!S_{2,\frac{\lambda}{z}}(m,n)$$

$$= \sum_{m=1}^{\infty}a_{m}z^{m}(-1)^{m}\left(\frac{\lambda}{z}\right)^{m-1}(m-1)!$$

$$= z\sum_{m=1}^{\infty}a_{m}(-1)^{m}\lambda^{m-1}(m-1)!.$$

Therefore, by (22), we obtain the following theorem.

Theorem 6. *For $f_{\lambda}(t) = \sum_{k=0}^{\infty}a_{k}(t)_{k,\lambda} \in \mathbb{C}[[t]]$, the following identity holds.*

$$\sum_{n=1}^{\infty}\frac{1}{n}\left\{\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}f_{\lambda}(zk)\right\} = -z\sum_{m=1}^{\infty}a_{m}(-1)^{m}\lambda^{m-1}(m-1)!$$

We observe that

(23) $$f'_{\lambda}(0) = \frac{d}{dt}f_{\lambda}(t)\Big|_{t=0} = \sum_{k=1}^{\infty}a_{k}\frac{d}{dt}(t)_{k,\lambda}\Big|_{t=0} = \sum_{k=1}^{\infty}a_{k}(-1)^{k-1}\lambda^{k-1}(k-1)!.$$

By Theorem 6 and (23), we get

(24) $$\sum_{n=1}^{\infty}\frac{1}{n}\left\{\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}f_{\lambda}(zk)\right\} = -z\sum_{m=1}^{\infty}a_{m}(-1)^{m-1}\lambda^{m-1}(m-1)! = -zf'_{\lambda}(0).$$

Therefore, we obtain the following theorem.

Theorem 7. *Let $f_{\lambda}(t) = \sum_{k=0}^{\infty}a_{k}(t)_{k,\lambda} \in \mathbb{C}[[t]]$. Then the following identity is valid.*

$$\sum_{n=1}^{\infty}\frac{1}{n}\left\{\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}f_{\lambda}(zk)\right\} = -zf'_{\lambda}(0),$$

where $f'_{\lambda}(0) = \frac{d}{dt}f_{\lambda}(t)\big|_{t=0}$.

Let us take $f_\lambda(t) = \binom{t}{p}_\lambda = \frac{(t)_{p,\lambda}}{p!}$, $(p \in \mathbb{N} \cup \{0\})$. Then we have $a_m = \frac{1}{m!}\delta_{m,p}$, $(m, p \geq 0)$, where $\delta_{m,n}$ is the Kronecker's symbol.

Then, from Theorem 3, we have

$$(25) \qquad \sum_{k=0}^{n} \binom{n}{k}(-1)^k \binom{zk}{p}_\lambda = (-1)^n \frac{n!}{p!} S_{2,\frac{\lambda}{z}}(p,n)z^p.$$

Thus, we have the following lemma.

Lemma 8. *For $n \geq 0$, we have the following identity.*

$$(26) \qquad \sum_{k=0}^{n} \binom{n}{k}(-1)^k \binom{zk}{p}_\lambda = (-1)^n \frac{n!}{p!} S_{2,\frac{\lambda}{z}}(p,n)z^p.$$

Note that $\sum_{k=0}^{n} \binom{n}{k}(-1)^k \binom{zk}{n}_\lambda = (-1)^n z^n$. It is well known that the Bell polynomials are defined by

$$\sum_{n=0}^{\infty} \phi_n(x)\frac{t^n}{n!} = \lim_{\lambda \to 0} e^{x(e_\lambda(t)-1)} = e^{x(e^t-1)}.$$

The compositional inverse of $e_\lambda(t)$ is denoted by $\log_\lambda t$. So $e_\lambda(\log_\lambda t) = \log_\lambda(e_\lambda(t)) = t$. They are called the degenerate logarithms and given by

$$(27) \qquad \log_\lambda t = \frac{1}{\lambda}(t^\lambda - 1), \quad \log_\lambda(1+t) = \frac{1}{\lambda}\big((1+t)^\lambda - 1\big) = \sum_{n=1}^{\infty} \lambda^{n-1}(1)_{n,\frac{1}{\lambda}}\frac{t^n}{n!}.$$

Now, we observe that

$$(28) \qquad \frac{1}{n!}\big((1+t)^z - 1\big)^n = \frac{1}{n!}\big(e^{z\log(1+t)} - 1\big)^n$$
$$= \sum_{m=n}^{\infty} S_2(m,n)\frac{z^m}{m!}\big(\log(1+t)\big)^m = \sum_{m=n}^{\infty} S_2(m,n)z^m \sum_{p=m}^{\infty} S_1(p,m)\frac{t^p}{p!}$$
$$= \sum_{p=n}^{\infty} \left(\sum_{m=n}^{p} S_2(m,n)S_1(p,m)z^m\right)\frac{t^n}{p!},$$

where $S_2(m,n) = \lim_{\lambda \to 0} S_{2,\lambda}(m,n)$ and $S_1(m,n) = \lim_{\lambda \to 0} S_{1,\lambda}(m,n)$ are respectively the Stirling numbers of the second kind and the Stirling numbers of the first kind.

