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LOCALLY BOUNDED AUTOMORPHISMS OF CONNECTED LIE GROUPS WITHOUT NONTRIVIAL COMPACT CONNECTED SUBGROUPS

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Abstract. It is proved that every locally bounded automorphism of a connected Lie group without nontrivial compact connected subgroups is continuous. In particular, every locally bounded automorphism of the universal covering group of the real unimodular group of 2×2 matrices is continuous.

§ 1. Introduction

Recall that a (not necessarily continuous) homomorphism π of a topological group G into a topological group H is said to be *relatively compact* if there is a neighborhood $U = U_{e_G}$ of the identity element e_G in G whose image $\pi(U)$ has compact closure in H. Obviously, a homomorphism into a locally compact group is relatively compact if and only if it is *locally bounded*, i.e., there is a neighborhood U_e whose image is contained in some element of the filter $\mathfrak{V}$ of neighborhoods of e_V having compact closure.

Let us also recall the notion of discontinuity group of a homomorphism π of a topological group G into a topological group H, which was introduced in [1] and used in [2] and [3]. Let $\mathfrak{U} = \mathfrak{U}_G$ be the filter of neighborhoods of

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e_G in G. For every (not necessarily continuous) locally relatively compact homomorphism π of G into H, the set

$$\mathrm{DG}(\pi) = \bigcap_{U \in \mathfrak{U}} \overline{\pi(U)}$$

is called the discontinuity group of π. Here and further, the bar stands for the closure in the corresponding topology (here the closure is taken in the topology of H). (See Definition 1.1.1 in [2].)

The discontinuity group of a homomorphism has many important properties. Under the above conditions, the set $\mathrm{DG}(\pi)$ is a compact subgroup of the topological group H and a compact normal subgroup of the closed subgroup $\overline{\pi(G)}$ of H. Moreover, the filter basis $\{\overline{\pi(U)} \mid U \in \mathfrak{U}\}$ converges to $\mathrm{DG}(\pi)$, and the homomorphism π is continuous if and only if $\mathrm{DG}(\pi) = \{e_H\}$. (See Theorem 1.1.2 of [2].) If G is a connected Lie group, then $\mathrm{DG}(\pi)$ is a compact connected subgroup of H. (See Lemma 1.1.6 of [2].) Let G be a connected Lie group, let N be a closed normal subgroup of G, and let π be a locally bounded homomorphism of G into a locally compact group H. Let $M = \mathrm{DG}(\pi)$ be the discontinuity group of the restriction $\pi|_N$. Then M is a closed normal subgroup of the compact discontinuity group $\mathrm{DG}(\pi)$, and the corresponding quotient group $\mathrm{DG}(\pi)/M$ is isomorphic to the discontinuity group $\mathrm{DG}(\psi)$ of the homomorphism ψ of G obtained as the composition of the homomorphism π and the canonical homomorphism $\overline{\pi(G)} \to \overline{\pi(G)}/M$. (See Lemma 1.1.7 of [2].)

§ 2. Preliminaries

According to Chap. 3, Exercise 43 in [4], a real analytic group is said to be reductive if it has a faithful representation and all its representations are semisimple, and the following conditions for a real analytic group G are equivalent: (i) G is reductive; (ii) the commutator subgroup DG of G is closed in G, the center Z of G is compact, and DG is a semisimple group with a faithful representation. As was proved in [5], every locally bounded automorphism of a reductive Lie group is continuous. The universal covering group $\mathcal{G} = \widetilde{\mathrm{SL}(2,\mathbb{R})}$ of the real unimodular group of 2×2 matrices is not reductive (according to the above definition), and thus the result cited in the introduction does not work for this group.

However, as is well known, $\mathcal{G}$ has no nontrivial compact subgroups. In this note we prove that every locally bounded automorphism of a connected

Lie group having no nontrivial connected compact subgroups is automatically continuous. In particular, every locally bounded automorphism of $\mathcal{G}$ is continuous.

§ 3. Main theorem

Theorem 1. *Let G be a connected Lie group without nontrivial compact connected subgroups, and let π be a locally bounded automorphism of G. Then π is continuous.*

Proof. Recall that, since G is a connected Lie group, it follows that the discontinuity group $\mathrm{DG}(\pi)$ is a compact connected subgroup of G. (See Lemma 1.1.6 in [2].) Since G has no nontrivial compact connected groups by assumption, it follows that the discontinuity group $\mathrm{DG}(\pi)$ of π is the identity subgroup of G. Hence π is continuous. (See Theorem 1.1.2 of [2].)

§ 4. Concluding remarks

Corollary 1. *Every locally bounded automorphism of the universal covering group of the real unimodular group of 2×2 matrices is continuous.*

This follows immediately from Theorem 1 and from the fact that the universal covering group of the real unimodular group of 2×2 matrices has no nontrivial compact subgroups.

It is still unclear whether or not the universal covering group of the real unimodular group of 2×2 matrices has not locally bounded automorphisms.

Using [1]–[3], we can extend the result of Theorem 1 to connected locally compact groups without nontrivial connected compact subgroups.

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