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NEW FORMS OF COMPACTNESS IN BITOPOLOGICAL SPACES

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Abstract. The aim of the present paper is to introduce some new types of pairwise compactness in bitopological spaces. In this paper we define pairwise $\rho\mathcal{I}$-compactness and pairwise $\sigma\mathcal{I}$-compactness. We study some of their properties and also prove that these two compactness are the strong forms of pairwise compactness modulo an ideal.

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1. Introduction and Preliminaries

In 1967, Newcomb [9] introduced a novel concept, compactness modulo an ideal. This concept provided a new field of research in general topology. After that many mathematicians gave their contribution to define the properties of general topology such as compactness, connectedness, C-compactness, C-α-compactness with respect to an ideal. Hamlett and Jancovic [4], Rancin [12], Gupta and Kaur [5], Lal and Gupta [8], Gupta and Noiri [3], Gaur and Gupta [2] are some renowned names among them. Bitopological

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space is mentioned as a great idea given by Kelly [6]. In 1998, Lal and Gupta [8] introduced the concept of pairwise compactness modulo an ideal in bitopological spaces which is a generalization of pairwise compactness given by Fletcher, Hoyle and Patty [1]. Recently, Pachon [10] introduced new types of strong compactness with respect to an ideal and named them as $\rho\mathcal{I}$-compactness and $\sigma\mathcal{I}$-compactness. Inspiring by these concepts, we introduce pairwise $\rho\mathcal{I}$-compactness and pairwise $\sigma\mathcal{I}$-compactness in bitopological spaces. The purpose of this paper is a simultaneous generalization of pairwise compactness with respect to an ideal by Lal and Gupta [8] and $\rho\mathcal{I}$-compactness and $\sigma\mathcal{I}$-compactness by Pachon [10]. We name them as pairwise $\rho\mathcal{I}$-compactness and pairwise $\sigma\mathcal{I}$-compactness. We study some of their properties and also compare them with pairwise compactness with respect to an ideal. The nature of pairwise $\rho\mathcal{I}$-compact space and pairwise $\sigma\mathcal{I}$-compact space under certain type of functions is also studied.

An ideal $\mathcal{I}$ on a non-empty set X is a collection of subsets of X which is closed for subsets and finite unions. A topological space (X, τ) with an ideal $\mathcal{I}$ on X is denoted by $(X, \tau, \mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I}, \tau)$ (called the adherence of A modulo an ideal $\mathcal{I}$) or $A^*(\mathcal{I})$ or just A^* is the set $\{x \in X : A \cap U \notin \mathcal{I}$ for every neighborhood U of $x\}$. $A^*(\mathcal{I}, \tau)$ has been called the local function of A with respect to $\mathcal{I}$. The operator cl*: $P(X) \rightarrow P(X)$ defined by cl$^*(A) = A \cup A^*$ is a Kuratowski [7] closure operator on X and hence generates a topology $\tau^*(\mathcal{I})$ or just τ^* on X finer than τ. Also the set $\mathcal{B} = \{U - I : U \in \tau$ and $I \in \mathcal{I}\}$ is a base for the topology τ^*.