On the other hand,

$$(29) \qquad \frac{1}{n!}\big((1+t)^z - 1\big)^n = \frac{1}{n!}\big(e_\lambda^z(\log_\lambda(1+t)) - 1\big)^n$$
$$= \frac{1}{n!}\big(e_{\frac{\lambda}{z}}(z\log_\lambda(1+t)) - 1\big)^n$$
$$= \sum_{m=n}^{\infty} S_{2,\frac{\lambda}{z}}(m,n)\frac{z^m}{m!}\big(\log_\lambda(1+t)\big)^m$$
$$= \sum_{m=n}^{\infty} S_{2,\frac{\lambda}{z}}(m,n)z^m \sum_{p=m}^{\infty} S_{1,\lambda}(p,m)\frac{t^p}{p!}$$
$$= \sum_{p=n}^{\infty} \left(\sum_{m=n}^{p} S_{2,\frac{\lambda}{z}}(m,n)S_{1,\lambda}(p,m)z^m\right)\frac{t^p}{p!}.$$

Therefore, by (27) and (29), we obtain the following theorem.

Theorem 9. *For n, p, with $n \geq p \geq 0$, the following holds true.*

$$\sum_{m=n}^{p} S_2(m,n)S_1(p,m)z^m = \sum_{m=n}^{p} S_{2,\frac{\lambda}{z}}(m,n)S_{1,\lambda}(p,m)z^m.$$

From Theorem 3, we note that

(30)
$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \left\{ \sum_{k=0}^{n} \binom{n}{k}(-1)^k f_\lambda(y+kz) \right\}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}(-1)^n n! \sum_{m=0}^{\infty} a_m \left\{ \sum_{p=0}^{m} \binom{m}{p} S_{2,\frac{\lambda}{z}}(p,n)z^p(y)_{m-p,\lambda} \right\}$$

$$= \sum_{m=0}^{\infty} a_m \sum_{p=0}^{m} \binom{m}{p}(y)_{m-p,\lambda} z^p \left\{ \sum_{n=0}^{\infty} x^n(-1)^n S_{2,\frac{\lambda}{z}}(p,n) \right\}$$

$$= \sum_{m=0}^{\infty} a_m \sum_{p=0}^{m} \binom{m}{p}(y)_{m-p,\lambda} z^p \sum_{n=0}^{p} x^n(-1)^n S_{2,\frac{\lambda}{z}}(p,n)$$

$$= \sum_{m=0}^{\infty} a_m \sum_{p=0}^{m} \binom{m}{p}(y)_{m-p,\lambda} z^p \phi_{p,\frac{\lambda}{z}}(-x).$$

From (30), we obtain the following theorem.

Theorem 10. *Let $f_\lambda(t) = \sum_{k=0}^{\infty} a_k(t)_{k,\lambda} \in \mathbb{C}[[t]]$. Then we have the following identity:*

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \left\{ \sum_{k=0}^{n} \binom{n}{k}(-1)^k f_\lambda(y+kz) \right\} = \sum_{m=0}^{\infty} a_m \sum_{p=0}^{m} \binom{m}{p}(y)_{m-p,\lambda} z^p \phi_{p,\frac{\lambda}{z}}(-x).$$

In particular, for $y = 0$,

(31)
$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \left\{ \sum_{k=0}^{n} \binom{n}{k}(-1)^k f_\lambda(kz) \right\} = \sum_{m=0}^{\infty} a_m z^m \phi_{m,\frac{\lambda}{z}}(-x).$$

Let us take $x = -1$ in (31). Then we have

(32)
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\{ \sum_{k=0}^{n} \binom{n}{k}(-1)^k f_\lambda(kz) \right\} = \sum_{m=0}^{\infty} a_m z^m \phi_{m,\frac{\lambda}{z}}.$$

Let us take $f_\lambda(t) = \binom{t}{p}_\lambda = \frac{(t)_{p,\lambda}}{p!}$. Then, from (31) and Theorem 3, we get

(33)
$$\sum_{n=0}^{p} \frac{x^n}{n!} \left\{ \sum_{k=0}^{n} \binom{n}{k}(-1)^k \binom{zk}{p}_\lambda \right\} = \frac{z^p}{p!} \phi_{p,\frac{\lambda}{z}}(-x).$$

Here we observe that $\sum_{k=0}^{n} \binom{n}{k}(-1)^k \binom{zk}{p}_\lambda = \frac{(-1)^n n!}{p!} S_{2,\frac{\lambda}{z}}(p,n)z^p = 0$, if $n > p$.

