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SOME IDENTITIES OF THE POLY-CHANGHEE AND UNIPOLY CHANGEE POLYNOMIALS

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Abstract. Recently, several authors studied the degenerate q-special polynomials related to L. Carlitz's degenerate polynomials(see [11-25]). In this paper, we introduced the Carlitz's type degenerate q-Changhee numbers and polynomials. Also, we study some explicit identities and properties for the Carlitz's type degenerate q-Changhee numbers and polynomials arising from p-adic invariant q-integral on $\mathbb{Z}_p$.

1. Introduction

In Ars Conjectandi [2] published in 1713, J. Bernoulli introduced the Bernoulli numbers in connection to the study of the sums of powers of consecutive integers, and those numbers and polynomials have been generalized by many researchers. Luo and Srivastava defined the Apostol-Bernoulli polynomials and investigated the relationship between Gaussian hypergeometric function, Hurwitz function and those polynomials in [26]. In [6], Frappier defined a generalized Bernoulli polynomials and found a generalization of a Fourier series representation of generalized Bernoulli polynomials. Furthermore, Natalini and Bernardini defined another generalized Bernoulli polynomials and showed that if a differential equation with those polynomials is of order n, then all the considered families of polynomials are solutions of differential operators of infinite order (see [27]). Arakawa, Ibukiyama and Kaneko defined the poly-Bernoulli polynomials and found explicit formula of those polynomials in [1]. Khan, Araci, Acikgoz and Esi defined Laguerre-based Hermit-Bernoulli polynomials in [8] and derived summation formulas and related bilateral series associated with the newly introduced generating function. Kim, Kim, Jang and Kim defined type 2 degenerate polynomials and studied relationships between the Stirling numbers, Bernoulli polynomials, Daehee polynomials and those polynomials in [19].

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The *Bernoulli polynomials of the second kind* are defined by the generating function to be

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}, \quad \text{(see [11, 29, 30])}. \tag{1.1}$$

In particular, $x = 0$, $b_n = b_n(0)$ are called the *Bernoulli numbers of the second kind*.

From (1.1), we note that

$$\left(\frac{t}{\log(1+t)}\right)^r (1+t)^{x-1} = \sum_{n=0}^{\infty} B_n^{(n-r+1)}(x)\frac{t^n}{n!}, \quad \text{(see [11])}. \tag{1.2}$$

Where $B_n^{(r)}(x)$ are the *the higher-order Bernoulli polynomials* which are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}, \quad \text{(see [12])}. \tag{1.3}$$

When $x = 0$, $B_n^{(k)} = B_n^{(k)}(0)$ are called the *Bernoulli numbers of order* r. In particular, when $x = 1$ and $r = 1$, $b_n = B_n^{(n)}(1)$.

As is well-known, the *Changhee polynomias* are defined by the generating function to be

$$\sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!} = \frac{2}{2+t}(1+t)^x, \quad \text{(see [5, 12, 13, 15, 16, 17, 24])}. \tag{1.4}$$

In the special case $x = 0$, $Ch_n = Ch_n(0)$ called the *Changhee numbers*, and Kim and Kim defined the *type* 2 *Changhee polynomials* to be

$$\sum_{n=0}^{\infty} c_n(x)\frac{t^n}{n!} = \frac{2}{(1+t)+(1+t)^{-1}}(1+t)^x \text{ (see [16, 17, 24])}. \tag{1.5}$$

In the special case $x = 0$, $c_n = c_n(0)$ are called the *type* 2 *Changhee numbers*.

The *Daehee polynomials* are defined by the generating function to be

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, \quad \text{(see [14, 16, 17, 24])}. \tag{1.6}$$

When $x = 0$, $D_n = D_n(0)$ are called the *Daehee numbers*, and the *type* 2 *Daehee polynomials* are defined by the generating function to be

$$\sum_{n=0}^{\infty} d_n(x)\frac{t^n}{n!} = \frac{\log(1+t)}{(1+t)-(1+t)^{-1}}(1+t)^x, \quad \text{(see [14, 17, 24])}. \tag{1.7}$$

In the special case $x = 0$, $c_n = c_n(0)$ are called the *type* 2 *Daehee numbers*.

For a nonnegative integer k, the *poly-logarithm* is defined by

$$\mathrm{Li}_k(x)=\sum_{m=1}^{\infty}\frac{x^m}{m^k},\quad \text{(see [7, 9])}. \tag{1.8}$$

Recently, Kim and Kim introduced the *modified poly-exponential function* as

$$\mathrm{Ei}_k(x)=\sum_{n=1}^{\infty}\frac{x^n}{(n-1)!n^k},\quad \text{(see [9])}. \tag{1.9}$$

From (1.9), we get $\mathrm{Ei}_1(x)=e^x-1$.

