Advanced Studies in Contemporary Mathematics
31 (2021), No. 4, pp. 485 - 500
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2021.31.4.485

RECIPROCITY RELATIONS OF UNIPOLY-DEDEKIND SUMS

HYE KYUNG KIM[1]

ABSTRACT. Recently, Kim et al. introduced the poly-Dedekind sums associated with the type 2 poly-Bernoulli functions of index k and the poly-Dedekind sums associated with the poly-Bernoulli functions of index k. In this paper, as generalizations of the Apostol's generalized Dedekind sums, we introduce two kinds of unipoly-Dedekind sums. One is the unipoly-Dedekind sums that replaces the type 2 poly-Bernoulli functions of arbitrary indices in poly-Dedekind sums by the type 2 unipoly-Bernoulli functions of arbitrary indices, and we derive the reciprocity relation for these. The other is the unipoly-Dedekind sums that replaces the poly-Bernoulli functions of arbitrary indices in poly-Dedekind sums by the unipoly-Bernoulli functions of arbitrary indices, and we derive the reciprocity relation for these.

1. INTRODUCTION

The Dedekind sums were studied by Dedekind in 1930 [7], after which Rademacher and Whiteman studied these sums related to the theory of modular functions that appear in partition theory [20, 21]. After that Apostol considered the generalized Dedekind sums by replacing the first Bernoulli function appearing in Dedekind sums by any Bernoulli functions and derived a reciprocity relation for them [3]. Dedekind sums have been studied in various fields of mathematics related to number theory, topology, combinatorial geometry, and algorithmic complexity, etc [1-7, 9-12, 14-21]. In combinatorial number theory, one is interested in partitions of an integer n from a finite set of positive integers. That is, one writes n as a nonnegative integer linear combination of a given finite set of positive integers. In [4], Beck et al. showed that the number of such partitions of n from a finite set is a quasi-polynomial in n, whose coefficients are built up from some generalization of Dedekind sums.

Recently, as a generalization of the Apostol's generalized Dedekind sums, Kim et al. introduced the poly-Dedekind sums associated with the type 2 poly-Bernoulli functions of index k [15]. In addition, Ma et al. introduced the poly-Dedekind sums associated with the poly-Bernoulli functions of index k [17].

This paper is divided into two sections. In section 2, we consider the unipoly-Dedekind sums that replaced the type 2 poly-Bernoulli functions of index k in poly-Dedekind sums by the type 2 unipoly-Bernoulli functions of index k and derive the reciprocity relation for these. In section 3, we consider the unipoly-Dedekind sums replaced any poly-Bernoulli functions of index k in poly-Dedekind sums by associated the unipoly-Bernoulli functions of index k and derive the reciprocity relation for these.

Now, we give some notations, definitions and properties that are needed in this paper.

$$(1)\qquad ((x)) = \begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \notin \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases} \qquad (\text{see } [1,5])$$

2010 *Mathematics Subject Classification.* 11F20; 11B68; 11B83 .

Key words and phrases. the Dedekind sums; the type 2 poly-Bernoulli polynomials; the poly-Bernoulli polynomials; the unipoly function; the polylogarithm function.

* is corresponding author.

where $[\,\cdot\,]$ denotes the greatest integer not exceeding x.

The Dedekind sums are defined by

$$S(l,k)=\sum_{\mu=1}^{k}\left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{l\mu}{k}\right)\right),\quad (\text{see } [1-7,9-11,14-21]), \tag{2}$$

where l is any integer.

From (2), we note that s

$$S(l,k)=\sum_{\mu=1}^{k}\left(\frac{\mu}{k}-\frac{1}{2}\right)\left(\left(\frac{l\mu}{k}\right)\right)=\sum_{\mu=1}^{k}\frac{\mu}{k}\left(\left(\frac{l\mu}{k}\right)\right),\quad (\text{see } [6,7]). \tag{3}$$

The Bernoulli polynomials are given by the generating function to be

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad (\text{see } [4,5,8-14]). \tag{4}$$

When $x=0$, $B_n=B_n(0)$, $(n\geq 0)$, are called the Bernoulli numbers. It is well known that $B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6},\cdots,$ and $\mathrm{B}_{2n+1}=0$.

From (4), we note that

$$B_n(x)=\sum_{l=0}^{n}\binom{n}{l}B_lx^{n-l}=(B+x)^n,\quad (n\geq 0),\quad (\text{see } [8-10,12-14]), \tag{5}$$

with the usual convention about replacing B^n by B_n.

We observe that

$$\sum_{l=0}^{n-1}e^{lt}=\frac{t}{t(e^t-1)}\left(e^{nt}-1\right)=\sum_{j=0}^{\infty}\left(\frac{B_{j+1}(n)-B_{j+1}}{j+1}\right)\frac{t^j}{j!},\quad (n\in\mathbb{N}). \tag{6}$$

Thus, by (6), we get

$$\sum_{l=0}^{n-1}l^j=\frac{1}{j+1}\left(B_{j+1}(x)-B_j(x)\right),\quad (n\in\mathbb{N},\ j\geq 0). \tag{7}$$

Let the fractional function $\langle\cdot\rangle$ be given by

$$\langle x\rangle=x-[x]. \tag{8}$$

The Bernoulli function is defined by

$$\overline{B}_n(x)=B_n(\langle x\rangle),\quad (n\geq 0),\quad (\text{see } [1,5,21]). \tag{9}$$

Thus, by (3) and (9), we get

$$\begin{aligned} S(h,m) &= \sum_{\mu=1}^{m-1}\frac{\mu}{m}\left(\frac{h\mu}{m}-\left[\frac{h\mu}{k}\right]-\frac{1}{2}\right) \\ &= \sum_{\mu=1}^{m-1}\frac{\mu}{m}\overline{B}_1\left(\frac{h\mu}{m}\right) = \sum_{\mu=1}^{m-1}\overline{B}_1\left(\frac{\mu}{m}\right)\overline{B}_1\left(\frac{h\mu}{m}\right), \end{aligned} \tag{10}$$

where h,k are positive integers [22].

We note [15]that

$$\sum_{i=0}^{l-1} B_n\left(\frac{x+i}{l}\right) = l^{1-n}B_n(x), \tag{11}$$

$$\sum_{i=0}^{l-1} \overline{B}_n\left(\frac{x+i}{l}\right) = l^{1-n}\overline{B}_n(x), \tag{12}$$

$$\sum_{i=0}^{l-1} B_n\left(\frac{<x>+i}{l}\right) = \overline{B}_n\left(\frac{x+i}{l}\right). \tag{13}$$

Kim-Kim considered the polyexponential function defined by

$$\mathrm{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k(n-1)!}, \ (k \in \mathbb{Z}), \quad (\text{see } [8,12]). \tag{14}$$

Note that $\mathrm{Ei}_1(x) = e^x - 1$.

The type 2 poly-Bernoulli polynomials are given by

$$\frac{\mathrm{Ei}_k(\log(1+t))}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}, \quad (k \in \mathbb{Z}) \quad (\text{see } [8]). \tag{15}$$

When $x=0$, $B_n^{(k)} = B_n^{(k)}(0)$, $(n \geq 0)$, are called the type 2 poly-Bernoulli numbers.

The poly-Bernoulli polynomials of index k are defined by the generating function

$$\frac{Li_k(1-e^{-t})}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} \beta_n^{(k)}(x)\frac{t^n}{n!}, \quad (k \in \mathbb{Z}), \quad (\text{see } [9,11,21]), \tag{16}$$

where $Li_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}$ is the polylogarithm function of index k. When $x=0$, $\beta_n^{(k)} = \beta_n^{(k)}(0)$ $(n \geq 0)$ are called the poly-Bernoulli numbers.

