Proceedings of the Jangjeon Mathematical Society
24 (2021), No. 4, pp. 583 - 593
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2021.24.4.583

VARIOUS NEW FIXED POINT RESULTS IN G_b METRIC SPACES

[1]VISHAL GUPTA, [2]RAJANI SAINI, AND [3]NAVEEN SHARMA

Abstract. In this communication we present some new fixed point results in G_b-metric spaces. The main objective of this article is to prove some fixed point results for mapping that satisfied sufficient conditions on complete G_b-metric spaces.

2000 Mathematics Subject Classification: 47H10, 54H25.

Keywords and phrases: Fixed point, b-metric space, G-metric space, G_b-metric space.
Date of submission: 12 November 2020.

1. Introduction and preliminaries

Metric spaces are playing an increasing role in mathematics and in the applied sciences. Over the past two decades the development of fixed point theory in metric spaces has attracted considerable attention due to numerous applications in areas such as variational and linear inequalities, optimization and approximation theory. Mustafa and Sims [13] generalized the concept of a metric space and based on the notion of generalized metric space, Mustafa et al. [11] obtained some fixed point results for mapping satisfying different contractive conditions. Abbas et al. [3] initiated the study of common fixed point theory in generalized metric spaces. Since then, many authors obtained fixed and common fixed point results in the setup of G-metric spaces[8, 9, 10]. On the other hand, the concept of b-metric space was introduced by Czerwik[7]. After that, several interesting spaces were obtained. Aghajani et al. [1] introduced the concept of generalized b-metric space named as G_b-metric space and then they presented some basic properties of G_b-metric space. Many authors acquired fixed point and common fixed point results in the structure of G_b-metric space [2, 4, 5, 6, 7, 12]. In this paper, author's aim is to prove some fixed point theorems for G_b-metric spaces. The paper begin with known definitions and properties of G-metric spaces.

Definition 1.1. [13] *In a non-empty set P, a mapping $G : P\times P\times P \to R^+$ satisfying the following properties:*

(1) $G(p, q, r) = 0$ if and only if $p = q = r$;

(2) $0 < G(p, p, q)$ for all $p, q \in P$ with $p = q$;

Corresponding author: Vishal Gupta (vishal.gmn@gmail.com).

(3) $G(p,p,q) \leq G(p,q,r)$ *for all* $p,q,r \in P$ *with* $q \neq r$*;*

(4) $G(p,q,r) = G(p,r,q) = G(q,r,p) = G(q,r,p) = G(r,p,q) = G(r,q,p)$*;*

(5) $G(p,q,r) \leq G(p,a,a) + G(a,q,r)$ *for all* $p,q,r,a \in P$.

Then G *is called a* G*-metric on* P *and the pair* (P,G) *is called a* G*-metric space.*

Definition 1.2. [13] *Metric space* G *is called symmetric if* $G(p,q,q) = G(q,p,p)$ *for all* $p,q \in P$.

Definition 1.3. [13] *A sequence* $\{p_n\}$ *in* P *is called* G*-convergent to a point* $p \in P$ *if for every* $\epsilon > 0$*, there always occur a positive integer* n_1 *such that for all* $m,n \geq n_1$*,* $G(p_n,p_m,p) < \epsilon$.

Definition 1.4. [13] *A sequence* $\{p_n\}$ *in* P *is said to be* G*-Cauchy if for every* $\epsilon > 0$*, there always occur a positive integer* n_0 *such that for all* $m,n,l \geq n_0$*,* $G(p_n,p_m,p_l) < \epsilon$.

Definition 1.5. [13] *A* G*-metric space is called complete if every* G*-Cauchy sequence is* G*-convergent in* P.

Definition 1.6. [13] *In two* G*-metric spaces* (P,G) *and* (P',G') *a mapping* $f : P \to P'$ *is called* G*-continuous at a point* $p \in P$ *if and only if it is* G*-sequentially continuous at* p*, it means that whenever* $\{p_n\}$ *is* G*-convergent to* p*,* $\{fp_n\}$ *is* G'*-convergent to* $f(p)$.

Now we recollect some elementary definitions of b-metric spaces.

Definition 1.7. [7] *A function* $d : P \times P \to [0,+\infty)$ *is called a* b*-metric on* P *with constants* $s \geq 1$ *if it satisfies:*

(1) $d(p,q) = 0$ *if* $p = q$*;*

(2) $d(p,q) = d(q,p)$ *for all* $p,q \in P$*;*

(3) $d(p,q) \leq s[d(p,a) + d(a,q)]$ *for each* $p,q,a \in P$.