A bitopological space equipped with an ideal $\mathcal{I}$ on X is denoted by $(X, \tau_1, \tau_2, \mathcal{I})$. By [8], a space $(X, \tau_1, \tau_2, \mathcal{I})$ is said to be pairwise compact modulo an ideal $\mathcal{I}$ on X if every pairwise open cover $\mathcal{U}$ of X has a finite subcollection $\{U_1, U_2, ..., U_n\}$ of $\mathcal{U}$ such that $X - \cup_{i=1}^{n} U_{\alpha_i} \in \mathcal{I}$. By [10], if $(X, \tau, \mathcal{I})$ be an ideal space and $A \subseteq X$, A is said to be $\rho\mathcal{I}$-compact if for every family $\{U_\alpha\}_{\alpha\in\Lambda}$ of open subsets of X, if $A - \cup_{\alpha\in\Lambda} U_\alpha \in \mathcal{I}$, then there exists a finite subcollection $\{U_{\alpha_i} : i = 1, 2, ..., n\}$ such that $A - \cup_{i=1}^{n} U_{\alpha_i} \in \mathcal{I}$. The ideal space $(X, \tau, \mathcal{I})$ is said to be $\rho\mathcal{I}$-compact if X is $\rho\mathcal{I}$-compact. Also by [10], if $(X, \tau, \mathcal{I})$ be an ideal space and $A \subseteq X$, A is said to be $\sigma\mathcal{I}$-compact if for every family $\{U_\alpha\}_{\alpha\in\Lambda}$ of open subsets of X, if $A - \cup_{\alpha\in\Lambda} U_\alpha \in \mathcal{I}$, then there exists a finite subcollection $\{U_{\alpha_i} : i = 1, 2, ..., n\}$ such that $A \subseteq \cup_{i=1}^{n} U_{\alpha_i}$.The ideal space $(X, \tau, \mathcal{I})$ is said to be $\sigma\mathcal{I}$-compact if X is $\sigma\mathcal{I}$-compact. By [8], a collection $\{F_\alpha : \alpha \in \Lambda\}$ of subsets of $(X, \tau_1, \tau_2\}$ is said to be pairwise closed if $\{F_\alpha : \alpha \in \Lambda\}$ is a subfamily of τ_1-closed sets and τ_2-closed sets such that it contains at least one proper τ_1-closed set and at least one proper τ_2-closed set. Also a family $\mathcal{F}$ of subsets of X is said to have finite intersection property modulo $\mathcal{I}$, denoted $(\mathcal{I})FIP$, if the intersection of no finite subfamily of $\mathcal{F}$ is a member of $\mathcal{I}$. A mapping $f : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ is said to be pairwise continuous [11](resp. pairwise closed, pairwise homeomorphism) if the induced mappings $f : (X, \tau_1) \to (Y, \delta_1)$ and $f : (X, \tau_2) \to (Y, \delta_2)$ are continuous (resp. closed, homeomorphisms). If $\mathcal{I}$ is an ideal on X and $f : X \to Y$ then $f(\mathcal{I}) = \{f(I) : I \in \mathcal{I}\}$ is an ideal on Y. A function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \delta_1, \delta_2)$ is said to be pairwise pointwise $\mathcal{I}$-continuous [8] if $f : (X, \tau_1^*, \tau_2^*) \to (Y, \delta_1, \delta_2)$ is pairwise continuous.

2. Pairwise $\rho\mathcal{I}$-compact spaces

Definition 2.1. *A collection $\mathcal{U} \subset \tau_1 \cup \tau_2$ of subsets of X is said to be pairwise open* [1] *if $\mathcal{U}$ contains a non-empty member of τ_1 and a non-empty member of τ_2.*

Definition 2.2. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a bitopological space equipped with an ideal $\mathcal{I}$ on X. A subset A of X is said to be pairwise $\rho\mathcal{I}$-compact if for every pairwise open collection $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ of X satisfying $A - \cup_{\alpha\in\Lambda} U_\alpha \in \mathcal{I}$, there exists a finite subcollection $\{U_{\alpha_i} : i = 1, 2, ..., n\}$ of $\mathcal{U}$ such that $A - \cup_{i=1}^{n} U_{\alpha_i} \in \mathcal{I}$. The space $(X, \tau_1, \tau_2, \mathcal{I})$ is said to be pairwise $\rho\mathcal{I}$-compact if X is pairwise $\rho\mathcal{I}$-compact. It is clear that (X, τ_1, τ_2) is pairwise $\rho\mathcal{I}$-compact if and only if $(X, \tau_1, \tau_2, \{\phi\})$ is pairwise $\mathcal{I}$-compact.*

It can be easily checked that if (X, τ_1, τ_2) is pairwise $\rho\mathcal{I}$-compact and $\tau_1 \subset \tau_2$ then (X, τ_1) is $\rho\mathcal{I}$-compact.