Recently, as one generalization of the Apostol's generalized Dedekind sums, Kim et al. introduced the poly-Dedekind sums associated with the type 2 poly-Bernoulli functions of index k which is given by

$$S_p^{(k)}(h,m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m}\overline{B}_p^{(k)}(h\mu/m), \quad (h,m,p \in \mathbb{N}), \quad (\text{see } [14]). \tag{17}$$

where $\overline{B}_p(h\mu/m) = B_p\big(\langle h\mu/m\rangle\big)$.

Note that for any relatively prime positive integers h, m, we have

$$S_1^1(h,m) \ = \ \sum_{\mu=1}^{m-1}(\mu/m)\overline{B}_1(h\mu/m) = \sum_{\mu=1}^{m-1}((\mu/m))((h\mu/m)) \ = \ S(h,m).$$

Moreover, Ma et al. studied the poly-Dedekind sums associated with the poly-Bernoulli functions of index k which are given by

$$T_p^{(k)}(h,m) = \sum_{\mu=1}^{m-1} \frac{\mu}{m}\overline{\beta}_p^{(k)}(h\mu/m), \quad (h,m,p \in \mathbb{N}), \quad (\text{see } [16]). \tag{18}$$

where $\overline{\beta}_p(h\mu/m) = \beta_p\big(\langle h\mu/m\rangle\big)$.

2. Unipoly-Dedekind sums associated with the type 2 unipoly-Bernoulli functions of index k

In this section, as a generalization of the Apostol's generalized Dedekind sums, we consider unipoly-Dedekind sums associated with the type 2 unipoly-Bernoulli functions of index k and derive the reciprocity relation for these.

Let ρ be any arithmetic function which is real or complex valued and defined on the set of positive integers $\mathbb{N}$. Then Kim-Kim defined the unipoly function attached to polynomials ρ by

$$u_k(x|\rho) = \sum_{n=1}^{\infty} \frac{\rho(n)x^n}{n^k}, \quad (k \in \mathbb{Z}) \quad (\text{see [8]}). \tag{19}$$

When $\rho(n) = 1$, $u_k(x|1) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = Li_k(x)$ is the ordinary polylogarithm function.

From (19), we have

$$\frac{d}{dx} u_k(x|\rho) = \frac{1}{x} u_{k-1}(x|\rho). \tag{20}$$

The type 2 unipoly Bernoulli polynomials of index k are defined by

$$\frac{u_k(\log(1+t)|\rho)}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\rho}^{(k)}(x) \frac{t^n}{n!}, \quad (\text{see [8]}). \tag{21}$$

When $x = 0$, $B_{n,\rho}^{(k)} = B_{n,p}^{(k)}(0)$ are called the type 2 unipoly Bernoulli numbers.

It is easy to show that $B_{n,\rho}^{(k)}(x) = \sum_{m=0}^{n} \sum_{l=1}^{m+1} \binom{n}{m} \frac{\rho(l)(l-1)!S_1(m+1,l)}{l^{l-1}} B_{n-m}(x)$, $B_{0,\rho}^{(k)} = \rho(1)$ and $B_{0,\rho}^{(k)}(1) = \rho(1)$.

Now, as a generalization of the Apostol's generalized Dedekind sums, we consider unipoly-Dedekind sums associated with the type 2 unipoly-Bernoulli functions of index k as follows.

$$U_{q,\rho}^{(k)}(h,m) \ = \ \sum_{\mu=1}^{m-1} \left(\mu/m\right) \overline{B}_{q,\rho}^{(k)}\left(h\mu/m\right), \quad (h,m,q \in \mathbb{N}, k \in \mathbb{Z}), \tag{22}$$

where $\overline{B}_{q,\rho}^{(k)}(x) = B_{q,\rho}^{(k)}(<x>)$ are the type 2 unipoly-Bernoulli functions of index k.

For $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}$, let $\rho(n) = \frac{1}{\Gamma(n)} = \frac{1}{(n-1)!}$. Then, from (14) and (15), we have

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,\frac{1}{\Gamma}}^{(k)} \frac{t^n}{n!} &= \frac{u_k(\log(1+t) \mid \frac{1}{\Gamma})}{e^t - 1} e^{xt} \\ &= \frac{Ei_k(\log(1+t))}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}. \end{aligned} \tag{23}$$

Thus, from (17), (22) and (23), we have

$$B_{n,\frac{1}{\Gamma}}^{(k)}(x) = B_n^{(k)}(x) \ \text{ and } \ U_{q,\frac{1}{\Gamma}}^{(k)}(h,m) = S_q^{(k)}(h,m). \tag{24}$$

In addition, we note that

$$U_{q,\frac{1}{\Gamma}}^{(1)}(h,m) = \sum_{\mu=1}^{m-1} \left(\mu/m\right) \overline{B}_{q,\frac{1}{\Gamma}}^{(1)}\left(h\mu/m\right) = S_q(h,m).$$

From (21), we note that

$$\frac{u_k(\log(1+t)|\rho)}{e^t-1}e^{xt} = \left(\sum_{l=0}^{\infty} B_{l,p}^{(k)}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\frac{x^m}{m!}t^m\right) = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}B_{l,\rho}^{(k)}x^{n-l}\right)\frac{t^n}{n!}. \tag{25}$$

Thus, by (25), we get

$$B_{n,\rho}^{(k)}(x) = \sum_{l=0}^{n}\binom{n}{l}B_{l,\rho}^{(k)}x^{n-l}, \quad (n\geq 0). \tag{26}$$

By (26), we get

$$\frac{d}{dx}B_{n,\rho}^{(k)}(x) = nB_{n-1,\rho}^{(k)}(x), \quad (n\geq 1). \tag{27}$$

In this section, we show that most of the properties in [15] can obtained from our definition (22).

Theorem 1. *For $n\geq 1$, we have*

$$B_{n,\rho}^{(k)}(1)-B_{n,\rho}^{(k)} = \sum_{m=1}^{n}\frac{\rho(m)\,m!}{m^k}S_1(n,m), \quad (k\in\mathbb{Z}),$$

where $S_1(n,m)$ are the Stirling numbers of the first kind.