From (8), we note that

(34)
$$\sum_{n=0}^{\infty} F_{n,\lambda}(x) \frac{t^n}{n!} = \frac{1}{1 - x(e_\lambda(t) - 1)} = \sum_{k=0}^{\infty} x^k(e_\lambda(t) - 1)^k$$

$$= \sum_{k=0}^{\infty} x^k k! \frac{1}{k!}(e_\lambda(t) - 1)^k = \sum_{k=0}^{\infty} x^k k! \sum_{n=k}^{\infty} S_{2,\lambda}(n,k) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} x^k k! S_{2,\lambda}(n,k) \right) \frac{t^n}{n!}.$$

Thus, we have

$$(35) \qquad F_{n,\lambda}(x) = \sum_{k=0}^{n} k! S_{2,\lambda}(n,k) x^k, \quad (n \geq 0).$$

From Theorem 3, we have

$$(36) \qquad \sum_{n=0}^{\infty} x^n \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_\lambda(y+zk) \right\}$$

$$= \sum_{n=0}^{\infty} x^n (-1)^n n! \sum_{m=0}^{\infty} a_m \left\{ \sum_{p=0}^{m} \binom{m}{p} (y)_{m-p,\lambda} z^p S_{2,\frac{\lambda}{z}}(p,n) \right\}$$

$$= \sum_{m=0}^{\infty} a_m \sum_{p=0}^{m} \binom{m}{p} (y)_{m-p,\lambda} z^p \left\{ \sum_{n=0}^{\infty} x^n (-1)^n n! S_{2,\frac{\lambda}{z}}(p,n) \right\}$$

$$= \sum_{m=0}^{\infty} a_m \sum_{p=0}^{m} \binom{m}{p} (y)_{m-p,\lambda} z^p \left\{ \sum_{n=0}^{p} x^n (-1)^n n! S_{2,\frac{\lambda}{z}}(p,n) \right\}$$

$$= \sum_{m=0}^{\infty} a_m \sum_{p=0}^{m} \binom{m}{p} (y)_{m-p,\lambda} z^p F_{p,\frac{\lambda}{z}}(-x).$$

Therefore, by (36), we obtain the following theorem.

Theorem 11. *Let* $f_\lambda(t) = \sum_{k=0}^{\infty} a_k (t)_{k,\lambda} \in \mathbb{C}[[t]]$. *Then the following identity holds true.*

$$\sum_{n=0}^{\infty} x^n \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_\lambda(y+zk) \right\} = \sum_{m=0}^{\infty} a_m \sum_{p=0}^{m} \binom{m}{p} (y)_{m-p,\lambda} z^p F_{p,\frac{\lambda}{z}}(-x).$$

In particular, for $y = 0$,

$$\sum_{n=0}^{\infty} x^n \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_\lambda(zk) \right\} = \sum_{m=0}^{\infty} a_m z^m F_{m,\frac{\lambda}{z}}(-x).$$

Let us take $f_\lambda(t) = \binom{t}{p}_\lambda = \frac{(t)_{p,\lambda}}{p!}$. Then we have $a_m = \frac{1}{m!} \delta_{m,p}$, $(m,p \geq 0)$. Hence we have

$$\sum_{n=0}^{\infty} x^n \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{zk}{p}_\lambda \right\} = \frac{1}{p!} z^p F_{p,\frac{\lambda}{z}}(-x).$$

From Theorem 11, we note that

$$(37) \qquad \int_0^x \sum_{n=0}^{\infty} t^n \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_\lambda(y+zk) \right\} dt = \sum_{m=0}^{\infty} a_m \sum_{p=0}^{m} \binom{m}{p} (y)_{m-p,\lambda} z^p \int_0^x F_{p,\frac{\lambda}{z}}(-t) dt.$$

By (35), we easily get

$$(38) \qquad \int_0^x F_{p,\frac{\lambda}{z}}(-t) dt = \sum_{j=0}^{p} (-1)^j j! \frac{x^{j+1}}{j+1} S_{2,\frac{\lambda}{z}}(p,j).$$

From (37) and (38), we have

$$(39) \qquad \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_\lambda(y+zk) \right\}$$

$$= \sum_{m=0}^{\infty} a_m \sum_{p=0}^{m} \binom{m}{p} (y)_{m-p,\lambda} z^p \sum_{j=0}^{p} (-1)^j j! \frac{x^{j+1}}{j+1} S_{2,\frac{\lambda}{z}}(p,j).$$

Thus, we see that

$$
(40) \qquad \sum_{n=0}^{\infty} \frac{x^n}{n+1} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_\lambda(y+zk) \right\}
$$

$$
= \sum_{m=0}^{\infty} a_m \sum_{p=0}^{m} \binom{m}{p} (y)_{m-p,\lambda} z^p \sum_{j=0}^{p} (-1)^j j! \frac{x^j}{j+1} S_{2,\frac{\lambda}{z}}(p,j).
$$