Theorem 2.3. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be pairwise $\rho\mathcal{I}$-compact. If $\delta_i, i = 1, 2$ are topologies on X such that $\delta_i \subset \tau_i$ and $\mathcal{I}^*$ is an ideal on X with $\mathcal{I} \subset \mathcal{I}^*$, then $(X, \delta_1, \delta_2, \mathcal{I}^*)$ is pairwise $\rho\mathcal{I}^*$-compact.*

Proof. Easy to prove. □

Theorem 2.4. *Every pairwise $\rho\mathcal{I}$-compact space is pairwise $\mathcal{I}$-compact.*

Proof. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be any pairwise $\rho\mathcal{I}$-compact space. Now we have to show that $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\mathcal{I}$-compact. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ be an pairwise open cover of X. So $\mathcal{U}$ is a pairwise open collection of X and $X - \cup_{\alpha\in\Lambda} U_\alpha = \phi \in \mathcal{I}$. By hypothesis, there exists a finite subcollection $\{U_{\alpha_i} : i = 1, 2, ..., n\}$ such that $X - \cup_{i=1}^{n} U_{\alpha_i} \in \mathcal{I}$. This proves the result. □

Converse of the above theorem, in general, is not true. This can be prove by following example.

Example 2.5. *Let X be the set of real numbers, $\tau_1 = \{\phi, R\} \cup \{(-\infty, n) : n \in Z\}$ and $\tau_2 = \{\phi, R\} \cup \{(n, \infty) : n \in Z\}$. Let $\mathcal{I} = \mathcal{I}_f$ be the ideal of finite subsets of X. Then this space is pairwise $\mathcal{I}$-compact but not pairwise $\rho\mathcal{I}$-compact as $X - \{\cup_{r>0}(r,\infty) \bigcup \cup_{r<0}(-\infty, r)\} = \{0\} \in \mathcal{I}$. Let $0 < r_1 < r_2 < ... < r_n$ where $n \in Z^+$, then $X - \{\cup_{i=1}^{n}(r_i,\infty) \bigcup \cup_{i=1}^{n}(-\infty, r_i)\} = [-r_i, r_i] \notin \mathcal{I}$.*

Theorem 2.6. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a space. Then the following are equivalent:*

(a) *$(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\rho\mathcal{I}$-compact;*
(b) *For any family $\{F_\alpha\}_{\alpha\in\Lambda}$ of pairwise closed sets such that $\cap_{\alpha\in\Lambda}F_\alpha \in \mathcal{I}$, there exists a finite subfamily $\{F_{\alpha_i} : i = 1, 2, ..., n\}$ such that $\cap_{i=1}^{n}F_{\alpha_i} \in \mathcal{I}$;*
(c) *For any family $\{F_\alpha\}_{\alpha\in\Lambda}$ of pairwise closed sets with $(\mathcal{I})$FIP, we have $\cap_{\alpha\in\Lambda}F_\alpha \notin \mathcal{I}$.*