Proof. From (21), we have

$$u_k(\log(1+t)|\rho) = \left(\sum_{l=0}^{\infty}B_{l,\rho}^{(k)}\frac{t^l}{l!}\right)\left(e^t-1\right) = \sum_{n=0}^{\infty}\left(B_{n,\rho}^{(k)}(1)-B_{n,\rho}^{(k)}\right)\frac{t^n}{n!} = \sum_{n=1}^{\infty}\left(B_{n,\rho}^{(k)}(1)-B_{n,\rho}^{(k)}\right)\frac{t^n}{n!}. \tag{28}$$

On the other hand, from (19), we have

$$u_k(\log(1+t)|\rho) = \sum_{m=1}^{\infty}\frac{\rho(m)}{m^k}(\log(1+t))^m = \sum_{m=1}^{\infty}\frac{\rho(m)\,m!}{m^k}\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!} = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{\rho(m)\,m!}{m^k}S_1(n,m)\right)\frac{t^n}{n!}. \tag{29}$$

Therefore, by (28) and (29), we obtain the desired result. □

From (4), (21) and (29), we observe that

$$\sum_{n=0}^{\infty}B_{n,\rho}^{(k)}(x)\frac{t^n}{n!} = \frac{u_k(\log(1+t)|\rho)}{e^t-1}e^{xt} = \frac{t}{e^t-1}e^{xt}\frac{1}{t}\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{\rho(l)\,(l-1)!}{l^{k-1}}S_1(m,l)\right)\frac{t^m}{m!} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{\rho(l)\,(l-1)!}{l^{k-1}(m+1)}S_1(m+1,l)B_{n-m}(x)\right)\frac{t^n}{n!}. \tag{30}$$

Thus, from (30), we get

$$B_{n,\rho}^{(k)}(x)=\sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{\rho(l)\ (l-1)!}{l^{k-1}(m+1)}S_1(m+1,l)B_{n-m}(x). \tag{31}$$

From (31), we note that

$$B_{0,\rho}^{(k)}(1)=\rho(1)B_0(1)=\rho(1)B_0=B_{0,\rho}^{(k)}=\rho(1). \tag{32}$$

Lemma 2. *For $l,\ q\in\mathbb{N}$, we have*

$$\frac{1}{q+1}\binom{q+1}{l}B_{q-l+1,\rho}^{(k)}(1)+\frac{(l-1)}{(q+1)(q+2)}\binom{q+2}{l}B_{q-l+2,\rho}^{(k)}(1)=\sum_{j=0}^{q}\frac{1}{q-j+2}\binom{q}{j}\binom{q-j+2}{l}B_{j,\rho}^{(k)}.$$

Proof. By (27), we get

$$\frac{d}{dx}\left(xB_{q,\rho}^{(k)}(x)\right)=B_{q,\rho}^{(k)}(x)+qxB_{q-1,\rho}^{(k)}(x),$$

where q is positive integer. Then we have

$$\begin{aligned}\left(\frac{d}{dx}\right)^{l-1}\left(xB_{q,\rho}^{(k)}(x)\right)\Big|_{x=1} &= \sum_{k=0}^{l-1}\binom{l-1}{k}\left(\left(\frac{d}{dx}\right)^{k}x\right)\left(\left(\frac{d}{dx}\right)^{l-1-k}B_{q,\rho}^{(k)}(x)\right)\Big|_{x=1}\\ &= \frac{l!}{q+1}\binom{q+1}{l}B_{q-l+1}^{(k)}(1)+\frac{(l-1)l!}{(q+1)(q+2)}\binom{q+2}{l}B_{q-l+2,\rho}^{(k)}(1).\end{aligned} \tag{33}$$

On the other hand, by (26), we get

$$\begin{aligned}\left(\frac{d}{dx}\right)^{l-1}\left(xB_{q,\rho}^{(k)}(x)\right)\Big|_{x=1} &=\sum_{\nu=0}^{q}\binom{q}{\nu}B_{\nu,\rho}^{(k)}\left(\left(\frac{d}{dx}\right)^{l-1}x^{q-\nu+1}\right)\Big|_{x=1}\\ &=\sum_{j=0}^{q}\binom{q}{j}B_{j,\rho}^{(k)}(q-j+1)\cdots(q-j-l+3)\\ &=\sum_{j=0}^{q}\binom{q}{j}\frac{l!}{q-j+2}\binom{q-j+2}{l}B_{j,\rho}^{(k)}.\end{aligned} \tag{34}$$

Therefore, by (33) and (34), we obtain the theorem. □

Now, we observe that

$$\begin{aligned}\sum_{j=0}^{q}\frac{1}{q-j+2}\binom{q}{j}\binom{q-j+2}{l}B_{j,\rho}^{(k)} &=\sum_{j=0}^{q-l+2}\frac{1}{q-j+2}\binom{q}{j}\binom{q-j+2}{l}B_{j,\rho}^{(k)}\\ &=\sum_{j=0}^{q-l+1}\frac{1}{q-j+2}\binom{q}{j}\binom{q-j+2}{l}B_{j,\rho}^{(k)}+\frac{1}{l}\binom{q}{l-2}B_{q-l+2,\rho}^{(k)}.\end{aligned} \tag{35}$$

Therefore, by Lemma 2 and (35), we obtain the following corollary.

Corollary 3. *For $s,q \in \mathbb{N}$, we have*

$$\frac{1}{q+1}\binom{q+1}{l}B^{(k)}_{q-l+1,\rho}(1)+\frac{(l-1)}{(q+1)(q+2)}\binom{q+2}{l}B^{(k)}_{q-l+2,\rho}(1)-\frac{1}{l}\binom{q}{l-2}B^{(k)}_{q-l+2,\rho} = \sum_{j=0}^{q-l+1}\binom{q}{j}\binom{q-j+2}{l}\frac{B^{(k)}_{j,\rho}}{q-j+2}.$$

Lemma 4. *For $q \in \mathbb{N}$, we have*

$$\frac{B^{(k)}_{q+1,\rho}(1)}{q+1}-\frac{B^{(k)}_{q+2,\rho}(1)}{(q+1)(q+2)}+\frac{B^{(k)}_{q+2,\rho}}{(q+1)(q+2)}=\sum_{l=0}^{q}\binom{q}{l}B^{(k)}_{l,\rho}\frac{1}{q+2-l}.$$

Proof. From (27), we have

$$\begin{aligned}\int_0^1 xB^{(k)}_{q,\rho}(x)dx &= \left[x\frac{B^{(k)}_{q+1,\rho}(x)}{q+1}\right]_0^1-\frac{1}{q+1}\int_0^1 B^{(k)}_{q+1,\rho}(x)dx \\ &= \frac{B^{(k)}_{q+1,\rho}(1)}{q+1}-\frac{B^{(k)}_{q+2,\rho}(1)}{(q+1)(q+2)}+\frac{B^{(k)}_{q+2,\rho}}{(q+1)(q+2)}.\end{aligned} \tag{36}$$

On the other hand, by (26), we get

$$\int_0^1 xB^{(k)}_{q,\rho}(x)dx=\sum_{l=0}^{q}\binom{q}{l}B^{(k)}_{l,\rho}\int_0^1 x^{q-l+1}dx=\sum_{l=0}^{q}\binom{q}{l}B^{(k)}_{l,\rho}\frac{1}{q+2-l}. \tag{37}$$

Therefore, by (36) and (37), we obtain the desired result. □

Lemma 5. *Let q be an odd positive integer ≥ 3, then we have*

$$\begin{aligned}m^qU^{(k)}_{q,\rho}(1,m)&=\sum_{j=0}^{q}\binom{q}{j}\frac{B^{(k)}_{j,\rho}}{q+2-j}m^{q+1} \\ &\quad+\sum_{i=1}^{q-1}\sum_{j=0}^{q+1-i}\binom{q}{j}\binom{q+2-j}{i}\frac{B^{(k)}_{j,\rho}}{q+2-j}B_i\, m^{q+1-i}+\rho(1)B_{q+1}.\end{aligned}$$

Proof. From (5), we note that

$$\begin{aligned}B_{q+2-j}(m)-B_{q+2-j} &= \sum_{i=0}^{q+2-j}\binom{q+2-j}{i}B_im^{q+2-j-i}-B_{q+2-j} \\ &= \sum_{i=0}^{q+1-j}\binom{q+2-j}{i}B_im^{q+2-j-i}.\end{aligned} \tag{38}$$