Now, we consider $(r-1)$-times iterated integration with respect to x as follows:

$$
(41) \qquad \frac{1}{x} \underbrace{\int_0^x \frac{1}{x} \int_0^x \cdots \frac{1}{x} \int_0^x}_{(r-1)-\text{times}} \sum_{n=0}^{\infty} \frac{x^n}{n+1} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_\lambda(y+zk) \right\} dx \cdots dx
$$

$$
= \sum_{m=0} a_m \sum_{p=0}^{m} \binom{m}{p} (y)_{m-p,\lambda} z^p \sum_{j=0}^{p} (-1)^j \frac{j!}{j+1} S_{2,\frac{\lambda}{z}}(p,j)
$$

$$
\times \frac{1}{x} \underbrace{\int_0^x \frac{1}{x} \int_0^x \cdots \frac{1}{x} \int_0^x}_{(r-1)-\text{times}} x^j dx \cdots dx
$$

$$
= \sum_{m=0}^{\infty} a_m \sum_{p=0}^{m} \binom{m}{p} (y)_{m-p,\lambda} z^p \sum_{j=0}^{p} (-1)^j \frac{j!}{(j+1)^r} x^j S_{2,\frac{\lambda}{z}}(p,j).
$$

By (41), we get

$$
(42) \qquad \sum_{n=0}^{\infty} \frac{x^n}{(n+1)^r} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_\lambda(y+zk) \right\}
$$

$$
= \sum_{m=0}^{\infty} a_m \left\{ \sum_{p=0}^{m} \binom{m}{p} (y)_{m-p,\lambda} z^p \sum_{j=0}^{p} (-1)^j j! S_{2,\frac{\lambda}{z}}(p,j) \right\} \frac{x^j}{(j+1)^r}
$$

Let us take $y=0$ in (42). Then we have

$$
(43) \qquad \sum_{n=0}^{\infty} \frac{x^n}{(n+1)^r} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_\lambda(zk) \right\} = \sum_{m=0}^{\infty} a_m z^m \sum_{j=0}^{m} (-1)^j j! S_{2,\frac{\lambda}{z}}(m,j) \frac{x^j}{(j+1)^r}.
$$

By Theorem 11 and (35), we see that

$$
(44) \qquad \sum_{n=0}^{\infty} x^n \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_\lambda(zk) \right\}
$$

$$
= \sum_{m=0}^{\infty} a_m z^m F_{m,\frac{\lambda}{z}}(-x) = \sum_{m=0}^{\infty} a_m z^m \left\{ \sum_{p=0}^{m} S_{2,\frac{\lambda}{z}}(m,p)(-x)^p p! \right\}.
$$

Thus, we have

$$
(45) \qquad \sum_{n=1}^{\infty} x^{n-1} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_\lambda(zk) \right\} = \sum_{m=1}^{\infty} a_m z^m \sum_{p=1}^{m} S_{2,\frac{\lambda}{z}}(m,p)(-1)^p p! x^{p-1}.
$$

From (45), we note that

$$
(46) \qquad \sum_{n=1}^{\infty} \frac{x^n}{n} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_\lambda(zk) \right\} = \sum_{m=1}^{\infty} a_m z^m \sum_{p=1}^{m} S_{2,\frac{\lambda}{z}}(m,p)(-1)^p p! \int_0^x x^{p-1} dx
$$

$$
= \sum_{m=1}^{\infty} a_m z^m \sum_{p=1}^{m} S_{2,\frac{\lambda}{z}}(m,p)(-1)^p \frac{p!}{p} x^p.
$$

Therefore, we obtain the following theorem.

Theorem 12. *Let* $f_\lambda(t) = \sum\limits_{k=0}^{\infty} a_k(t)_{k,\lambda} \in \mathbb{C}[[t]]$. *Then the following identity holds.*

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_\lambda(zk) \right\} = \sum_{m=1}^{\infty} a_m z^m \sum_{p=1}^{m} S_{2,\frac{\lambda}{z}}(m,p)(-1)^p(p-1)!x^p.$$

We define the degenerate poly-Bernoulli polynomials which are given by

$$(47) \qquad\qquad \frac{\mathrm{Li}_k(1 - e_\lambda(-t))}{1 - e_\lambda(-t)} e_\lambda^x(-t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}.$$

When $x = 0$, $\beta_{n,\lambda}^{(k)} = \beta_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Bernoulli numbers.
By (47), we see that

$$(48) \qquad \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^n}{n!} = \frac{1}{1 - e_\lambda(-t)} \sum_{j=1}^{\infty} \frac{(1 - e_\lambda(-t))^j}{j^k} = \sum_{j=1}^{\infty} \frac{(1 - e_\lambda(-t))^{j-1}}{j^k}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j j!}{(j+1)^k} \frac{1}{j!} \left(e_\lambda(-t) - 1 \right)^j = \sum_{n=0}^{\infty} \left((-1)^n \sum_{j=0}^{n} \frac{(-1)^j j!}{(j+1)^k} S_{2,\lambda}(n,j) \right) \frac{t^n}{n!}.$$