Proof. $(a) \Rightarrow (b)$ Let $\{F_\alpha\}_{\alpha\in\Lambda}$ be a family of pairwise closed sets such that $\cap_{\alpha\in\Lambda}F_\alpha \in \mathcal{I}$. Then $\{X - F_\alpha : \alpha \in \Lambda\}$ is a pairwise open collection of X such that $X - \cup_{\alpha\in\Lambda}(X - F_\alpha) \in \mathcal{I}$. So by assumption there exists a finite subcollection $\{X - F_{\alpha_i} : i = 1, 2, ..., n\}$ such that $X - \cup_{i=1}^{n}(X - F_{\alpha_i}) \in \mathcal{I}$. This implies $\cap_{i=1}^{n}F_{\alpha_i} \in \mathcal{I}$.
$(b) \Rightarrow (c)$ Easy to prove.
$(c) \Rightarrow (a)$ Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ be a pairwise open collection of X such that $X - \cup_{\alpha\in\Lambda}U_\alpha \in \mathcal{I}$. If there exists a finite subcollection $\{U_{\alpha_i} : i = 1, 2, ..., n\}$ such that $X - \cup_{i=1}^{n}U_{\alpha_i} \in \mathcal{I}$ then the theorem is proved. If not, then $\{X - U_\alpha : \alpha \in \Lambda\}$ is a family of pairwise closed sets such that $\cap_{i=1}^{n}\{X - U_{\alpha_i}\} \notin \mathcal{I}$ for any finite subcollection $\{U_{\alpha_i} : i = 1, 2, ..., n\}$. So by (c) $\cap_{\alpha\in\Lambda}\{X - U_\alpha\} \notin \mathcal{I} \Rightarrow X - \cup_{\alpha\in\Lambda}U_\alpha \notin \mathcal{I}$, a contradiction. Hence $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\rho\mathcal{I}$-compact.

□

Theorem 2.7. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a space. Then (X, τ_1, τ_2) is pairwise $\rho\mathcal{I}$-compact if and only if (X, τ_1^*, τ_2^*) is pairwise $\rho\mathcal{I}$-compact.*

Proof. We know that $\tau_1^* \supseteq \tau_1$ and $\tau_2^* \supseteq \tau_2$ so we have to prove only that (X, τ_1^*, τ_2^*) is pairwise $\rho\mathcal{I}$-compact whenever (X, τ_1, τ_2) is pairwise $\rho\mathcal{I}$-compact. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ be a pairwise open collection of X by basic open sets in τ_1^* and τ_2^* such that $X - \cup_{\alpha\in\Lambda} U_\alpha \in \mathcal{I}$. Then $U_\alpha = G_\alpha - I_\alpha$, where $G_\alpha \in \tau_i$ and $I_\alpha \in \mathcal{I}$ for each $U_\alpha \in \tau_i^*$ where $i = 1, 2$. Then $\{G_\alpha\}_{\alpha\in\Lambda}$ is a pairwise open collection of X by basic open sets in τ_1 and τ_2 such that $X - \cup_{\alpha\in\Lambda} G_\alpha \in \mathcal{I}$. By hypothesis, there exists a subfamily $\{G_{\alpha_i} : i = 1, 2, ..., n\}$ such that $X - \cup_{i=1}^n G_{\alpha_i} \in \mathcal{I}$. Since ideal $\mathcal{I}$ is closed under the operation of finite unions, we have $(X - \cup_{i=1}^n G_{\alpha_i}) \cup (\cup_{i=1}^n I_{\alpha_i}) \in \mathcal{I}$. This implies $X - \cup_{i=1}^n U_{\alpha_i} \in \mathcal{I}$. Thus (X, τ_1^*, τ_2^*) is pairwise $\rho\mathcal{I}$-compact.

□

Theorem 2.8. *Let A and B be pairwise $\rho\mathcal{I}$-compact subsets of a space $(X, \tau_1, \tau_2, \mathcal{I})$, then $A \cup B$ is also pairwise $\rho\mathcal{I}$-compact.*

Proof. Can be proved easily. □

Theorem 2.9. *Let $f : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ be a pairwise continuous surjection and $\mathcal{I}$ be an ideal on X. If (X, τ_1, τ_2) is pairwise $\rho\mathcal{I}$-compact then (Y, δ_1, δ_2) is pairwise $\rho f(\mathcal{I})$-compact.*