The pair (P,d) *is named as* b*-metric space with coefficient* s.

Obviously, every metric space is a b-metric space with $s = 1$. But there are certain b-metric spaces which are not metric spaces.

Definition 1.8. [7] *A sequence $\{p_n\}$ in b-metric space is called convergent if and only if there exists $p \in P$ such that $d(p_n, p) \to 0$ as $n \to +\infty$.*

Definition 1.9. [7] *A sequence $\{p_n\}$ in b-metric space is called Cauchy if and only if $d(p_n, p_m) \to 0$ as $m, n \to +\infty$.*

Definition 1.10. [7] *A b-metric space is said to be continuous if every Cauchy sequence in P converges.*

Definition 1.11. [7] *Let (P, d) and (P_1, d_1) be b-metric spaces with constant s and $s_1 \geq 1$ respectively. Then the mapping $h : P \to P_1$ is said to be continuous if sequence $\{p_n\}$ converges to some $p \in P$ with respect to d, then the sequence $\{hp_n\}$ converges to $h(p)$ in P_1 with respect to d_1.*

Next we recall some basic notions of G_b-metric spaces.

Definition 1.12. [1] *A mapping $G_b : X \times X \times X \to R^+$ is defined as a generalized b-metric space if it satisfies following conditions:*

(Gb1) $G_b(p, q, r) = 0$ *if* $p = q = r$;

(Gb2) $0 < G_b(p, p, q)$ *for all* $p, q \in P$ *with* $p \neq q$;

(Gb3) $G_b(p, p, q) \leq G_b(p, q, r)$ *for all* $p, q, r \in P$ *with* $q \neq r$;

(Gb4) $G_b(p, q, r) = G_b(p, r, q) = G_b(q, r, p) = G_b(q, r, p) = G_b(r, p, q) = G_b(r, q, p)$;

(Gb5) $G_b(p, q, r) \leq s\{G_b(p, a, a) + G_b(a, , q, r)\}$ *for all* $p, q, r \in P$.

The pair (G_b, P) is called a generalized b-metric space or a G_b-metric space. Every G-metric space is a G_b-metric space with $s = 1$. But converse is not always true.

Example 1. [1] *Let (P, G) be a G-metric space and $G_1(p, q, r) = G(p, q, r)^t$, where $t > 1$ is a real number. It is a G_b-metric with $s = 2^{t-1}$. Clearly G_1 satisfies condition (Gb1)-(Gb4) of the definition of G_b-metric space. We will check for the (Gb5).*
If $1 < t < \infty$, as the function $f(x) = x^t$ $(x > 0)$ is convex. Therefore we have $(a + b)^t \leq 2^{t-1}(a^t + b^t)$.
For each $p, q, r \in P$, we have
$G_1(p, q, r) = G(p, q, r)^t \leq [G(p, a, a) + G(a, q, r)]^t \leq 2^{t-1}[G(p, a, a) + G(a, q, r)]^t$
$= 2^{t-1}[G_1(p, a, a) + G_1(a, q, r)]$
Here $s = 2^{t-1}$. So G_1 is a G_b-metric space.

Next example shows that a G_b-metric on P need not to be a G-metric on P.

Example 2. [1] *$P = R$, $G_1(p,q,r) = G(p,q,r)^2$ and G-metric be defined by $G(p,q,r) = \frac{1}{3}(|p-q| + |q-r| + |r-p|)$ for all $p,q,r \in P$*

$G_1(p,q,r) = G(p,q,r)^2 = \frac{1}{9}(|p-q| + |q-r| + |r-p|)^2$ is a G_b-metric space with $s = 2^{2-1} = 2$, but is not a G-metric on R. To prove this choose $p = 3, q = 5, r = 7$ and $a = \frac{7}{2}$, we get $G_1(3,5,7) = \frac{64}{9}, G_1(3,\frac{7}{2},\frac{7}{2}) = \frac{1}{9}, G_1(\frac{7}{2},5,7) = \frac{49}{9}$,
so $G_1(3,5,7) = \frac{64}{9} \geq \frac{50}{9} = G_1(3,\frac{7}{2},\frac{7}{2}) + G_1(\frac{7}{2},5,7)$.