Thus, by (47) and (48), we have

$$(49) \qquad\qquad \beta_{n,\lambda}^{(k)} = \sum_{j=0}^{n} \frac{(-1)^{n-j} j!}{(j+1)^k} S_{2,\lambda}(n,j),$$

$$(50) \qquad\qquad \beta_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} \beta_{l,\lambda}^{(k)}(x)_{n-l,\lambda}.$$

From (42), we note that

$$(51) \qquad \sum_{n=0}^{\infty} \frac{1}{(n+1)^r} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_\lambda(y + zk) \right\}$$

$$= \sum_{m=0}^{\infty} a_m \sum_{p=0}^{m} \binom{m}{p} z^p (y)_{m-p,\lambda} \sum_{j=0}^{p} (-1)^j j! S_{2,\frac{\lambda}{z}}(p,j) \frac{1}{(j+1)^r}$$

$$= \sum_{m=0}^{\infty} a_m \sum_{p=0}^{m} \binom{m}{p} z^p (y)_{m-p,\lambda} (-1)^p \beta_{p,\frac{\lambda}{z}}^{(r)}$$

$$= \sum_{m=0}^{\infty} a_m \sum_{p=0}^{m} \binom{m}{p} z^m z^{p-m} \left(\frac{y}{z} \right)_{m-p,\frac{\lambda}{z}} (-1)^p \beta_{p,\frac{\lambda}{z}}^{(r)}$$

$$= \sum_{m=0}^{\infty} a_m (-1)^m z^m \sum_{p=0}^{m} \binom{m}{p} \left(\frac{y}{z} \right)_{m-p,\frac{\lambda}{z}} (-1)^{m-p} \beta_{p,\frac{\lambda}{z}}^{(r)}$$

$$= \sum_{m=0}^{\infty} a_m (-1)^m z^m \beta_{m,\frac{\lambda}{z}}^{(r)} \left(\frac{y}{z} \right).$$

Therefore, by (51), we obtain the following theorem.

Theorem 13. *Let* $f_\lambda(t) = \sum\limits_{k=0}^{\infty} a_k(t)_{k,\lambda} \in \mathbb{C}[[t]]$. *Then the following identity is valid.*

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^r} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_\lambda(y + zk) \right\} = \sum_{m=0}^{\infty} a_m (-1)^m z^m \beta_{m,\frac{\lambda}{z}}^{(r)} \left(\frac{y}{z} \right), \quad (r \in \mathbb{Z}).$$

In particular, for $y = 0$, we have

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^r} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_\lambda(zk) \right\} = \sum_{m=0}^{\infty} a_m (-1)^m z^m \beta_{m,\frac{\lambda}{z}}^{(r)}, \quad (r \in \mathbb{Z}).$$

Let $f_\lambda(t) = (t)_{m,\lambda}$, and let $z = 1$ in Theorem 13. Then we have

(52) $$\sum_{n=0}^{m} \frac{1}{(n+1)^r} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k (y+k)_{m,\lambda} \right\} = (-1)^m \beta_{m,\lambda}^{(r)}(y),$$

in view of Remark 2.

Now, we observe that

(53) $$\sum_{n=0}^{\infty} \frac{(-\lambda)^n (1)_{n+1,\frac{1}{\lambda}}}{(n+1)!} \sum_{k=0}^{n} \binom{n}{k} (-1)^k e_\lambda^k(z) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n (1)_{n+1,\frac{1}{\lambda}}}{(n+1)!} (1 - e_\lambda(z))^n$$

$$= \frac{1}{1 - e_\lambda(z)} \sum_{n=1}^{\infty} (1 - e_\lambda(z))^n \frac{(-\lambda)^{n-1}}{n!} (1)_{n,\frac{1}{\lambda}}$$

$$= \frac{1}{1 - e_\lambda(z)} \left(-\log_\lambda(1 - (1 - e_\lambda(z))) \right)$$

$$= \frac{z}{e_\lambda(z) - 1} = \sum_{n=0}^{\infty} \beta_{n,\lambda} \frac{z^n}{n!}.$$

Thus, we have

(54) $$\sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \frac{(-\lambda)^m (1)_{m+1,\frac{1}{\lambda}}}{(m+1)!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k (k)_{n,\lambda} \right) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \beta_{n,\lambda} \frac{z^n}{n!}.$$

Therefore, by comparing the coefficients on both sides of (54), we obtain the following theorem.