Proof. Let $\{G_\alpha\}_{\alpha\in\Lambda}$ is a pairwise open collection of Y such that $Y - \cup_{\alpha\in\Lambda} G_\alpha \in f(\mathcal{I})$. Then $\{f^{-1}(G_\alpha) : \alpha \in \Lambda\}$ is a pairwise open collection of X and $X - \cup_{\alpha\in\Lambda} f^{-1}(G_\alpha) \in \mathcal{I}$. By hypothesis, there exists a finite subcollection $\{f^{-1}(G_{\alpha_i}) : i = 1, 2, ..., n\}$ such that $X - \cup_{i=1}^n f^{-1}(G_{\alpha_i}) \in \mathcal{I}$. This implies $f(X - \cup_{i=1}^n f^{-1}(G_{\alpha_i})) \in f(\mathcal{I}) \Rightarrow Y - \cup_{i=1}^n G_{\alpha_i} \in f(\mathcal{I})$. Therefore (Y, δ_1, δ_2) is pairwise $\rho f(\mathcal{I})$-compact. □

Theorem 2.10. *Let* $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \delta_1, \delta_2)$ *be a pairwise pointwise* $\mathcal{I}$*-continuous surjection. Then* (Y, δ_1, δ_2) *is pairwise* $\rho f(\mathcal{I})$*-compact if* $(X, \tau_1, \tau_2, \mathcal{I})$ *is pairwise* $\rho\mathcal{I}$*-compact.*

Proof. This theorem can be easily proved by using Theorem 2.7 and Theorem 2.9. □

Theorem 2.11. *If* (X, τ_1, τ_2) *is pairwise Hausdorff and* $A \subset X$ *is* $\rho\mathcal{I}$*-compact subset of* (X, τ_i)*, where* $\mathcal{I}$ *is an ideal on* X*, then* A *is* τ_j^**-closed for* $i \neq j$*,* $i, j = 1, 2$*.*

Proof. Let A be a $\rho\mathcal{I}$-compact subset of (X, τ_1). We have to show that A is τ_2^*-closed. Let $x \in X - A$. Then by pairwise Hausdorff property, for each $y \in A$, there exists $U_y \in \tau_2$ and $V_y \in \tau_1$ with $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \phi$. Therefore $\{V_y : y \in A\}$ is a τ_2-open collection of A such that $A - \cup_{y \in A} V_y \in \mathcal{I}$. So, by assumption, there exists a finite subcollection $\{V_{y_i} : i = 1, 2, ..., n\}$ such that $A - \cup_{i=1}^n V_{y_i} \in \mathcal{I}$. Let $U = \cap_{i=1}^n U_{y_i}$, where U_{y_i} are the corresponding τ_2-open sets containing x. Since $x \notin A$, $x \in U - (A - \cup_{i=1}^n V_{y_i}) \subset X - A$. But $U - (A - \cup_{i=1}^n V_{y_i}) \in \mathcal{B}(\tau_2, \mathcal{I})$, the basis for τ_2^*. Hence $X - A$ is τ_2^*-open. This implies A is τ_2^*-closed. Similarly we can prove the other part. □

Theorem 2.12. *Let* $(X, \tau_1, \tau_2, \mathcal{I})$ *be a pairwise* $\rho\mathcal{I}$*-compact space. If* A *is proper* τ_i^**-closed subset of* X *then* A *is a* $\rho\mathcal{I}$*-compact subset of* (X, τ_j)*,* $i \neq j$*,* $i, j = 1, 2$*.*

Proof. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a pairwise $\rho\mathcal{I}$-compact space and A be a proper τ_1^*-closed subset of X then we have to prove that A is a $\rho\mathcal{I}$-compact subset of (X, τ_2). Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ be any τ_2-open collection of A. Then $\mathcal{U} \cup \{X - A\}$ is a pairwise open collection of (X, τ_1^*, τ_2^*). By theorem 2.7, there exists a finite subcollection $\{U_{\alpha_i} : i =$

$1, 2, ..., n\} \cup \{X - A\}$ such that $X - \cup_{i=1}^{n} U_{\alpha_i} \cup \{X - A\} \in \mathcal{I}$. This means $A - \cup_{i=1}^{n} U_{\alpha_i} \in \mathcal{I}$. Hence A is a $\rho\mathcal{I}$-compact subset of (X, τ_2). Similarly we can prove that if A is proper τ_2^*-closed subset of X then A is a $\rho\mathcal{I}$-compact subset of (X, τ_1). □

Using above two theorems, we obtain the following result.