Definition 1.13. [1] *A G_b-metric is known as symmetric if $G_b(p,q,q) = G_b(q,p,p)$ for all $p,q \in P$.*

Proposition 1.1. [1] *In a G_b-metric space following properties holds for all $p,q,r,a \in P$:*

(1) If $G_b(p,q,r) = 0$ then $p = q = r$;

(2) $G_b(p,q,r) \leq s[G_b(p,p,q) + G_b(p,p,r)]$;

(3) $G_b(p,q,q) \leq 2sG_b(q,p,p)$;

(4) $G_b(p,q,r) \leq s\{G_b(p,a,r) + G_b(a,p,r)\}$.

Definition 1.14. [1] *A sequence $\{p_n\}$ in P is called G_b-Cauchy if for each $\epsilon > 0$, there exists a positive integer n_1 such that for all $m,n,l \geq n_1, G_b(p_n,p_m,p_l) < \epsilon$.*

Definition 1.15. [1] *A sequence $\{p_n\}$ in P is G_b-convergent to a point $p \in P$ if for each $\epsilon > 0$, there exist a positive integer n_1 such that for all $m,n \geq n_1, G_b(p_n,p_m,p) < \epsilon$.*

Definition 1.16. [1] *A G_b-metric space P is said to be complete, if every G_b-Cauchy sequence is G_b-convergent in P.*

Definition 1.17. [1] *In two G_b-metric spaces (P,G) and $(P^{'},G^{'})$ a mapping $f : P \to P^{'}$ is called G_b-continuous at a point $p \in P$ if and only if it is G_b-sequentially continuous at p, it means that if sequence $\{p_n\}$ is G_b-convergent to p in P then $\{f(p_n)\}$ is G_b-convergent to $f(p)$ in $G^{'}$.*

2. Main Results

Our aim in this paper is to prove fixed point theorems in G_b-metric space.

Theorem 2.1. *In a complete G_b-metric space with $s \geq 1$, a mapping $f : P \to P$ satisfies:*

(1)

$$G_b(f(p), f(q), f(r)) \leq \alpha G_b(p, q, r) + \beta G_b(p, f(p), f(p)) + \gamma G_b(q, f(q), f(q)) + \delta G_b(r, f(r), f(r)) + \eta max[G_b(p, f(q), f(q)), G_b(q, f(p), f(p)), G_b(q, f(r), f(r)), G_b(r, f(q), f(q)), G_b(r, f(p), f(p)), G_b(p, f(r), f(r))]$$

for all $p, q, r \in P$, where $\alpha, \beta, \gamma, \delta, \eta \geq 0$ with $0 \leq (\alpha + \beta + \gamma + \delta) + 2\eta s < 1$, then there exist a unique fixed point v of f i.e. $f(v) = v$. Moreover f is G_b-continuous at v.

Proof. Consider an arbitrary point $p_0 \in P$ and a sequence $\{p_n\}$ be in P such that $P_n = f^n(p_0)$ and $p_n \neq p_{n+1}$, using (1)

$G_b(p_n, p_{n+1}, p_{n+1}) \leq \alpha G_b(p_{n-1}, p_n, p_n) + \beta G_b(p_{n-1}, p_n, p_n) + \gamma G_b(p_n, p_{n+1}, p_{n+1}) + \delta G_b(p_n, p_{n+1}, p_{n+1}) + \eta max[G_b(p_{n-1}, p_{n+1}, p_{n+1}), G_b(p_n, p_n, p_n), G_b(p_n, p_{n+1}, p_{n+1}), G_b(p_n, p_{n+1}, p_{n+1}), G_b(p_n, p_n, p_n), G_b(p_{n-1}, p_{n+1}, p_{n+1})]$,
this gives,

(2) $$G_b(p_n, p_{n+1}, p_{n+1}) \leq \alpha G_b(p_{n-1}, p_n, p_n) + \beta G_b(p_{n-1}, p_n, p_n) + \gamma G_b(p_n, p_{n+1}, p_{n+1}) + \delta G_b(p_n, p_{n+1}, p_{n+1}) + \eta max[G_b(p_{n-1}, p_{n+1}, p_{n+1}), G_b(p_n, p_{n+1}, p_{n+1})].$$

By rectangle inequality,

$$G_b(p_{n-1}, p_{n+1}, p_{n+1}) \leq s\left\{G_b(p_{n-1}, p_n, p_n) + G_b(p_n, p_{n+1}, p_{n+1})\right\}.$$