From (7), (22), (26) and (38), we have

$$
\begin{aligned}
(39)\qquad U_{q,\rho}^{(k)}(1,m) &= \sum_{\mu=1}^{m-1}\big(\mu/m\big)\overline{B}_{q,\rho}^{(k)}\big(\mu/m\big),\\
&= \sum_{\mu=1}^{m-1}\big(\mu/m\big)\sum_{j=0}^{q}\binom{q}{j}B_{j,\rho}^{(k)}\big(\mu/m\big)^{q-j}\\
&= \sum_{j=0}^{q}\binom{q}{j}B_{j,\rho}^{(k)}m^{-(q-j+1)}\frac{1}{q+2-j}\big(B_{q+2-j}(m)-B_{q+2-j}\big)\\
&= \frac{1}{m^q}\sum_{j=0}^{q}\binom{q}{j}\frac{B_{j,\rho}^{(k)}}{q+2-j}\sum_{i=0}^{q+1-j}\binom{q+2-j}{i}B_i\, m^{q+1-i}.
\end{aligned}
$$

Since q is an odd positive integer ≥ 3, $B_q=0$. In addition, from (32) and (39), we have

$$
\begin{aligned}
(40)\qquad m^qU_{q,\rho}^{(k)}(1,m) &= \sum_{j=0}^{q}\binom{q}{j}\frac{B_{j,\rho}^{(k)}}{q+2-j}\sum_{i=0}^{q+1-j}\binom{q+2-j}{i}B_im^{q+1-i}\\
&= \sum_{j=0}^{q}\binom{q}{j}\frac{B_{j,\rho}^{(k)}}{q+2-j}m^{q+1}+\sum_{i=1}^{q+1}\sum_{j=0}^{q+1-i}\binom{q}{j}\binom{q+2-j}{i}\frac{B_{j,\rho}^{(k)}}{q+2-j}\,B_i\,m^{q+1-i}\\
&= \sum_{j=0}^{q}\binom{q}{j}\frac{B_{j,\rho}^{(k)}}{q+2-j}m^{q+1}+\sum_{i=1}^{q-1}\sum_{j=0}^{q+1-i}\binom{q}{j}\frac{B_{j,\rho}^{(k)}}{q+2-j}\binom{q+2-j}{i}B_im^{q+1-i}\\
&\quad+\frac{B_{0,\rho}^{(k)}}{q+2}\binom{q+2}{q+1}B_{q+1}+\sum_{j=0}^{1}\binom{q}{j}\frac{B_{j,\rho}^{(k)}}{q+2-j}\binom{q+2-j}{q}B_qm\\
&= \sum_{j=0}^{q}\binom{q}{j}\frac{B_{j,\rho}^{(k)}}{q+2-j}m^{q+1}+\sum_{i=1}^{q-1}\sum_{j=0}^{q+1-i}\binom{q}{j}\frac{\binom{q+2-j}{i}}{q+2-j}B_{j,\rho}^{(k)}B_i\,m^{q+1-i}+\rho(1)B_{q+1}.
\end{aligned}
$$

Therefore, by (40), we arrive at the desired result. □

To simplify the proofs from now on, we employ the symbolic notation as

$$
(41)\qquad B_n(x)=(B+x)^n,\quad B_{n,\rho}^{(k)}(x)=\big(B_\rho^{(k)}+x\big)^n,\quad (n\geq 0),
$$

with the usual convention about replacing B^m and $(B_\rho^(k))^m$ respectively by B_m and $B_{m,\rho}^{(k)}$.

Lemma 6. *Let q be an odd positive integer ≥ 3. Then we have*

$$
\begin{aligned}
(q+1)m^qU_{q,\rho}^{(k)}(1,m) &= \sum_{i=0}^{q+1}\binom{q+1}{i}m^{q+1-i}B_iB_{q+1-i,\rho}^{(k)}\\
&\quad+\frac{1}{q+2}\sum_{i=0}^{q+1}\binom{q+2}{i}(i-1)\,m^{q+1-i}B_i\big(B_{q+2-i,\rho}^{(k)}(1)-B_{q+2-i,\rho}^{(k)}\big).
\end{aligned}
$$

Proof. Since q is an odd positive integer ≥ 3, $B_q=0$. In addition, from Theorem 1 and (32), we have $B_{1,\rho}^{(k)}(1)-B_{1,\rho}^{(k)}=\rho(1)$ and $B_{0,\rho}^{(k)}(1)=B_{0,\rho}^{(k)}=\rho(1)$. In addition, we observe that $\binom{q}{i-2}\frac{q+1}{i}=\frac{1}{q+2}\binom{q+2}{i}(i-1)$, for $i\geq 1$.

From Corollary 3, Lemma 4 and Lemma 5, we have

(42)

$$
\begin{aligned}
&(q+1)m^qU^{(k)}_{q,\rho}(1,m)\\
&=\sum_{j=0}^{q}\binom{q}{j}\frac{B^{(k)}_{j,\rho}}{q+2-j}(q+1)m^{q+1}+\sum_{i=1}^{q-1}\sum_{j=0}^{q+1-i}\binom{q}{j}\frac{\binom{q+2-j}{i}}{q+2-j}(q+1)B^{(k)}_{j,\rho}B_i\, m^{q+1-i}+(q+1)\rho(1)B_{q+1}\\
&=\left(B^{(k)}_{q+1,\rho}(1)-\frac{B^{(k)}_{q+2,\rho}(1)}{q+2}+\frac{B^{(k)}_{q+2,\rho}}{q+2}\right)m^{q+1}\\
&\qquad+\sum_{i=1}^{q-1}\binom{q+1}{i}B_iB^{(k)}_{q+1-i,\rho}(1)\, m^{q+1-i}+(q+1)\rho(1)B_{q+1}\\
&\qquad+\sum_{i=1}^{q-1}\binom{q+2}{i}B_iB^{(k)}_{q+2-i,\rho}(1)\, m^{q+1-i}\frac{(i-1)}{q+2}-\sum_{i=1}^{q-1}\binom{q}{i-2}\frac{(q+1)}{i}B^{(k)}_{q+2-i,\rho}B_i\, m^{q+1-i}\\
&=\sum_{i=0}^{q-1}\binom{q+1}{i}B_iB^{(k)}_{q+1-i,\rho}(1)\, m^{q+1-i}+\rho(1)B_{q+1}+\frac{1}{q+2}(-1)\left(B^{(k)}_{q+2,\rho}(1)-B^{(k)}_{q+2,\rho}\right)m^{q+1}\\
&\qquad+\frac{1}{q+2}\sum_{i=1}^{q-1}\binom{q+2}{i}B_i\left(B^{(k)}_{q+2-i,\rho}(1)-B^{(k)}_{q+2-i,\rho}\right)(i-1)\, m^{q+1-i}+q\rho(1)B_{q+1}\\
&=\sum_{i=0}^{q+1}\binom{q+1}{i}B_iB^{(k)}_{q+1-i,\rho}(1)\, m^{q+1-i}\\
&\qquad+\frac{1}{q+2}\sum_{i=0}^{q+1}\binom{q+2}{i}B_i(B^{(k)}_{q+2-i,\rho}(1)-B^{(k)}_{q+2-i,\rho})(i-1)\, m^{q+1-i}.
\end{aligned}
$$