Corollary 2.13. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a pairwise $\rho\mathcal{I}$-compact and pairwise Hausdorff. Then a subset A of X is $\rho\mathcal{I}$-compact in (X, τ_i) if and only if A is τ_j^*-closed, $i \neq j$, $i, j = 1, 2$.*

Theorem 2.14. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a pairwise $\rho\mathcal{I}$-compact space. If both (X, τ_1) and (X, τ_2) are Hausdorff then $\tau_1^* = \tau_2^*$.*

Proof. Easy to prove. □

Theorem 2.15. *Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \delta_1, \delta_2)$ be a pairwise pointwise $\mathcal{I}$- continuous surjection with (X, τ_1, τ_2) pairwise $\rho\mathcal{I}$-compact and (Y, δ_1, δ_2) pairwise Hausdorff. Then $f : (X, \tau_1^*, \tau_2^*) \to (Y, \delta_1^*(f(\mathcal{I})), \delta_2^*(f(\mathcal{I})))$ is pairwise closed.*

Proof. Let A be any τ_1^*-closed subset of X. If $A = X$ or ϕ then $f(A)$ is clearly closed in $\delta_1^*(f(\mathcal{I}))$. So, let A be a proper τ_1^*-closed subset of X. Then by Theorem 2.12, A is a $\rho\mathcal{I}$-compact subset of (X, τ_2). Now because of pairwise pointwise $\mathcal{I}$- continuity of f, we have by Theorem 2.10 $f(A)$ is $\rho(f(\mathcal{I}))$ compact subset of pairwise Hausdorff space (Y, δ_2). Now using Corollary 2.13, we have $f(A)$ is $\delta_1^*(f(\mathcal{I}))$-closed. Similarly it can be proved that if A is τ_2^*-closed then $f(A)$ is $\delta_2^*(f(\mathcal{I}))$-closed. Hence proved. □

Theorem 2.16. *If $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \delta_1, \delta_2)$ be a pairwise pointwise $\mathcal{I}$- continuous bijection with (X, τ_1, τ_2) pairwise $\rho\mathcal{I}$-compact and (Y, δ_1, δ_2) pairwise Hausdorff. Then*

$f : (X, \tau_1^*, \tau_2^*) \to (Y, \delta_1^*(f(\mathcal{I})), \delta_2^*(f(\mathcal{I})))$ is pairwise homeomorphism.

Proof. We have to show only that f is pairwise continuous from (X, τ_1^*, τ_2^*) to $(Y, \delta_1^*(f(\mathcal{I})), \delta_2^*(f(\mathcal{I})))$. Let $F \cup f(I)$ be a basic $\delta_1^*(f(\mathcal{I}))$-closed subset of Y where F is closed in δ_i and $I \in \mathcal{I}$. Then $f^{-1}(F \cup f(I)) = f^{-1}(F) \cup I$ is $\tau_i^*(I)$-closed for $i = 1, 2$. Hence proved. □

3. Pairwise $\sigma\mathcal{I}$-compact spaces

Definition 3.1. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be a bitopological space equipped with an ideal $\mathcal{I}$ on X. A subset A of X is said to be pairwise $\sigma\mathcal{I}$-compact if for every pairwise open collection $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ of X satisfying $A - \cup_{\alpha \in \Lambda} U_\alpha \in \mathcal{I}$, there exists a finite subcollection $\{U_{\alpha_i} : i = 1, 2, ..., n\}$ of $\mathcal{U}$ such that $A \subseteq \cup_{i=1}^n U_{\alpha_i}$. The space $(X, \tau_1, \tau_2, \mathcal{I})$ is said to be pairwise $\sigma\mathcal{I}$-compact if X is pairwise $\sigma\mathcal{I}$-compact. It is clear that (X, τ_1, τ_2) is pairwise $\sigma\mathcal{I}$-compact if and only if $(X, \tau_1, \tau_2, \{\phi\})$ is pairwise $\mathcal{I}$-compact.*