For a nonnegative integer n, the *Stirling numbers of the first kind* are defined by

$$(x)_n=\sum_{l=0}^{n}S_1(n,l)x^l,\ \text{(see [3, 10, 22, 28])}, \tag{1.10}$$

where $(x)_0=x$, $(x)_n=x(x-1)\cdots(x-n+1)$, $(n\geq 0)$. From (1.10), we easily derive the following equation:

$$\frac{1}{n!}\left(\log(1+t)\right)^n=\sum_{k=n}^{\infty}S_1(k,n)\frac{t^k}{k!},\ \text{(see [3, 10, 22, 28])}. \tag{1.11}$$

For given nonnegative integer n, the *Stirling numbers of the second kind* are defined by

$$x^n=\sum_{l=0}^{n}S_2(n,l)(x)_l,,\ \text{(see [3, 10, 22, 28])}. \tag{1.12}$$

By (1.12), it is to see easily that

$$\left(e^t-1\right)^n=n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!},,\ \text{(see [3, 10, 22, 28])}. \tag{1.13}$$

Recently, many authors investigated Changhee polynomials or type 2 Changhee polynomials which are related closely to the Euler polynomials, type 2 Euler polynomials, the Stirling numbers and special polynomials and numbers (see [5, 12, 13, 16, 17, 24]). As a generalization of some special polynomials, authors used the poly-exponential function, poly-logarithm function and unipoly function, and showed that those polynomials were represented a linear combinations of some special numbers and polynomials (see [4, 7, 9, 18, 20, 21, 23, 25]).

In this paper, we define the poly-Changhee polynomials and unipoly Changhee polynomials by using poly-exponential function and unipoly function, and derive some identities and properties of those polynomials.

2. Poly-Changhee polynomials and numbers

In this section, we consider the *poly-Changhee polynomials* by means of the poly-exponential functions as follows:

$$\frac{\mathrm{Ei}_k\left(\log(1+2t)\right)}{t(2+t)}(1+t)^x=\sum_{n=0}^{\infty}Ch_n^{(k)}(x)\frac{t^n}{n!}. \tag{2.1}$$

When $x=0$,$Ch_n^{(k)}=Ch_n^{(k)}(0)$ are called the *poly-Changhee numbers*.

From (1.9), we have

$$\begin{aligned}\mathrm{Ei}_k\left(\log(1+2t)\right)&=\sum_{n=1}^{\infty}\frac{1}{(n-1)!n^k}\left(\log(1+2t)\right)^n\\&=\sum_{n=1}^{\infty}\frac{1}{n^{k-1}}\frac{1}{n!}n!\sum_{l=n}^{\infty}S_1(l,n)\frac{2^lt^l}{l!}\\&=\sum_{n=0}^{\infty}\sum_{m=1}^{n+1}\frac{2^{n+1}}{m^{k-1}}S_1(n+1,m)\frac{t^{n+1}}{(n+1)!}.\end{aligned} \tag{2.2}$$

By (2.1) and (2.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty}Ch_n^{(k)}(x)\frac{t^n}{n!}=&\frac{1}{t}\left(\frac{1}{2+t}(1+t)^x\right)\mathrm{Ei}_k\left(\log(1+2t)\right)\\=&\frac{1}{t}\left(\frac{(1+t)^x}{2+t}\right)\left(\sum_{n=0}^{\infty}\sum_{m=1}^{n+1}\frac{2^{n+1}}{m^{k-1}}S_1(n+1,m)\frac{t^{n+1}}{(n+1)!}\right)\\=&\left(\frac{2}{2+t}(1+t)^x\right)\left(\sum_{n=0}^{\infty}\sum_{m+1}^{n+1}\frac{2^{n}}{m^{k-1}}S_1(n+1,m)\frac{t^{n}}{(n+1)!}\right)\\=&\left(\sum_{n=0}^{\infty}Ch_n(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{m+1}^{n+1}\frac{2^{n}}{m^{k-1}}S_1(n+1,m)\frac{t^{n}}{(n+1)!}\right)\\=&\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{2^lS_1(l+1,m)}{(l+1)m^{k-1}}Ch_{n-l}(x)\right)\frac{t^n}{n!},\end{aligned} \tag{2.3}$$

and thus by (2.3), we obtain the following theorem.