Therefore, by (42), we obtain what we want. □

Lemma 7. *For $m,n,q\in\mathbb{N}$ with $(h,m)=1$, and q be an odd positive integer ≥ 3, we have*

$$
\begin{aligned}
&\sum_{j=0}^{q+1}\binom{q+1}{j}B_jB^{(k)}_{q+1-j,\rho}(1)(mh)^{q+1-j}\\
&\qquad=m^q\sum_{\mu=0}^{m-1}\sum_{j=0}^{q+1}\binom{q+1}{j}h^jB^{(k)}_{j,\rho}(\mu/m)B_{q+1-j}\left(h-\left[\frac{h\mu}{m}\right]\right).
\end{aligned}
$$

Proof. As the index μ ranges through the values $\mu=0,1,2,\ldots,m-1$, the product $h\mu$ ranges through a complete residue system modulo on such that $(h,m)=1$ and we may replace $<\frac{h\mu}{m}>=$

$\frac{h\mu}{m}-[\frac{h\mu}{m}]$ by $<\frac{h\mu}{m}>$ without alternating the sum over μ. Therefore, from (11), we have

$$
\begin{aligned}
(43)\qquad & m^q\cdot\sum_{\mu=0}^{m-1}\sum_{j=0}^{q+1}\binom{q+1}{j}h^jB_{j,\rho}^{(k)}(\mu/m)B_{q+1-j}\big(h-[h\mu/m]\big)\\
&=m^q\sum_{\mu=0}^{m-1}\sum_{s=0}^{q+1}\binom{q+1}{j}h^j\big(B_\rho^{(k)}+\mu m^{-1}\big)^j\big(B+h-[h\mu/m]\big)^{q+1-j}\\
&=m^q\sum_{\mu=0}^{m-1}\left(h\big(B_\rho^{(k)}+\mu m^{-1}\big)+B+h-[h\mu/m]\right)^{q+1}\\
&=m^q\sum_{\mu=0}^{m-1}\left(h(B_\rho^{(k)}+1)+B+\frac{\mu}{m}\right)^{q+1}\\
&=m^q\sum_{\mu=0}^{m-1}\sum_{j=0}^{q+1}\binom{q+1}{j}B_j\left(\frac{\mu}{m}\right)h^{q+1-j}B_{q+1-j,\rho}^{(k)}(1)\\
&=\sum_{j=0}^{q+1}\binom{q+1}{j}m^{q-j+1}\left(m^{j-1}\sum_{\mu=0}^{m-1}B_j\left(\frac{\mu}{m}\right)\right)h^{q+1-j}B_{q+1-j,\rho}^{(k)}(1)\\
&=\sum_{j=0}^{q+1}\binom{q+1}{j}B_jm^{q-j+1}h^{q+1-j}B_{q+1-j,\rho}^{(k)}(1).
\end{aligned}
$$

Therefore, from (43), we obtain what we want. □

Theorem 8. *For $n\in\mathbb{N}\cup\{0\}$ and $k\in\mathbb{Z}$, we have*

$$B_{n,\rho}^{(k)}(x)=\sum_{j=0}^{n}\sum_{i=0}^{d-1}\sum_{l=1}^{n-j+1}\binom{n}{j}\frac{d^{j-1}\,\rho(l)\,l!}{(n-j+1)\,l^k}\,S_1(n-j+1,l)B_j\left(\frac{x+i}{d}\right).$$

Proof. From (4), (19) and (21), we have

$$
\begin{aligned}
(44)\qquad \sum_{n=0}^{\infty}B_{n,\rho}^{(k)}(x)\frac{t^n}{n!}&=\frac{u_k(\log(1+t)|\rho)}{e^t-1}e^{xt}=\frac{u_k(\log(1+t)|\rho)}{e^{dt}-1}\sum_{i=0}^{d-1}e^{(i+x)t}\\
&=\frac{u_k(\log(1+t)|\rho)}{dt}\sum_{i=0}^{d-1}e^{(\frac{i+x}{d})dt}\frac{dt}{e^{dt}-1}\\
&=\sum_{j=0}^{\infty}d^{j-1}\sum_{i=0}^{d-1}B_j\left(\frac{x+i}{d}\right)\frac{t^j}{j!}\frac{1}{t}\sum_{l=1}^{\infty}\frac{\rho(l)(\log(1+t))^l}{l^k}\frac{l!}{l!}\\
&=\sum_{j=0}^{\infty}d^{j-1}\sum_{i=0}^{d-1}B_j\left(\frac{x+i}{d}\right)\frac{t^j}{j!}\frac{1}{t}\sum_{\nu=1}^{\infty}\sum_{l=1}^{\nu}\frac{\rho(l)\,l!}{l^k}S_1(\nu,l)\frac{t^\nu}{\nu!}\\
&=\sum_{j=0}^{\infty}d^{j-1}\sum_{i=0}^{d-1}B_j\left(\frac{x+i}{d}\right)\frac{t^j}{j!}\sum_{\nu=0}^{\infty}\sum_{l=1}^{\nu+1}\frac{\rho(l)\,l!}{l^k}S_1(\nu+1,l)\frac{t^\nu}{(\nu+1)!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{i=0}^{d-1}\sum_{l=1}^{n-j+1}\binom{n}{j}d^{j-1}\,B_j\left(\frac{x+i}{d}\right)\frac{\rho(l)\,l!}{(n-j+1)l^k}S_1(n-j+1,l)\right)\frac{t^n}{n!}.
\end{aligned}
$$

Therefore, by comparing the coefficients on both sides of (44), we get the desired result. □

Thus, we obtain the following reciprocity theorem for the unipoly-Dedekind sums associated with the type2 unipoly-Bernoulli function with index k.

Theorem 9. *For $m,h,q\in\mathbb{N}$ and $k\in\mathbb{Z}$, we have*

$$\begin{aligned}&hm^qU^{(k)}_{q,\rho}(h,m)+mh^qU^{(k)}_{q,\rho}(m,h)\\&=\sum_{\zeta=0}^{m-1}\sum_{j=0}^{q}\sum_{\eta=0}^{h-1}\sum_{l=1}^{q-j+1}\frac{(mh)^{j-1}\binom{q}{j}\rho(l)\,l!\,S_1(p-j+1,l)}{(p-j+1)l^k}\\&\qquad\times\left((\zeta h)m^{q-j}+(m\eta)h^{q-j}\right)\overline{B}_j\left(\frac{\eta}{h}+\frac{\zeta}{m}\right).\end{aligned}$$

Proof. From (13), (22) and Theorem 8, we have

(45)