Hence (2) turn into

$G_b(p_n, p_{n+1}, p_{n+1}) \leq \alpha G_b(p_{n-1}, p_n, p_n) + \beta G_b(p_{n-1}, p_n, p_n) + \gamma G_b(p_n, p_{n+1}, p_{n+1}) + \delta G_b(p_n, p_{n+1}, p_{n+1}) + \eta s\left\{G_b(p_{n-1}, p_n, p_n) + G_b(p_n, p_{n+1}, p_{n+1})\right\}$,
implies
$(1-\gamma-\delta-\eta s)G_b(p_n, p_{n+1}, p_{n+1}) \leq \alpha G_b(p_{n-1}, p_n, P_n) + \beta G_b(p_{n-1}, p_n, p_n) + \eta s G_b(p_{n-1}, p_n, p_n)$,
we get
$G_b(p_n, p_{n+1}, p_{n+1}) \leq \frac{(\alpha+\beta+\eta s) G_b(p_{n-1}, p_n, p_n)}{1-\gamma-\delta-\eta s}$.

Let $k = \frac{(\alpha+\beta+\eta s)}{1-\gamma-\delta-\eta s} < 1$, as $0 \leq (\alpha + \beta + \gamma + \delta) + 2\eta s < 1$.

$G_b(p_n, p_{n+1}, p_{n+1}) \leq k G_b(p_{n-1}, p_n, p_n)$, by repeating the same argument, we have

$$G_b(p_n, p_{n+1}, p_{n+1}) \leq k^n G_b(p_0, p_1, p_1). \tag{3}$$

By rectangle inequality, we have

$G_b(p_n, p_m, p_m) \leq s\,\{G_b(p_n, p_{n+1}, p_{n+1}) + G_b(p_{n+1}, p_m, p_m)\}$

$\leq\ sG_b(p_n, p_{n+1}, p_{n+1}) + s^2G_b(p_{n+1}, p_{n+2}, p_{n+2}) + s^3G_b(p_{n+2}, p_{n+3}, p_{n+3})$
$+ ...$

$\leq sk^nG_b(p_0, p_1, p_1) + s^2k^{n+1}G_b(p_0, p_1, p_1) + s^3k^{n+2}G_b(p_0, p_1, p_1) + s^4k^{n+3}G_b(p_0, p_1, p_1)$
$+ ...$

$$G_b(p_n, p_m, p_m) \leq \frac{sk^n}{1 - sk} G_b(p_0, p_1, p_1), \tag{4}$$

and $G_b(p_n, p_m, p_m) = 0$ as $m, n \to \infty$. Hence $\{p_n\}$ is a G_b-Cauchy sequence.
Completeness property of G_b-metric space enable us to find a sequence $\{p_n\}$ which is G_b convergent to point $v \in P$.

If $f(v) \neq v$, then

$$G_b(p_n, f(v), f(v)) \tag{5}$$

$$\leq \alpha G_b(p_{n-1}, v, v) + \beta G_b(p_{n-1}, p_n, p_n) + \gamma G_b(v, f(v), f(v)) + \delta G_b(v, f(v), f(v))$$

$$+\eta max[G_b(p_{n-1}, f(v), f(v)), G_b(v, p_n, p_n), G_b(v, f(v), f(v)),$$

$$G_b(v, f(v), f(v)), G_b(v, p_n, p_n), G_b(p_{n-1}, f(v), f(v))],$$

taking limit as $n \to \infty$ in (5), we get

$G_b(v, f(v), f(v)) \leq (\gamma + \beta + \eta)G_b(v, f(v), f(v))$, this is not possible as $0 \leq \gamma + \delta + \eta < 1$, hence $f(v) = v$.

Now, we shall show that v is unique fixed point. Suppose that there exist $w \in P$ such that $f(w) = w$ and $v \neq w$.

$$\begin{aligned} G_b(v, w, w) \quad = \quad G_b(f(v), f(w), f(w)) \quad \leq \quad \alpha G_b(v, w, w) \\ + \beta G_b(v, v, v) + \gamma G_b(w, w, w) + \delta G_b(w, w, w) + \eta max[G_b(v, w, w), \\ G_b(w, v, v), G_b(w, w, w), G_b(w, w, w), G_b(w, v, v), G_b(v, w, w)]. \end{aligned} \tag{6}$$

This gives,

$$\begin{aligned} G_b(v,w,w) &\leq \alpha G_b(v,w,w) + \eta max\left\{G_b(v,w,w), G_b(w,v,v)\right\} \\ &\leq \alpha G_b(v,w,w) + \eta max\left\{2sG_b(w,v,v), G_b(w,v,v)\right\} \\ &\leq \alpha G_b(v,w,w) + 2\eta s G_b(w,v,v) \\ G_b(v,w,w) &\leq \frac{2\eta s}{1-\alpha} G_b(w,v,v) \end{aligned}$$

Similarly, $G_b(w,v,v) \leq \frac{2\eta s}{1-\alpha} G_b(v,w,w)$.