Theorem 2.1. *For each nonnegative integer n and each integer k, we have*

$$Ch_n^{(k)}(x)=\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{2^l}{(l+1)m^{k-1}}S_1(l+1,m)Ch_{n-l}(x).$$

In the special case of the Theorem 2.1, if we put $x = 0$, then we obtain the following corollary.

Corollary 2.2. *For each nonnegative integer n and each integer k, we have*

$$Ch_n^{(k)} = \sum_{l=0}^{n} \sum_{m=1}^{l+1} \binom{n}{l} \frac{2^l}{(l+1)m^{k-1}} S_1(l+1, m) Ch_{n-l}.$$

Since $Ch_n^{(1)}(x) = Ch_n(x)$, by the Corollary 2.2, we obtain the following corollary.

Corollary 2.3. *For each nonnegative integer n and each integer k, we have*

$$Ch_n^{(1)} = \sum_{l=0}^{n} \sum_{m=1}^{l+1} \binom{n}{l} \frac{2^l}{l+1} S_1(l+1, m) Ch_{n-l}.$$

Moreover,

$$\sum_{l=1}^{n} \sum_{m=1}^{l+1} \binom{n}{l} \frac{2^l}{l+1} S_1(l+1, m) Ch_{n-l} = 0.$$

By (2.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} Ch_n^{(k)}(x) \frac{t^n}{n!} &= \left(\frac{\mathrm{Ei}_k\left(\log(1+2t)\right)}{t(2+t)} \right) (1+t)^x \\ &= \left(\sum_{n=0}^{\infty} Ch_n^{(k)} \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} (x)_n \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} Ch_{n-m}^{(k)}(x)_m \right) \frac{t^n}{n!}. \end{aligned} \tag{2.4}$$

Therefore, by (2.4), we obtain the following theorem.

Theorem 2.4. *For each nonnegative integer n and each integer k, we have*

$$Ch_n^{(k)}(x) = \sum_{m=0}^{n} \binom{n}{m} Ch_{n-m}^{(k)}(x)_m.$$

By the definition of poly-Changhee polynomials and (2.2), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} Ch_n^{(k)}(x)\frac{t^n}{n!} = \frac{\mathrm{Ei}_k\left(\log(1+2t)\right)}{t(2+t)}(1+t)^x \\
=&\frac{(1+t)^x}{t\left((1+t)^2-1\right)}\left((1+t)-1\right)\mathrm{Ei}_k\left(\log(1+2t)\right) \\
=&\frac{(1+t)^{x-1}\log(1+t)}{(1+t)-(1+t)^{-1}}\frac{1}{\log(1+t)}\left(\sum_{n=0}^{\infty}\sum_{m=1}^{n+1}\frac{2^{n+1}}{m^{k-1}}S_1(n+1,m)\frac{t^{n+1}}{(n+1)!}\right) \\
=&\frac{(1+t)^{x}\log(1+t)}{(1+t)-(1+t)^{-1}}\frac{t}{\log(1+t)}\frac{1}{1+t}\left(\sum_{n=0}^{\infty}\sum_{m=1}^{n+1}\frac{2^{n+1}}{m^{k-1}}\frac{S_1(n+1,m)}{n+1}\frac{t^{n}}{n!}\right) \\
=&\left(\sum_{n=0}^{\infty}d_n(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}b_n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(-1)^n t^n\right)\left(\sum_{n=0}^{\infty}\sum_{m=1}^{n+1}\frac{2^{n+1}}{m^{k-1}}\frac{S_1(n+1,m)}{n+1}\frac{t^{n}}{n!}\right) \\
=&\sum_{n=0}^{\infty}\left(\sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{l=0}^{r}\sum_{m=1}^{s-r+1}\binom{n}{s}\binom{s}{r}\binom{r}{l}\frac{(-1)^{n-s}2^{s-r+1}(n-s)!}{(s-r+1)m^{k-1}}S_1(s-r+1,m)b_{r-l}d_l(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.5}$$

Therefore, by (2.5), we obtain the following theorem.

Theorem 2.5. *For each nonnegative integer n and each integer k, we have*

$$Ch_n^{(k)}(x) = \sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{l=0}^{r}\sum_{m=1}^{s-r+1}\binom{n}{s}\binom{s}{r}\binom{r}{l}\frac{(-1)^{n-s}2^{s-r+1}(n-s)!}{(s-r+1)m^{k-1}}S_1(s-r+1,m)b_{r-l}d_l(x)$$

By the Theorem 2.5, we obtain the following corollary.