Theorem 14. *For $n \geq 0$, we have the identity.*

$$\beta_{n,\lambda} = \sum_{k=0}^{\infty} \left(\sum_{m=k}^{\infty} \binom{m}{k} \frac{(-\lambda)^m (1)_{m+1,\frac{1}{\lambda}}}{(m+1)!} (-1)^k \right) (k)_{n,\lambda}$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \frac{(-\lambda)^m (1)_{m+1,\frac{1}{\lambda}}}{(m+1)!} (k)_{n,\lambda}.$$

From (8), we note that

(55) $$\sum_{n=0}^{\infty} F_{n,\lambda}\left(-\frac{1}{2} \right) \frac{t^n}{n!} = \frac{1}{1 + \frac{1}{2}(e_\lambda(t) - 1)} = \frac{2}{e_\lambda(t) + 1}$$

$$= 2 \left(\frac{1}{e_\lambda(t) - 1} - \frac{2}{e_\lambda^2(t) - 1} \right)$$

$$= \frac{2}{t} \left(\frac{t}{e_\lambda(t) - 1} - \frac{2t}{e_{\frac{\lambda}{2}}(2t) - 1} \right)$$

$$= \frac{2}{t} \sum_{n=1}^{\infty} \left(\beta_{n,\lambda} - 2^n \beta_{n,\frac{\lambda}{2}} \right) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{2}{n+1} \left(\beta_{n+1,\lambda} - 2^{n+1} \beta_{n+1,\frac{\lambda}{2}} \right) \frac{t^n}{n!}.$$

In addition, we also have

(56)
$$\frac{1}{1+\frac{1}{2}(e_\lambda(t)-1)} = \sum_{p=0}^{\infty} \frac{(-1)^p}{2^p}(e_\lambda(t)-1)^p$$

$$= \sum_{p=0}^{\infty} \frac{(-1)^p p!}{2^p}\frac{1}{p!}(e_\lambda(t)-1)^p$$

$$= \sum_{n=0}^{\infty} \left(\sum_{p=0}^{n} \frac{(-1)^p p!}{2^p} S_{2,\lambda}(n,p) \right) \frac{t^n}{n!}.$$

Therefore, by (55) and (56), we obtain the following theorem.

Theorem 15. *For $n \geq 0$, the following identity holds.*

$$F_{n,\lambda}\left(-\frac{1}{2}\right) = \frac{2}{n+1}\left(\beta_{n+1,\lambda} - 2^{n+1}\beta_{n+1,\frac{\lambda}{2}}\right)$$

$$= \sum_{p=0}^{n} \frac{(-1)^p p!}{2^p} S_{2,\lambda}(n,p).$$

Let us take $x = \frac{1}{2}$ in Theorem 11. Then we have

(57)
$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\left\{\sum_{k=0}^{n}\binom{n}{k}(-1)^k f_\lambda(zk)\right\} = \sum_{m=0}^{\infty} a_m z^m F_{m,\frac{\lambda}{z}}\left(-\frac{1}{2}\right)$$

$$= \sum_{m=0}^{\infty} a_m z^m \frac{2}{m+1}\left(\beta_{m+1,\frac{\lambda}{z}} - 2^{m+1}\beta_{m+1,\frac{\lambda}{2z}}\right).$$

By (8), we get

(58)
$$\sum_{m=0}^{\infty} F_{m,\lambda}\left(\frac{x}{1-x}\right)\frac{t^m}{m!} = \frac{1}{1-\frac{x}{1-x}(e_\lambda(t)-1)} = \frac{1-x}{1-x-x(e_\lambda(t)-1)}$$

$$= \frac{1-x}{1-xe_\lambda(t)}.$$

On the other hand,

(59)
$$\frac{1-x}{1-xe_\lambda(t)} = (1-x)\sum_{n=0}^{\infty} x^n e_\lambda^n(t) = (1-x)\sum_{n=0}^{\infty} x^n \sum_{m=0}^{\infty}(n)_{m,\lambda}\frac{t^m}{m!}$$

$$= \sum_{m=0}^{\infty}\left((1-x)\sum_{n=0}^{\infty} x^n(n)_{m,\lambda}\right)\frac{t^m}{m!}.$$

Therefore, by (62) and (63), we obtain the following theorem.

Theorem 16. *For $m \geq 0$, the following holds true.*

$$\frac{1}{1-x}F_{m,\lambda}\left(\frac{x}{1-x}\right) = \sum_{n=0}^{\infty} x^n(n)_{m,\lambda}.$$

Let

(60)
$$h_\lambda(t) = \frac{1}{\mu e_\lambda(\gamma t)+1} = \frac{1}{1-(-\mu e_\lambda(\gamma t))} = \sum_{n=0}^{\infty}(-\mu)^n e_\lambda^n(\gamma t).$$