Thus, we have $G_b(v,w,w) \leq (\frac{2\eta s}{1-\alpha})^2 G_b(v,w,w)$,

this implies that $v = w$, as $0 \leq \frac{2\eta s}{1-\alpha} < 1$.

Next is to show that f is G_b-continuous at v. Let $q_n \subseteq P$ be a sequence such that $lim_{n\to\infty} q_n = v$.

(7)

$$\begin{gathered} G_b(f(q_n), f(v), f(q_n)) \leq \alpha G_b(q_n, v, q_n) + \beta G_b(q_n, f(q_n), f(q_n)) + \gamma G_b(v,v,v) \\ +\delta G(q_n, f(q_n), f(q_n)) + \eta max[G_b(q_n,v,v), G_b(v, f(q_n), f(q_n)), G_b(v, f(q_n), f(q_n)), \\ G_b(q_n,v,v), G_b(q_n, f(q_n), f(q_n)), G_b(q_n, f(q_n), f(q_n))] \end{gathered}$$

Also by rectangle inequality,

(8) $$G_b(q_n, f(q_n), f(q_n)) \leq s[G_b(q_n,v,v) + G_b(v, f(q_n), f(q_n))].$$

Using (8) in (7)

$$\begin{gathered} G_b(f(q_n), f(v), f(q_n)) \leq \alpha G_b(q_n, q_n, v) + (\beta+\delta) s G_b(q_n, v, v) \\ +sG_b(v, f(q_n), f(q_n)) + \eta s[G_b(q_n,v,v) + G_b(v, f(q_n), f(q_n))], \end{gathered}$$

this gives,

$$\begin{gathered} [(1 - s(\beta+\delta+\eta))] G_b(v, f(q_n), f(q_n)) \leq \\ \alpha G_b(q_n, q_n, v) + s(\beta+\delta+\eta)(G_b(q_n, v, v)), \end{gathered}$$

we have

$$G_b(v, f(q_n), f(q_n)) \leq \frac{\alpha}{(1-s(\beta+\delta+\eta))} G_b(q_n, q_n, v) + \frac{s(\beta+\delta+\eta}{(1-s(\beta+\delta+\eta))} G_b(q_n, v, v),$$

proceeding the limit $n \to \infty$ on both sides, gives us $G(v, f(q_n), f(q_n)) \to 0$, implies $f(q_n) \to v = f(v)$. This shows that f is G_b-continuous at v and hence the result.

□

Corollary 2.2. *In a complete G_b-metric space with $s \geq 1$, a mapping $f : P \to P$ fulfills the following conditions:*

$$G_b(f^a(p), f^a(q), f^a(r)) \leq \alpha G_b(p,q,r) + \beta G_b(p, f^a(p), f^a(p))$$
$$+\gamma G_b(q, f^a(q), f^a(q)) + \delta G_b(r, f^a(r), f^a(r)) +$$
$$\eta max[G_b(p,(f^q),(f^q)), G_b(q, f^a(p), f^a(p)),$$
$$G_b(q, f^a(r), f^a(r)), G_b(r, f^a(q), f^a(q)), G_b(r, f^p, f^a(p)), G_b(p, f^a(r), f^r)]$$

for all $p,q,r \in P$ *and* $a \in N$*, where* $\alpha,\beta,\gamma,\delta,\eta \geq 0$ *with* $0 \leq (\alpha+\beta+\gamma+\delta)+2\eta s < 1$*, then there exist a unique fixed point* v *of* f *i.e.* $f(v) = v$ *and* f^a *is* G_b*-continuous at* v*.*

Proof. By Theorem 2.1, there exist a unique fixed point v of f^a and $f^a(v) = v$.
Now $f(v) = f(f^a(v) = f^{a+1}(v) = f^a(f(v)))$, so $f(v)$ is one more fixed point for f^a and due to uniqueness $f(v)$=v.