Theorem 3.2. *Every pairwise $\sigma\mathcal{I}$-compact space is pairwise $\rho\mathcal{I}$-compact.*

Proof. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be any pairwise $\sigma\mathcal{I}$-compact. Now we have to show that $(X, \tau_1, \tau_2, \mathcal{I})$ is pairwise $\rho\mathcal{I}$-compact. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ be an pairwise open collection of X and $X - \cup_{\alpha \in \Lambda} U_\alpha \in \mathcal{I}$. By hypothesis, there exists a finite subcollection $\{U_{\alpha_i} : i = 1, 2, ..., n\}$ such that $X = \cup_{i=1}^n U_{\alpha_i}$. Hence $X - \cup_{i=1}^n U_{\alpha_i} = \phi \in \mathcal{I}$. This proves the result. □

Converse of the above theorem, in general, is not true. This can be prove by following example.

Example 3.3. *Let $X = R$, $\mathcal{I} = \{A : A \subset (-2, 2)\}$, τ_1 =usual topology for R and τ_2 =discrete topology for R. Let $\mathcal{U} = \{(-\infty, -1), (-1, 0), (0, 1), (1, \infty)\}$. Then $X -$*

$\cup_{U\in\mathcal{U}}U = \{-1,0,1\} \in \mathcal{I}$. Now consider a finite subcollection $\mathcal{V}$ of $\mathcal{U}$, where $\mathcal{V} = \{(-\infty,-1),(1,\infty)\}$. Here X is pairwise $\rho\mathcal{I}$-compact as $X - \cup_{V\in\mathcal{V}}V = [-1,1] \in \mathcal{I}$, but X is not pairwise $\sigma\mathcal{I}$-compact as X is not contained in $\cup_{V\in\mathcal{V}}V$.

So we have the following implications:
Pairwise $\sigma\mathcal{I}$-compact $\Rightarrow$ Pairwise $\rho\mathcal{I}$-compact $\Rightarrow$ Pairwise $\mathcal{I}$-compact.

Theorem 3.4. *Let $(X,\tau_1,\tau_2,\mathcal{I})$ be pairwise $\sigma\mathcal{I}$-compact. If $\delta_i, i=1,2$ are topologies on X such that $\delta_i \subset \tau_i$ and $\mathcal{I}^*$ is an ideal on X with $\mathcal{I} \subset \mathcal{I}^*$, then $(X,\delta_1,\delta_2,\mathcal{I}^*)$ is pairwise $\sigma\mathcal{I}^*$-compact.*

Proof. Easy to prove. □

Theorem 3.5. *Let $(X,\tau_1,\tau_2,\mathcal{I})$ be a space. Then the following are equivalent:*

- (a) *$(X,\tau_1,\tau_2,\mathcal{I})$ is pairwise $\sigma\mathcal{I}$-compact;*
- (b) *For any family $\{F_\alpha\}_{\alpha\in\Lambda}$ of pairwise closed sets such that $\cap_{\alpha\in\Lambda}F_\alpha \in \mathcal{I}$, there exists a finite subfamily $\{F_{\alpha_i} : i=1,2,...,n\}$ such that $\cap_{i=1}^{n}F_{\alpha_i} \neq \phi$;*
- (c) *For any family $\{F_\alpha\}_{\alpha\in\Lambda}$ of pairwise closed sets with $(\mathcal{I})$FIP, we have $\cap_{\alpha\in\Lambda}F_\alpha \notin \mathcal{I}$.*

Proof. Similar to that of Theorem 2.6. □

Theorem 3.6. *Let $(X,\tau_1,\tau_2,\mathcal{I})$ be a space. Then (X,τ_1,τ_2) is pairwise $\sigma\mathcal{I}$-compact if and only if (X,τ_1^*,τ_2^*) is pairwise $\sigma\mathcal{I}$-compact.*

