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A DECOMPOSITION OF THE SPACE OF PSEUDOCHARACTERS ON A GROUP WITH RESPECT TO A NORMAL SUBGROUP

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Abstract. We prove that the vector space of nontrivial pseudocharacters on a group having a normal subgroup admits a natural direct decomposition.

§ 1. Introduction

For the definitions, notation, and generalities concerning pseudocharacters and their specific properties related to normal subgroups, see [1–7].

Let G be a group and let $\mathrm{BD}(G)$ be the vector space of real-valued functions f on G for which the function $\delta f\colon G\times G$ defined by the rule

$$(\delta f)(hk) = f(hk) - f(h) - f(k), \qquad h, k \in G, \tag{1}$$

is bounded. Let $\mathrm{PCH}(G)$ be the quotient space of $\mathrm{BD}(G)$ by the subspace $\mathrm{RCH}(G)$ of real characters of G, i.e., the real functions f on G for which δf is identically zero. Obviously, the space $\mathrm{PCH}(G)$ is linearly isomorphic to the image $\delta(\mathrm{BD}(G))$ of $\mathrm{BD}(G)$ in the space of real functions of two variables on $G\times G$. We call the space $\mathrm{PCH}(G)$ the *space of* (*nontrivial*) *pseudocharacters* of G.

Let N be a normal subgroup of G. Since every pseudocharacter on a group is constant on every conjugacy class in the group [1], it follows that the restriction to N of every pseudocharacter f on G satisfies the condition

$$f(g^{-1}ng) = f(n), \qquad n \in N, \quad g \in G. \tag{2}$$

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A pseudocharacter f on N is said to be G-*invariant* if (2) holds. A pseudocharacter f on N can be extended to a pseudocharacter on G only if it is G-invariant. Denote by $\mathrm{PCH}(N,G)$ the vector subspace of $\mathrm{PCH}(N)$ formed by G-invariant pseudocharacters on N. Examples show that the family $\mathrm{PCH}(N,G)$ can be larger in general than the vector space of restrictions of the elements of $\mathrm{PCH}(G)$ to N (see [4]). Denote the vector subspace of $\mathrm{PCH}(N,G)$ formed by these restrictions by $\mathrm{PCH}(G)_N$.

The objective of this note is to describe the structure of the space $\mathrm{PCH}(G)$ using the spaces $\mathrm{PCH}(G/N)$ and $\mathrm{PCH}(G)_N$.

§ 2. Preliminaries

We need an embedding ι_N of the space $\mathrm{PCH}(G)_N$ in the space $\mathrm{PCH}(G)$. Choose a basis in the vector space $\mathrm{PCH}(G)_N$; for every element F of this basis, we choose an element f_F of $\mathrm{PCH}(G)$ whose restriction coincides with F; such an f_F exists by the definition of $\mathrm{PCH}(G)_N$. Consider the linear span of the elements f_F for all basis elements $F \in \mathrm{PCH}(G)_N$ and, for every $f_1 \in \mathrm{PCH}(G)_N$, define the corresponding element of $\mathrm{PCH}(G)$ by linearity (if $f_1 = \sum_k \lambda_k F_k$, where F_k belong to the chosen basis, then the corresponding element $\iota(f_1)$ of $\mathrm{PCH}(G)$ is $\sum_k \lambda_k f_{F_k}$, and the sum is finite); the restriction of $\iota(f_1)$ to N is f_1; this correspondence well defines a vector subspace of $\mathrm{PCH}(G)$, which we denote by $\mathrm{PCH}(G)^b_N$, where the superscript b stands for the choice of the elements f_F for the basis elements F.