Invoking Theorem 16, we obtain

$$(61) \qquad \left(\frac{d}{dt}\right)^m \left(\frac{1}{\mu e_\lambda(\gamma t)+1}\right) = \left(\frac{d}{dt}\right)^m \sum_{n=0}^{\infty} (-\mu)^n e_\lambda^n(\gamma t)$$

$$= \sum_{n=0}^{\infty} (-\mu)^n (n)_{m,\lambda} \gamma^m e_\lambda^{n-m\lambda}(\gamma t)$$

$$= \gamma^m \sum_{n=0}^{\infty} \left(-\mu e_\lambda(\gamma t)\right)^n (n)_{m,\lambda} e_\lambda^{-m\lambda}(\gamma t)$$

$$= \frac{\gamma^m}{(1+\lambda \gamma t)^m} \sum_{n=0}^{\infty} \left(-\mu e_\lambda(\gamma t)\right)^n (n)_{m,\lambda}$$

$$= \frac{\gamma^m}{(1+\lambda \gamma t)^m} \frac{1}{1+\mu e_\lambda(\gamma t)} F_{m,\lambda} \left(\frac{-\mu e_\lambda(\gamma t)}{1+\mu e_\lambda(\gamma t)}\right).$$

From (60) and (61), we get the following equation:

$$(62) \qquad \left(\frac{d}{dt}\right)^m h_\lambda(t)\bigg|_{t=0} = h_\lambda^{(m)}(0) = \gamma^m \frac{1}{1+\mu} F_{m,\lambda}\left(\frac{-\mu}{1+\mu}\right), \quad (m \geq 0).$$

Thus we have

$$(63) \qquad h_\lambda(t) = \frac{1}{\mu e_\lambda(\gamma t)+1} = \sum_{m=0}^{\infty} \frac{h_\lambda^{(m)}(0)}{m!} t^m = \frac{1}{1+\mu} \sum_{m=0}^{\infty} \gamma^m F_{m,\lambda}\left(\frac{-\mu}{1+\mu}\right)\frac{t^m}{m!}.$$

Let us take $\mu = 1$ and $\gamma = 1$ in (63). Then we have

$$(64) \qquad \frac{1}{e_\lambda(t)+1} = \frac{1}{2} \sum_{m=0}^{\infty} F_{m,\lambda}\left(-\frac{1}{2}\right)\frac{t^m}{m!}.$$

As is well known, Carlitz's degenerate Euler polynomials are defined by

$$(65) \qquad \frac{2}{e_\lambda(t)+1} e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathscr{E}_{n,\lambda}(x)\frac{t^n}{n!}.$$

When $x = 0$, $\mathscr{E}_{n,\lambda} = \mathscr{E}_{n,\lambda}(0)$ are called the degenerate Euler numbers.

From (64), we note that

$$(66) \qquad \frac{2}{e_\lambda(t)+1} e_\lambda^x(t) = \left(\sum_{m=0}^{\infty} F_{m,\lambda}\left(-\frac{1}{2}\right)\frac{t^m}{m!}\right) e_\lambda^x(t)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} F_{m,\lambda}\left(-\frac{1}{2}\right)(x)_{n-m,\lambda}\right)\frac{t^n}{n!}.$$

Therefore, by comparing the coefficients on both sides of (65) and (66), we obtain the folloiwnwg theorem.

Theorem 17. *For $n \geq 0$, the following identity is valid.*

$$\mathscr{E}_{n,\lambda}(x) = \sum_{m=0}^{n} \binom{n}{m} F_{m,\lambda}\left(-\frac{1}{2}\right)(x)_{n-m,\lambda}.$$

In particular, for $x = 0$, we have

$$\mathscr{E}_{n,\lambda} = F_{n,\lambda}\left(-\frac{1}{2}\right) = \frac{2}{n+1}\left(\beta_{n+1,\lambda} - 2^{n+1}\beta_{n+1,\frac{\lambda}{2}}\right).$$

3. Further Remarks

Here we obtain an expression for $\mathscr{E}_{m,\lambda}\left(\frac{1}{2}\right)$ and a general operational formula (76).

Taking $x = \frac{1}{2}$ and $z = 1$ in Theorem 11, we have

$$(67) \qquad \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left\{ \sum_{k=0}^{n} \binom{n}{k}(-1)^k f_\lambda(y+k) \right\} = \sum_{m=0}^{\infty} a_m \sum_{p=0}^{m} \binom{m}{p}(y)_{m-p,\lambda} F_{p,\lambda}\left(-\frac{1}{2}\right).$$

By Theorem 17, we get

$$(68) \qquad \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left\{ \sum_{k=0}^{n} \binom{n}{k}(-1)^k f_\lambda(y+k) \right\} = \sum_{m=0}^{\infty} a_m \mathscr{E}_{m,\lambda}(y).$$

In particular, for $y = \frac{1}{2}$,

$$(69) \qquad \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left\{ \sum_{k=0}^{n} \binom{n}{k}(-1)^k f_\lambda\left(\frac{1}{2}+k\right) \right\} = \sum_{m=0}^{\infty} a_m \mathscr{E}_{m,\lambda}\left(\frac{1}{2}\right).$$

Let us take $f_\lambda(t) = (t)_{m,\lambda}$. Then, $a_k = \delta_{k,m}, \ (k \geq 0)$. From (69), we have

$$
\begin{aligned}
(70) \qquad \mathscr{E}_{m,\lambda}\left(\frac{1}{2}\right) &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left\{ \sum_{k=0}^{n} \binom{n}{k}(-1)^k \left(\frac{1}{2}+k\right)_{m,\lambda} \right\} \\
&= \sum_{n=0}^{m} \left(\frac{1}{2}\right)^m \sum_{k=0}^{n} \binom{n}{k}(-1)^k \left(\frac{1}{2}+k\right)_{m,\lambda},
\end{aligned}
$$

by invoking Remark 2.