Proof. It can be proved on the lines of Theorem 2.7. □

Theorem 3.7. *Let A and B be pairwise $\sigma\mathcal{I}$-compact subsets of a space $(X,\tau_1,\tau_2,\mathcal{I})$, then $A \cup B$ is also pairwise $\sigma\mathcal{I}$-compact.*

Proof. Can be proved easily. □

Theorem 3.8. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ *be a pairwise continuous surjection and* $\mathcal{I}$ *be an ideal on* X*. If* (X, τ_1, τ_2) *is pairwise* $\sigma\mathcal{I}$*-compact then* (Y, δ_1, δ_2) *is pairwise* $\sigma f(\mathcal{I})$*-compact.*

Proof. Can be proved easily as in Theorem 2.9. □

Theorem 3.9. *Let* $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \delta_1, \delta_2)$ *be a pairwise pointwise* $\mathcal{I}$*-continuous surjection. Then* (Y, δ_1, δ_2) *is pairwise* $\sigma f(\mathcal{I})$*-compact if* $(X, \tau_1, \tau_2, \mathcal{I})$ *is pairwise* $\sigma\mathcal{I}$*-compact.*

Proof. This theorem can be easily proved by using Theorem 3.6 and Theorem 3.8. □

Theorem 3.10. *If* (X, τ_1, τ_2) *is pairwise Hausdorff and* $A \subset X$ *is* $\sigma\mathcal{I}$*-compact subset of* (X, τ_i)*, where* $\mathcal{I}$ *is an ideal on* X*, then* A *is* τ_j^**-closed for* $i \neq j$*,* $i, j = 1, 2$*.*

Proof. It can be proved on the lines of Theorem 2.11. □

Theorem 3.11. *Let* $(X, \tau_1, \tau_2, \mathcal{I})$ *be a pairwise* $\sigma\mathcal{I}$*-compact space. If* A *is proper* τ_i^**-closed subset of* X *then* A *is a* $\sigma\mathcal{I}$*-compact subset of* (X, τ_j)*,* $i \neq j$*,* $i, j = 1, 2$*.*

Proof. Omitted as it is similar to that of Theorem 2.12. □

Corollary 3.12. *Let* $(X, \tau_1, \tau_2, \mathcal{I})$ *be a pairwise* $\sigma\mathcal{I}$*-compact and pairwise Hausdorff. Then a subset* A *of* X *is* $\sigma\mathcal{I}$*-compact in* (X, τ_i) *if and only if* A *is* τ_j^**-closed,* $i \neq j$*,* $i, j = 1, 2$*.*

Theorem 3.13. *Let* $(X, \tau_1, \tau_2, \mathcal{I})$ *be a pairwise* $\sigma\mathcal{I}$*-compact space. If both* (X, τ_1) *and* (X, τ_2) *are Hausdorff then* $\tau_1^* = \tau_2^*$*.*

Proof. Easy to prove. □

Theorem 3.14. *Let* $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \delta_1, \delta_2)$ *be a pairwise pointwise* $\mathcal{I}$*- continuous surjection with* (X, τ_1, τ_2)

pairwise $\sigma\mathcal{I}$-compact and (Y, δ_1, δ_2) pairwise Hausdorff. Then $f : (X, \tau_1^*, \tau_2^*) \to (Y, \delta_1^*(f(\mathcal{I})), \delta_2^*(f(\mathcal{I})))$ is pairwise closed.

Proof. It can be proved on the lines of Theorem 2.15. □

Theorem 3.15. *If $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \delta_1, \delta_2)$ be a pairwise pointwise $\mathcal{I}$- continuous bijection with (X, τ_1, τ_2) pairwise $\sigma\mathcal{I}$-compact and (Y, δ_1, δ_2) pairwise Hausdorff. Then $f : (X, \tau_1^*, \tau_2^*) \to (Y, \delta_1^*(f(\mathcal{I})), \delta_2^*(f(\mathcal{I})))$ is pairwise homeomorphism.*

Proof. Similar to that of Theorem 2.16. □

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