$$\begin{aligned}&hm^qU^{(k)}_{q,\rho}(h,m)+mh^qU^{(k)}_{q,\rho}(m,h)\\&=hm^q\sum_{\zeta=0}^{m-1}\frac{\zeta}{m}\overline{B}^{(k)}_{q,\rho}\left(\frac{h\zeta}{m}\right)+mh^q\sum_{\eta=0}^{h-1}\frac{\eta}{h}\overline{B}^{(k)}_{q,\rho}\left(\frac{m\eta}{h}\right)\\&=hm^q\sum_{\zeta=0}^{m-1}\frac{\zeta}{m}\sum_{j=0}^{q}h^{j-1}\binom{q}{j}\sum_{\eta=0}^{h-1}\sum_{l=1}^{q-j+1}\frac{\rho(l)\,l!\,S_1(q-j+1,l)}{(q-j+1)l^k}\overline{B}_j\left(\frac{\zeta}{m}+\frac{\eta}{h}\right)\\&+mh^q\sum_{\eta=0}^{h-1}\frac{\nu}{h}\sum_{j=0}^{q}m^{j-1}\binom{q}{j}\sum_{\zeta=0}^{m-1}\sum_{l=1}^{q-j+1}\frac{\rho(l)\,l!\,S_1(q-j+1,l)}{(q-j+1)l^k}\overline{B}_j\left(\frac{\eta}{h}+\frac{\zeta}{m}\right)\\&=\sum_{\zeta=0}^{m-1}\frac{\zeta}{m}\sum_{j=0}^{q}m^{q-j}(mh)^j\binom{q}{j}\sum_{\eta=0}^{h-1}\sum_{l=1}^{q-j+1}\overline{B}_j\left(\frac{\zeta}{m}+\frac{\eta}{h}\right)\frac{\rho(l)\,l!\,S_1(q-j+1,l)}{(q-j+1)l^k}\\&+\sum_{\eta=0}^{h-1}\frac{\eta}{h}\sum_{j=0}^{q}h^{q-j}(mh)^j\binom{q}{j}\sum_{\zeta=0}^{m-1}\sum_{l=1}^{q-j+1}\overline{B}_j\left(\frac{\eta}{h}+\frac{\zeta}{m}\right)\frac{\rho(l)\,l!\,S_1(q-j+1,l)}{(q-j+1)l^k}\\&=\sum_{\zeta=0}^{m-1}\sum_{j=0}^{q}\sum_{\eta=0}^{h-1}\sum_{l=1}^{q-j+1}(\zeta h)(mh)^{-1}m^{q-j}(mh)^j\binom{q}{j}\overline{B}_j\left(\frac{\zeta}{m}+\frac{\eta}{h}\right)\frac{\rho(l)\,l!\,S_1(q-j+1,l)}{(q-j+1)l^k}\\&+\sum_{\zeta=0}^{m-1}\sum_{j=0}^{q}\sum_{\eta=0}^{h-1}\sum_{l=1}^{q-j+1}(m\eta)(mh)^{-1}h^{q-j}(mh)^j\binom{q}{j}\overline{B}_j\left(\frac{\eta}{h}+\frac{\zeta}{m}\right)\frac{\rho(l)\,l!\,S_1(q-j+1,l)}{(q-j+1)l^k}\\&=\sum_{\zeta=0}^{m-1}\sum_{j=0}^{q}\sum_{\eta=0}^{h-1}\sum_{l=1}^{q-j+1}\frac{(mh)^{j-1}\binom{q}{j}\rho(l)\,l!\,S_1(q-j+1,l)}{(q-j+1)l^k}\left((\zeta h)m^{q-j}+(m\eta)h^{q-j}\right)\overline{B}_j\left(\frac{\eta}{h}+\frac{\zeta}{m}\right).\end{aligned}$$

Therefore, from (45) we arrive at what we want. □

When $k=1$, we get the following reciprocity relation for the generalized Dedekind sums defined by Apostol.

Corollary 10. *For $m,h,q\in\mathbb{N}$, we have*

$$hm^qU^{(1)}_{q,\frac{1}{\Gamma}}(h,m)+mh^qU^{(1)}_{q,\frac{1}{\Gamma}}(m,h)=mh^qS_q(h,m)+mh^qS_q(m,h)$$
$$=\sum_{\zeta=0}^{m-1}\sum_{\eta=0}^{h-1}(mh)^{q-1}(\zeta h+m\eta)\overline{B}_q\left(\frac{\eta}{h}+\frac{\zeta}{m}\right).$$

3. UNIPOLY-DEDEKIND SUMS ASSOCIATED WITH THE UNIPOLY-BERNOULLI FUNCTIONS OF INDEX k

In this section, as another generalization of the Apostol's generalized Dedekind sums, we consider unipoly-Dedekind sums associated with the unipoly-Bernoulli functions of index k and derive the reciprocity relation for these.

The unipoly Bernoulli polynomials of index k are defined by

$$\frac{u_k(1-e^{-t}|\rho)}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}\beta^{(k)}_{n,\rho}(x)\frac{t^n}{n!},\quad (\text{see [8]}). \tag{46}$$

When $x=0$, $\beta^{(k)}_{n,\rho}=\beta^{(k)}_{n,\rho}(0)$ is the unipoly Bernoulli numbers.

In this section, as another generalization of the Apostol's generalized Dedekind sums, we consider unipoly-Dedekind sums associated the unipoly-Bernoulli functions of index k as follows.

$$W^{(k)}_{q,\rho}(h,m)\ =\ \sum_{\mu=1}^{m-1}\big(\mu/m\big)\overline{\beta}^{(k)}_{q,\rho}\big(h\mu/m\big),\quad (h,m,q\in\mathbb{N},k\in\mathbb{Z}), \tag{47}$$

where $\overline{\beta}^{(k)}_{q,\rho}(x)=\beta^{(k)}_{q,\rho}(<x>)$ are called the unipoly-Bernoulli functions of index k.

We note that

$$\sum_{n=0}^{\infty}\beta^{(k)}_{n,1}\frac{t^n}{n!}=\frac{u_k(1-e^{-t}|1)}{e^t-1}e^{xt} \tag{48}$$
$$=\frac{Li_k(1-e^{-t})}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}\beta^{(k)}_{n}(x)\frac{t^n}{n!}.$$

Thus, we have

$$\beta^{(k)}_{n,1}(x)=\beta^{(k)}_{n}(x)\ \text{ and }\ W^{(k)}_{q,1}(h,m)=T^{(k)}_{q}(h,m). \tag{49}$$

In addition, we note that

$$W^{(1)}_{q,1}(h,m)=\sum_{\mu=1}^{m-1}\big(\mu/m\big)\overline{\beta}^{(1)}_{q,1}\big(h\mu/m\big)=T_q(h,m).$$

From (45), we note that

$$\frac{u_k(1-e^{-t}|\rho)}{e^t-1}e^{xt}\ =\ \left(\sum_{l=0}^{\infty}\beta^{(k)}_{l,p}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\frac{x^m}{m!}t^m\right) \tag{50}$$
$$=\ \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\beta^{(k)}_{l,\rho}x^{n-l}\right)\frac{t^n}{n!}.$$

Thus, by (50), we get

$$\beta_{n,\rho}^{(k)}(x) \;=\; \sum_{l=0}^{n}\binom{n}{l}\beta_{l,\rho}^{(k)}x^{n-l}, \quad (n\geq 0). \tag{51}$$

By (51), we get

$$\frac{d}{dx}\beta_{n,\rho}^{(k)}(x) = n\beta_{n-1,\rho}^{(k)}(x), \quad (n\geq 1). \tag{52}$$

Theorem 11. *For $n\geq 1$, we have*

$$\beta_{n,\rho}^{(k)}(1)-\beta_{n,\rho}^{(k)} = \sum_{m=1}^{n}\frac{(-1)^{n-m}\rho(m)\,m!}{m^k}S_2(n,m), \quad (k\in\mathbb{Z}),$$

where $S_2(n,m)$ is the Stirling number of the second kind.

From (4) and (46), we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\beta_{n,\rho}^{(k)}(x)\frac{t^n}{n!} &= \frac{u_k(1-e^t|\rho)}{e^t-1}e^{xt}\\ &= \frac{t}{e^t-1}e^{xt}\frac{1}{t}\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{\rho(l)\;(l-1)!(-1)^{l+m}}{l^{k-1}}S_2(m,l)\right)\frac{t^m}{m!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{\rho(l)\;(l-1)!(-1)^{l+m+1}}{l^{k-1}(m+1)}S_2(m+1,l)\beta_{n-m}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{53}$$

Thus, from (53), we get

$$\beta_{n,\rho}^{(k)}(x) = \sum_{m=0}^{n}\sum_{l=1}^{m+1}\binom{n}{m}\frac{\rho(l)\;(l-1)!}{l^{k-1}(-1)^{l+m+1}}S_2(m+1,l)\beta_{n-m}(x),$$

and

$$\beta_{0,\rho}^{(k)}(1) = \rho(1)B_0(1) = \rho(1)B_0 = \beta_{0,\rho}^{(k)} = \rho(1). \tag{54}$$

We can obtain the following lemmas in the same way as in section 2.