Let $D = \frac{d}{dx}$, and let $f(x) = \lim_{\lambda \to 0} f_\lambda(x) = \sum_{n=0}^{\infty} a_n x^n$. Then we note that

$$
\begin{aligned}
(71) \qquad f(x^{1-\lambda}D)e^x &= \sum_{n=0}^{\infty} a_n (x^{1-\lambda}D)^n \sum_{l=0}^{\infty} x^l \frac{1}{l!} \\
&= \sum_{n=0}^{\infty} a_n \sum_{l=0}^{\infty} \frac{(l)_{n,\lambda}}{l!} x^{l-n\lambda} \\
&= \sum_{n=0}^{\infty} a_n \left(\sum_{l=0}^{\infty} \frac{(l)_{n,\lambda}}{l!} x^l e^{-x} \right) e^x x^{-n\lambda} \\
&= \left(\sum_{n=0}^{\infty} a_n \phi_{n,\lambda}(x) x^{-n\lambda} \right) e^x.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(72) \qquad f(x^{1-\lambda}D)e^x &= \sum_{n=0}^{\infty} a_n (x^{1-\lambda}D)^n \sum_{k=0}^{\infty} \frac{x^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{n=0}^{\infty} a_n (k)_{n,\lambda} x^{-n\lambda} \right) x^k.
\end{aligned}
$$

Thus, we note that

$$\sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{n=0}^{\infty} a_n (k)_{n,\lambda} x^{-n\lambda} \right) x^k = e^x \sum_{n=0}^{\infty} a_n \phi_{n,\lambda}(x) x^{-n\lambda}.$$

Let $f(t) = \sum_{n=0}^{\infty} a_n t^n$, $g(x) = \sum_{k=0}^{\infty} c_k x^k$. Then we have

$$(73) \qquad f(x^{1-\lambda}D)g(x) = \sum_{n=0}^{\infty} a_n (x^{1-\lambda}D)^n \sum_{k=0}^{\infty} c_k x^k$$

$$= \sum_{k=0}^{\infty} c_k \left(\sum_{n=0}^{\infty} a_k(k)_{n,\lambda} x^{-n\lambda} \right) x^k.$$

By Taylor expansion, we get

$$(74) \qquad \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k = g(x) = \sum_{k=0}^{\infty} c_k x^k.$$

Thus, we have $c_k = \frac{g^{(k)}(0)}{k!}$, $(k \geq 0)$.

From (73) and (74), we note that

$$(75) \qquad f(x^{1-\lambda}D)g(x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} \left(\sum_{n=0}^{\infty} a_n(k)_{n,\lambda} x^{-n\lambda} \right) x^k$$

$$= \sum_{n=0}^{\infty} \left\{ a_n \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} (k)_{n,\lambda} x^k \right\} x^{-n\lambda}.$$

On the other hand,

$$(76) \qquad f(x^{1-\lambda}D)g(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (x^{1-\lambda}D)^n g(x)$$

$$= \sum_{n=0}^{\infty} \left(\frac{f^{(n)}(0)}{n!} \sum_{k=0}^{n} S_{2,\lambda}(n,k) x^k D^k g(x) \right) x^{-n\lambda},$$

where we used

$$(77) \qquad (x^{1-\lambda}D)^n f(x) = x^{-n\lambda} \sum_{k=0}^{n} S_{2,\lambda}(n,k) x^k D^k f(x).$$

The operational formula (77) follows by induction n from the following recurrence relation:

$$S_{2,\lambda}(n+1,k) = S_{2,\lambda}(n,k-1) + (k-n\lambda) S_{2,\lambda}(n,k).$$

4. CONCLUSION

The Rota's theory on umbral calculus is based on the linear functionals and the differential operators. The Sheffer sequences occupy the central position in the theory and are characterized by the generating functions involving the usual exponential function. The motivation for [8] started from the question that what if the usual exponential function is replaced by the degenerate exponential functions. It may be said that this question is very natural in view of the regained recent interests in degenerate special numbers and polynomials. As it turns out, it corresponds to replacing the linear functional by the λ-linear functionals and the differential operator by the λ-differential operators. In this way, we were led to introduce λ-umbral calculus and λ-Sheffer sequences.

As one of our future projects, we would like to continue to pursue our searches for λ-counterparts of some special polynomials, some special numbers, some transcendental functions and so on.

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