□

Theorem 2.3. *In a complete* G_b*-metric space with* $s \geq 1$*, a mapping* $f : P \to P$ *satisfies following condition:*

(9) $$G_b(f(p), f(q), f(r)) \leq \alpha[G_b(p, f(q), f(q)) + G_b(q, f(p), f(p))]$$
$$+\beta[G_b(q, f(r), f(r)) + G_b(r, f(q), f(q))]+\gamma[G_b(r, f(p), f(p)) + G_b(p, f(r), f(r))]$$
$$+\delta G_b(p,q,r) + \eta max[G_b(p, f(p), f(p)), G_b(q, f(q), f(q)), G_b(r, f(r), f(r))]$$

for all $p,q,r \in P$*, where* $\alpha,\beta,\gamma,\delta,\eta \geq 0$ *with* $0 \leq (2\alpha+2\gamma+2\eta)s+2\beta+\delta < 1$*, then there exist a unique fixed point* v *of* f*. i.e.* $f(v) = v$*, and* f *is* G_b*-continuous at* v*.*

Proof. Consider an arbitrary point $p_0 \in P$ and a sequence $\{p_n\}$ be in P such that $p_n = f^n(p_0)$ and p_n=p_{n+1}, using (9)

$$G_b(p_n, p_{n+1}, p_{n+1}) \leq \alpha[G_b(p_{n-1}, p_{n+1}, p_{n+1}) + G_b(p_n, p_n, p_n)]$$
$$+\beta[G_b(p_n, p_{n+1}, p_{n+1}) + G_b(p_n, p_{n+1}, p_{n+1})] +$$
$$\gamma[G_b(p_n, p_n, p_n) + G_b(p_{n-1}, p_{n+1}, p_{n+1})] + \delta G_b(p_{n-1}, p_n, p_n) +$$
$$\eta[max G_b(p_{n-1}, p_n, p_n), G_b(p_n, p_{n+1}, p_{n+1}), G_b(p_n, p_{n+1}, p_{n+1})]$$

$$\leq \alpha s[G_b(p_{n-1}, p_n, p_n) + G_b(p_n, p_{n+1}, p_{n+1})] + 2\beta G_b(p_n, p_{n+1}, p_{n+1})$$
$$+\gamma s[G_b(p_{n-1}, p_n, p_n) + G_b(p_n, p_{n+1}, p_{n+1})] + \delta G_b(p_{n-1}, p_n, p_n)$$
$$+\eta[sG_b(p_{n-1}, p_n, p_n) + G_b(p_n, p_{n+1}, p_{n+1})].$$

This gives,

$$(1-\alpha s-\gamma s-\eta s-2\beta)G_b(p_n, p_{n+1}, p_{n+1}) \leq (\alpha s+\gamma s+\eta s+\delta)G_b(p_{n-1}, p_n, p_n),$$

implies,

$$G_b(p_n, p_{n+1}, p_{n+1}) \leq \tfrac{\alpha s+\gamma s+\eta s+\delta}{1-\alpha s-\gamma s-\eta s-2\beta} G_b(p_{n-1}, p_n, p_n).$$

Let $k = \frac{\alpha s+\gamma s+\eta s+\delta}{1-\alpha s-\gamma s-\eta s-2\beta} < 1$, as $0 \leq (2\alpha + 2\gamma + 2\eta)s + 2\beta + \delta < 1$, therefore $G_b(p_n, p_{n+1}, p_{n+1}) \leq kG_b(p_{n-1}, p_n, p_n)$.

By repeating the same argument, we have

$$G_b(p_n, p_{n+1}, p_{n+1}) \leq k^n G_b(p_0, p_1, p_1) \tag{10}$$

by rectangle inequality, we have

$G_b(p_n, p_m, p_m) \leq$s$[G_b(p_n, p_{n+1}, p_{n+1}) + G_b(p_{n+1}, p_m, p_n)]$

$\leq$s$G_b(p_n, p_{n+1}, p_{n+1}) + s^2 G_b(p_{n+1}, p_{n+2}, p_{n+2}) + s^3 G_b(p_{n+2}, p_{n+3}, p_{n+3})$..........

$\leq$s$k^n G_b(p_0, p_1, p_1) + s^2 k^{n+1} G_b(p_0, p_1, p_1) + s^3 k^{n+2} G_b(p_0, p_1, p_1) + s^4 k^{n+3} G_b(p_0, p_1, p_1)$..........

$$G_b(p_n, p_m, p_m) \leq \frac{sk^n}{1 - sk} G_b(p_0, p_1, p_1) \tag{11}$$

and $G_b(p_n, p_m, p_m) = 0$ as $m, n \to \infty$. Hence $\{p_n\}$ is a G_b-Cauchy sequence.
Completeness property of G_b-metric space, enable us to find a sequence $\{p_n\}$ which is G_b convergent to point $v \in P$.