Lemma 12. *For $s,q\in\mathbb{N}$, we have*

(1) $$\sum_{j=0}^{q}\binom{q}{j}\binom{q-j+2}{l}\frac{\beta_{j,\rho}^{(k)}}{q-j+2} = \binom{q+1}{l}\frac{\beta_{q-l+1,\rho}^{(k)}(1)}{q+1}+\frac{(l-1)\beta_{q-l+2,\rho}^{(k)}(1)}{(q+1)(q+2)}\binom{q+2}{l}.$$

(2) $$\begin{aligned}&\sum_{j=0}^{q-l+1}\binom{q}{j}\binom{q-j+2}{l}\frac{\beta_{j,\rho}^{(k)}}{q-j+2}\\ &= \binom{q+1}{l}\frac{\beta_{q-l+1,\rho}^{(k)}(1)}{q+1}+\frac{(l-1)\beta_{q-l+2,\rho}^{(k)}(1)}{(q+1)(q+2)}\binom{q+2}{l}-\frac{1}{l}\binom{q}{l-2}\beta_{q-l+2,\rho}^{(k)}.\end{aligned}$$

(3) $$\sum_{l=0}^{q}\binom{q}{l}\beta_{l,\rho}^{(k)}\frac{1}{q+2-l} = \frac{\beta_{q+1,\rho}^{(k)}(1)}{q+1}-\frac{\beta_{q+2,\rho}^{(k)}(1)}{(q+1)(q+2)}+\frac{\beta_{q+2,\rho}^{(k)}}{(q+1)(q+2)}.$$

Lemma 13. *Let q be an odd positive integer ≥ 3. Then we have*

(1) $m^q W^{(k)}_{q,\rho}(1,m) = \sum_{j=0}^{q} \binom{q}{j} \frac{\beta^{(k)}_{j,\rho}}{q+2-j} m^{q+1}$

$+ \sum_{i=1}^{q-1} \sum_{j=0}^{q+1-i} \binom{q}{j} \binom{q+2-j}{i} \frac{\beta^{(k)}_{j,\rho}}{q+2-j} B_i\, m^{q+1-i} + \rho(1) B_{q+1}.$

(2) $(q+1) m^q W^{(k)}_{q,\rho}(1,m) = \sum_{i=0}^{q+1} \binom{q+1}{i} B_i \beta^{(k)}_{q+1-i,\rho}(1) m^{q+1-i}$

$+ \frac{1}{q+2} \sum_{i=0}^{q+1} \binom{q+2}{i} (i-1) B_i\, m^{q+1-i} \left(\beta^{(k)}_{q+2-i,\rho}(1) - \beta^{(k)}_{q+2-i,\rho} \right).$

Lemma 14. *For $m,n,q \in \mathbb{N}$ with $(h,m)=1$, and q be an odd positive integer ≥ 3, we have*

$$\sum_{i=0}^{q+1} \binom{q+1}{i} B_i \beta^{(k)}_{q+1-i,\rho}(1)(mh)^{q+1-i} = m^q \sum_{\mu=0}^{m-1} \sum_{i=0}^{q+1} \binom{q+1}{i} h^i \beta^{(k)}_{i,\rho}(\mu/m) B_{q+1-i}\left(h - \left[\frac{h\mu}{m} \right] \right).$$

Theorem 15. *For $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}$, we have*

$$\beta^{(k)}_{n,\rho}(x) = \sum_{j=0}^{n} \sum_{i=0}^{d-1} \sum_{l=1}^{n-j+1} \binom{n}{j} \frac{d^{j-1}\, \rho(l)\, l! (-1)^{l+n-j+1}}{(n-j+1)\, l^k}\, S_2(n-j+1,l) B_j\left(\frac{x+i}{d} \right).$$

Proof. From (4) and (45), using same method that of Theorem 8, we get the desired result.

□

Thus, we obtain the following reciprocity theorem for the unipoly-Dedekind sums associated with the unipoly-Bernoulli function with index k.

Theorem 16. *For $m,h,q \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have*

$$hm^q W^{(k)}_{q,\rho}(h,m) + mh^q W^{(k)}_{q,\rho}(m,h) = \sum_{\zeta=0}^{m-1} \sum_{j=0}^{q} \sum_{\eta=0}^{h-1} \sum_{l=1}^{q-j+1} \binom{q}{j} \frac{(mh)^{j-1} \rho(l)\, l!}{(q-j+1) l^k}$$
$$\times S_2(q-j+1,l)(-1)^{l+q-j+1} \big((\zeta h) m^{q-j} + (m\eta) h^{q-j} \big) \overline{B}_j\left(\frac{\eta}{h} + \frac{\zeta}{m} \right).$$

Proof. From (46), Lemma 13, Lemma 14 and Theorem 15, we have
(55)

$$
\begin{aligned}
&hm^q W_{q,\rho}^{(k)}(h,m) + mh^q W_{q,\rho}^{(k)}(m,h)\\
&= hm^q \sum_{\zeta=0}^{m-1} \frac{\zeta}{m} \overline{\beta}_{q,\rho}^{(k)}\left(\frac{h\zeta}{m}\right) + mh^q \sum_{\eta=0}^{h-1} \left(\frac{\eta}{h}\right) \overline{\beta}_{q,\rho}^{(k)}\left(\frac{m\eta}{h}\right)\\
&= hm^q \sum_{\zeta=0}^{m-1} \left(\frac{\zeta}{m}\right) \sum_{j=0}^{q} h^{j-1} \binom{q}{j} \sum_{\eta=0}^{h-1} \sum_{l=1}^{q-j+1} \frac{\rho(l)\ l!\ S_2(q-j+1,l)}{(q-j+1)l^k} (-1)^{l+q-j+1} \overline{B}_j\left(\frac{\zeta}{m}+\frac{\eta}{h}\right)\\
&+ mh^q \sum_{\eta=0}^{h-1} \left(\frac{\eta}{h}\right) \sum_{j=0}^{q} m^{j-1} \binom{q}{j} \sum_{\zeta=0}^{m-1} \sum_{l=1}^{q-j+1} \frac{\rho(l)\ l!\ S_2(q-j+1,l)}{(q-j+1)l^k} (-1)^{l+q-j+1} \overline{B}_j\left(\frac{\eta}{h}+\frac{\zeta}{m}\right)\\
&= \sum_{\zeta=0}^{m-1} \left(\frac{\zeta}{m}\right) \sum_{j=0}^{q} m^{q-j}(mh)^j \binom{q}{j} \sum_{\eta=0}^{h-1} \sum_{l=1}^{q-j+1} \overline{B}_j\left(\frac{\zeta}{m}+\frac{\eta}{h}\right) \frac{\rho(l)\ l!\ S_2(q-j+1,l)(-1)^{l+q-j+1}}{(q-j+1)l^k}\\
&+ \sum_{\eta=0}^{h-1} \left(\frac{\eta}{h}\right) \sum_{j=0}^{q} h^{q-j}(mh)^j \binom{q}{j} \sum_{\zeta=0}^{m-1} \sum_{l=1}^{q-j+1} \overline{B}_j\left(\frac{\eta}{h}+\frac{\zeta}{m}\right) \frac{\rho(l)\ l!\ S_2(q-j+1,l)(-1)^{l+q-j+1}}{(q-j+1)l^k}\\
&= \sum_{\zeta=0}^{m-1} \sum_{j=0}^{q} \sum_{\eta=0}^{h-1} \sum_{l=1}^{q-j+1} (\zeta h)(mh)^{-1} m^{q-j}(mh)^j \binom{q}{j} \overline{B}_j\left(\frac{\zeta}{m}+\frac{\eta}{h}\right) \frac{\rho(l)\ l!\ S_2(q-j+1,l)(-1)^{l+q-j+1}}{(q-j+1)l^k}\\
&+ \sum_{\zeta=0}^{m-1} \sum_{j=0}^{q} \sum_{\eta=0}^{h-1} \sum_{l=1}^{q-j+1} (m\eta)(mh)^{-1} h^{q-j}(mh)^j \binom{q}{j} \overline{B}_j\left(\frac{\eta}{h}+\frac{\zeta}{m}\right) \frac{\rho(l)\ l!\ S_2(q-j+1,l)(-1)^{l+q-j+1}}{(q-j+1)l^k}\\
&= \sum_{\zeta=0}^{m-1} \sum_{j=0}^{q} \sum_{\eta=0}^{h-1} \sum_{l=1}^{q-j+1} \frac{(mh)^{j-1}\binom{q}{j}\rho(l)\ l!\ S_2(q-j+1,l)(-1)^{l+q-j+1}}{(q-j+1)l^k}\\
&\qquad \times \left((\zeta h)m^{q-j} + (m\eta)h^{q-j}\right)\overline{B}_j\left(\frac{\eta}{h}+\frac{\zeta}{m}\right).
\end{aligned}
$$