§ 3. Main result

Theorem. *Let the spaces* $\mathrm{PCH}(G)$, $\mathrm{PCH}(G)_N$, $\mathrm{PCH}(G)^b_N$, *and* $\mathrm{PCH}(G/N)$ *be defined as above. Then every element of* $\mathrm{PCH}(G)$ *has a unique representation as the sum of an element in* $\mathrm{PCH}(G)^b_N$ *and an element of* $\mathrm{PCH}(G)$ *of the form* $\varphi \circ \rho$, *where* ρ *is the natural epimorphism of* G *onto* G/N *and* $\varphi \in \mathrm{PCH}(G/N)$. *According to this decomposition,* $\mathrm{PCH}(G)$ *is isomorphic to the direct sum of* $\mathrm{PCH}(G)^b_N$ *and* $\mathrm{PCH}(G/N)$.

Proof. Let $f_1 \in \mathrm{PCH}(G)$ and let $F_{f_1} \in \mathrm{PCH}(N)$ be the restriction of f_1 to N. By construction, $F_{f_1} \in \mathrm{PCH}(G)_N$. Consider the element $\iota_N(F_{f_1})$ of $\mathrm{PCH}(G)$. By construction, the restriction of $\iota_N(F_{f_1})$ to N is equal to F_{f_1}, and hence the difference $f_1 - \iota_N(F_{f_1})$ has zero restriction to N. By Theorem 2.5.2 (6) of [9], $f_1 - \iota_N(F_{f_1})$ has the form $\varphi \circ \rho$, where ρ is the natural epimorphism of G onto G/N and φ is a uniquely defined element of $\mathrm{PCH}(G/N)$.

§ 4. Comments

Definition 1. Let G be a group, let N be a normal subgroup of G, let $\sigma\colon G/N \to G$ be a section ($\sigma(gN) \in gN$ for every $g \in G$; we also assume that $\sigma(N) = e_N = e_G$), and let $\omega = \omega_\sigma\colon G \to N$ be defined by the rule

$$\omega(g) = n \quad \text{for} \quad g = \sigma(gN)n, \quad g \in G, \quad n \in N.$$

Let $\sigma(g_i N) = s_i \in g_i N$, $i = 1, 2$, and $\omega(g_i) = n_i$, $i = 1, 2$, and thus $g_i = s_i n_i$, $i = 1, 2$. Then there is an $n(g_1, g_2) = \omega(s_1 s_2) \in N$ such that

$$s_1 s_2 = \sigma(s_1 s_2 N)\omega(s_1 s_2).$$

As is known(see [5]), if G is a group, N is a normal subgroup of G, and F is a nontrivial pseudocharacter on N such that $F(gng^{-1}) = F(n)$ for all $n \in N$ and $g \in G$, and if there is a pseudocharacter on G extending F, then there is a section $\sigma\colon G/N \to G$ for which the set $\{F(n(g_1, g_2)), g_1, g_2 \in G\}$ defined above is bounded. Conversely, if the set $\{F(n(g_1, g_2)), g_1, g_2 \in G\}$ is bounded for some section σ, then the pseudocharacter F on N has an extension to N. This implies immediately the following corollary, giving a sufficient condition for a simpler decomposition of $\mathrm{PCH}(G)$.

Corollary 1. *Let G be a group and let N be a normal subgroup of G. If for every G-invariant pseudocharacter F on N there is a section $\sigma\colon G/N \to G$ for which the set $\{F(n(g_1, g_2)), g_1, g_2 \in G\}$ is bounded, where the set $\{n(g_1, g_2), g_1, g_2 \in G\}$ is introduced above in Definition 1 (in particular, if there is a section $\sigma\colon G/N \to G$ for which all pseudocharacters on N are bounded on the set $\{n(g_1, g_2), g_1, g_2 \in G\}$), then* $\mathrm{PCH}(G)$ *is the direct sum of some linear extension (similar to ι_N) of the vector subspace* $\mathrm{PCH}(N, G)$ *of* $\mathrm{PCH}(N)$ *to a vector subspace of* $\mathrm{PCH}(G)$ *and of the vector space naturally isomorphic to* $\mathrm{PCH}(G/N)$.

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