If $f(v) \neq v$, then

$G_b(p_n, f(v), f(v)) \leq \alpha[G_b(p_{n-1}, f(v), f(v)) + G_b(v, p_n, p_n)]$

$+\beta[G_b(v, f(v), f(v)) + G_b(v, f(v), f(v))]$

$+\gamma G_b(v, p_n, p_n) + G_b(p_{n-1}, f(v), f(v)) + \delta G_b(p_{n-1}, v, v)$

$+\eta max G_b(p_{n-1}, p_n, p_n), G_b(v, f(v), f(v)), G_b(v, f(v), f(v)),$

taking limit as $n \to \infty$, we get

$G_b(v, f(v), f(v)) \leq (\alpha + 2\beta + \gamma + \eta) G_b(v, f(v), f(v))$, this is not possible as $0 \leq \alpha + 2\beta + \gamma + \eta < 1$, hence $f(v) = v$.

Now we shall show that v is unique fixed point. Suppose that there exist $w \in P$ such that $f(w) = w$ and $v \neq w$.

$G_b(v, w, w) = G_b(f(v), f(w), f(w)) \leq \alpha G_b(v, w, w) + G_b(w, v, v) + 2\beta G_b(w, w, w)$
$+\gamma G_b(w, v, v) + G_b(v, w, w) + \delta G_b(v, w, w) + \eta max G_b(v, v, v), G_b(w, w, w), G_b(w, w, w)$

Therefore, $G_b(v, w, w) \leq (\alpha + \gamma)[G_b(v, w, w) + G_b(w, v, v)] + \delta G_b(v, w, w)$,

this gives $G_b(v, w, w) \leq \frac{\alpha+\gamma}{1-\alpha-\gamma-\delta} G_b(w, v, v)$.

Similarly, $G_b(w, v, v) \leq \frac{\alpha+\gamma}{1-\alpha-\gamma-\delta} G_b(v, w, w)$,

therefore, we have $G_b(v, w, w) \leq (\frac{\alpha+\gamma}{1-\alpha-\gamma-\delta})^2 G_b(v, w, w)$, this implies $v = w$, as $0 \leq \frac{\alpha+\gamma}{1-\alpha-\gamma-\delta} < 1$.

Next is to show that f is G_b-continuous at v. Let $\{q_n\} \subseteq P$ be a sequence such that $lim_{n\to\infty} q_n = v$.

$$G_b(f(q_n), f(v), f(v)) \leq \alpha[G_b(q_n, f(v), f(v)) + G_b(v, f(q_n), f(q_n))]$$

$$+\beta[G_b(v, f(v), f(v)) + G_b(v, f(v), f(v))]+\gamma[G_b(v, f(q_n), f(q_n)) + G_b(q_n, f(v), f(v))]$$

$$+\delta G_b(q_n, v, v)+\eta max[G_b(q_n, f(q_n), f(q_n)), G_b(v, f(v), f(v)), G_b(v, f(v), f(v))].$$

This gives, $G_b(f(q_n), v, v) \leq \alpha[G_b(q_n, v, v) + 2sG_b(f(q_n), v, v)]+2\beta G_b(v, v, v)$

$$+\gamma[2sG_b(f(q_n), v, v) + G_b(q_n, v, v)]+\delta G_b(q_n, v, v)+\eta s[G_b(q_n, v, v) + G_b(v, f(q_n), f(q_n))].$$

Therefore, $G_b(f(q_n), v, v) \leq \alpha[G_b(q_n, v, v) + 2sG_b(f(q_n), v, v)]+2\beta G_b(v, v, v)$

$$+\gamma[2sG_b(f(q_n), v, v) + G_b(q_n, v, v)] + \delta G_b(q_n, v, v)$$

$$+\eta s[G_b(q_n, v, v) + 2sG_b(f(q_n), v, v)],$$

implies $G_b(f(q_n), v, v) \leq (\alpha + \gamma + \eta s)/(1 - 2\alpha s - 2\gamma s - 2\eta s^2)G_b(q_n, v, v)$.

Proceeding the limit $n \to \infty$ both sides gives us $G_b(f(q_n), v, v) \to 0$, implies that $f(q_n) \to v = f(v)$. This shows that f is G_b-continuous at v and hence the result.