Therefore, from (55), we arrive at what we want. □

When $k = 1$, we get the following reciprocity relation for the generalized unipoly-Dedekind sums defined by Apostol.

Corollary 17. *For $m, h, q \in \mathbb{N}$, we have*

$$
\begin{aligned}
hm^q W_{q,1}^{(1)}(h,m) + mh^q W_{q,1}^{(1)}(m,h) &= hm^q T_q(h,m) + mh^q T_q(m,h)\\
&= \sum_{\zeta=0}^{m-1} \sum_{\eta=0}^{h-1} (mh)^{q-1}(\zeta h + m\eta)\overline{B}_q\left(\frac{\eta}{h}+\frac{\zeta}{m}\right).
\end{aligned}
$$

4. Conclusion

We introduced the unipoly-Dedekind sums associated the type 2 unipoly-Bernoulli functions of index k and the unipoly-Dedekind sums associated the unipoly-Bernoulli functions of index k. For each of these unipoly-Dedekind sums, various identities and the reciprocity relations were shown.

Following this study, we suggest the following open problem:

What is the reciprocity relations of the unpoly-DC Dedekind sums?

We would like to further study into properties of certain Dedekind sums and their applications to physics and engineering as well as mathematics.

Acknowledgments
The authors thank Jangjeon Research Institute for Mathematical Sciences for the support of this research.

Availability of data and material
Not applicable. This paper does not use data.

Funding This work was supported by the Basic Science Research Program, the National Research Foundation of Korea, (NRF-2021R1F1A1050151).

Ethics approval and consent to participate
The author declare that there is no ethical problem in the production of this paper.

REFERENCES

[1] Apostol, T. M.: Generalized Dedekind sums and transformation formulae of certain Lambert series, Duke Math. J. **17**, 147-157 (1950)
[2] Apostol, T.M.: Theorems on generalized Dedekind sums. Pac. J. Math. 2, 1-9 (1952)
[3] Bayad, A.: On values of Jacobi forms and elliptic Dedekind sums, Proc. Jangjeon Math. Soc. 12, no. 2, 119-139 (2009)
[4] Beck, M., Diaz, R., Robins, S.: The Frobenius problem, rational polytopes, and Fourier Dedekind sums, to appear in J. Number Th.
[5] Can, M., Cenkci, M., Kurt, V.i, Simsek, Y.: Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler l -functions, Adv. Stud. Contemp. Math. (Kyungshang) **18**, no. 2, 135-160 (2009)
[6] Carlitz, L.: Some theorems on generalized Dedekind-Rademacher sums, Pacific J. Math. **75**, no. 2, 347-358 (1978)
[7] Dedekind, R.: Erlauterungen zu Riemann's Fragmenten uber die Grenzfalle der elliptischen Funktionen, Gesammelte mathematische Werke, Braunschweig, vol. 1, 159-173 (1930)
[8] Hirzebruch, F., Zagier, D.: The Atiyah-Singer theorem and elementary number theory, Publish or Perish, Boston (1974)
[9] Kim, D. S., Kim, T.: A note on polyexponential and unipoly functions, Russ. J. Math. Phys. **26**, no. 1, 40-49 (2019)
[10] Kim, T.: Note on q-Dedekind-type sums related to q-Euler polynomials, Glasg. Math. J. **54**, no. 1, 121-125 (2012)
[11] Kim, T.: Note on Dedekind type DC sums, Adv. Stud. Contemp. Math. (Kyungshang) **18**, no. 2, 249-260 (2009)
[12] Kim, T.: A note on p-adic q-Dedekind sums, C. R. Acad. Bulgare Sci. **54**, no. 10, 37-42 (2001)
[13] Kim, T., Kim, D. S.: Degenerate polyexponential functions and degenerate Bell polynomials, J. Math. Anal. Appl. **487**, no. 2, 124017, 15 pp (2020)
[14] Kim, T., Kim, D. S., Kwon, J., Lee, H.: Degenerate polyexponential functions and type 2 degenerate poly-Bernoulli numbers and polynomials, Adv. Difference Equ., Paper No. 168, 12 pp (2020)
[15] Kim, T., Kim, D. S., Lee, H., Jang, L-C.: Identities on poly-Dedekind sums, Adv. Difference Equ. **2020:563**, 13 pp (2020)
[16] Knuth, D.: The art or computer programming, vol. 2, Addison-Wesley Reading, Mass., (1981)
[17] Ma, Y., Kim, D. S., Lee, H., Kim, T.: Poly-Dedekind sums with poly-bernoulli fuctions, J. Inequal. and Appl. **2020:248**, 10 pp (2020)
[18] Meyer, C.: Uber einige Anwendungen Dedekindscher Summen, J. reine angewandte Math. **198**, 143-203 (1957).
[19] Mordell, L. J.: Lattice points in a tetrahedron and generalized Dedekind sums, J. Indian Math. **15**, 41-46 (1951)
[20] Rademacher, H.: Zur Theorie der Modulfunktionen, J. Reine Angew. Math., **167**, 312-336 (1932)
[21] Rademacher, H., Whiteman, Theorems on Dedekind sum, Amer. J. Math., **63**, 377-407(1941)
[22] Simsek, Y.: Remarks on reciprocity laws of the Dedekind and Hardy sums, Adv. Stud. Contemp. Math. (Kyungshang) **12**, no. 2, 237-246 (2006)

DEPARTMENT OF MATHEMATICS EDUCATION, DAEGU CATHOLIC UNIVERSITY, GYEONGSAN 38430, REPUBLIC OF KOREA
E-mail address: hkkim@cu.ac.kr