□

Corollary 2.4. *In a complete G_b-metric space, a mapping $f : P \to P$ satisfies following conditions for $a \in N$:*

$$G_b(f^a(p), f^a(q), f^a(r)) \leq \alpha[G_b(p, f^a(q), f^a(q)) + G_b(q, f^a(p), f^a(p))]$$

$$+\beta[G_b(q, f^a(r), f^a(r)) + G_b(r, f^a(q), f^a(q))]$$

$$+\gamma[G_b(r, f^a(p), f^a(p)) + G_b(p, f^a(r), f^a(r))] + \delta G_b(p, q, r)$$

$$+\eta max[G_b(p, f^a(p), f^a(p)), G_b(q, f^a(q), f^a(q)), G_b(r, f^a(r), f^a(r))]$$

for all $p, q, r \in P$, where $\alpha, \beta, \gamma, \delta, \eta \geq 0$ with $0 \leq (2\alpha+2\gamma+2\eta)s+2\beta+\delta < 1$, and $s \geq 1$ then there exist a unique fixed point v of f i.e. $f(v) = v$ and f^a is G_b-continuous at v.

Proof. By Theorem 2.3, there exists a unique fixed point v of f^a and $f^a(v) = v$.
Now $f(v) = f(f^a(v) = f^{a+1}(v) = f^a(f(v)))$, so $f(v)$ is one more fixed point for f^a and due to uniqueness, $f(v) = v$.

□

References

[1] A. Aghajani, M. Abbas, J. R.Roshan, *Common Fixed point of generalized weak contractive mappings in partially ordered G_b-metric spaces*, Filomat, 2014, 1087-1101

[2] A. H. Ansari, O. Ege, S. Radenovic, *Some fixed point results on complex valued G_b-metric spaces*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales, Seria A. Matematicas, 112(2)(2018), 463-472.

[3] M. Abbas, T. Nazir, P. Vetro, *Common Fixed Point results for three maps in G-metric spaces*, Filomat, 23(4)(2011), 1-17.

[4] O. Ege, *Complex valued G_b-metric spaces*, Journal of Computational Analysis and Applications, 21(2)(2016), 363-368.

[5] O. Ege, *Some fixed point theorems in complex valued G_b-metric spaces*, Journal of Nonlinear and Convex Analysis, 18(11)(2017), 1997-2005.

[6] O. Ege and I. Karaca, *Common fixed point results on complex valued G_b-metric spaces*, Thai Journal of Mathematics, 16(3)(2018), 775-787.

[7] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Sem Mat Fis Univ Modena, 46(2)(1998), 263-276.

[8] Vishal Gupta, R.K. Saini, Ramandeep, *Some fixed point results in G-metric space involving generalised altering distances*, International Journal of Applied Nonlinear Science, 3(1)(2018), 66- 76.

[9] Vishal Gupta, Raman Deep, *Some Fixed Point Theorems in G-Metric and Fuzzy Metric Spaces using E.A Property*, Journal of Information and Computing Science, 10(2)(2015), 83-89.

[10] Vishal Gupta, Raman Deep, *Common Fixed Point Theorem Involving Two Mappings and Weakly Contractive Condition in G-Metric Spaces*, Proceedings of the Institute of Applied Mathematics, 3(1)(2014), 80-88.

[11] Z. Mustafa, H. Obiedat, F. Awawdeh, *Some common fixed point theorems for mapping on complete G-metric spaces*, Fixed Point Theory Appl,2008(2008), 1-12. DOI:10.1155/2008/189870

[12] Z. Mustafa, J. R. Roshan, V. Parvaneh, *Coupled coincidence point results for (ψ,ϕ)-weakly contractive mappings in Partially ordered G_b-metric spaces.* Fixed Point Theory App, 2013(2013). DOI:10.1186/1687-1812-2013-206.

[13] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J Nonlinear Convex Anal, 7(2006), 289-297.

[1]Department of Mathematics, Maharishi Markandeshwar(Deemed to be University), Mullana-133207, Haryana, India
E-mail address: vishal.gmn@gmail.com, vgupta@mmumullana.org

[2]Research Scholar, Department of Mathematics, Maharishi Markandeshwar(Deemed to be University), Mullana-133207, Haryana, India.

[2]Department of Mathematics, Government P.G. College, Ambala Cantt-133001, Haryana, India.
E-mail address: rajanisainiraj@gmail.com

[3]Department of Mathematics, D.A.V. College, Muzaffernagar, U.P., India
E-mail address: ns2000dav@gmail.com