Corollary 2.6. *For each nonnegative integer n and each integer k, we have*

$$Ch_n^{(k)} = \sum_{r=0}^{n}\sum_{l=0}^{r}\sum_{m=1}^{n-r+1}\binom{n}{r}\binom{r}{l}\frac{2^{n-r+1}}{(n-r+1)m^{k-1}}S_1(n-r+1,m)b_{r-l}d_l.$$

Note that

$$\begin{aligned}
\frac{d}{dt}Ei_k\left(\log(1+2t)\right) =&\frac{d}{dt}\sum_{n=1}^{\infty}\frac{\left(\log(1+2t)\right)^n}{(n-1)!n^k} \\
=&\frac{2}{1+2t}\sum_{n=1}^{\infty}\frac{n\left(\log(1+2t)\right)^{n-1}}{(n-1)!n^k} \\
=&\frac{2}{(1+2t)\log(1+2t)}\sum_{n=1}^{\infty}\frac{\left(\log(1+2t)\right)^n}{(n-1)!n^{k-1}} \\
=&\frac{2}{(1+2t)\log(1+2t)}e_{k-1}\left(\log(1+2t)\right).
\end{aligned} \tag{2.6}$$

Since the following identity is well-known fact that

$$\frac{t}{(1+t)\log(1+t)} = \sum_{n=0}^{\infty} B_n^{(n)} \frac{t^n}{n!}, \text{ (see [?])}, \tag{2.7}$$

by (2.6) and (2.7), we get

$$\begin{aligned}
&\mathrm{Ei}_k\left(\log(1+2t)\right) \\
&= \int_0^t \frac{2}{(1+2x)\log(1+2x)} \\
&\quad \times \underbrace{\int_0^x \frac{2}{(1+2x)\log(1+2x)} \cdots \int_0^x \frac{2}{(1+2x)\log(1+2x)}}_{(k-2)-times} \int_0^x \frac{4x}{(1+2x)\log(1+2x)} dx dx \cdots dx \\
&= \sum_{l=0}^{\infty} \sum_{l_1+\cdots+l_{k-1}=l} \frac{2^{l+1} l!}{l_1! l_2! \cdots l_{k-1}!} \frac{B_{l_1}^{(l_1)}}{l_1+1} \frac{B_{l_2}^{(l_2)}}{l_1+l_2+1} \cdots \frac{B_{l_{k-1}}^{(l_{k-1})}}{l_1+l_2+\cdots+l_{k-1}+1} \frac{t^{l+1}}{l!}.
\end{aligned} \tag{2.8}$$

By (2.8), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} Ch_n^{(k)} \frac{t^n}{n!} = \frac{\mathrm{Ei}_k\left(\log(1+2t)\right)}{t(2+t)} \\
&= \frac{2}{2+t} \sum_{l=0}^{\infty} \sum_{l_1+\cdots+l_{k-1}=l} 2^l \binom{l}{l_1, l_2 \ldots, l_{k-1}} \frac{B_{l_1}^{(l_1)}}{l_1+1} \frac{B_{l_2}^{(l_2)}}{l_1+l_2+1} \cdots \frac{B_{l_{k-1}}^{(l_{k-1})}}{l_1+\cdots+l_{k-1}+1} \frac{t^l}{l!} \\
&= \left(\sum_{n=0}^{\infty} Ch_n \frac{t^n}{n!}\right) \left(\sum_{l=0}^{\infty} \sum_{l_1+\cdots+l_{k-1}=l} 2^l \binom{l}{l_1, l_2 \ldots, l_{k-1}} \frac{B_{l_1}^{(l_1)}}{l_1+1} \frac{B_{l_2}^{(l_2)}}{l_1+l_2+1} \cdots \frac{B_{l_{k-1}}^{(l_{k-1})}}{l_1+\cdots+l_{k-1}+1} \frac{t^l}{l!}\right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{l_1+\cdots+l_{k-1}=l} 2^l \binom{n}{l} \binom{l}{l_1, l_2 \ldots, l_{k-1}} \frac{B_{l_1}^{(l_1)}}{l_1+1} \frac{B_{l_2}^{(l_2)}}{l_1+l_2+1} \cdots \frac{B_{l_{k-1}}^{(l_{k-1})}}{l_1+\cdots+l_{k-1}+1} Ch_{n-l}\right) \frac{t^n}{n!}.
\end{aligned} \tag{2.9}$$

Therefore, by (2.9), we obtain the following theorem.

Theorem 2.7. *For each $n \in \mathbb{N} \cup \{0\}$ and each $k \in \mathbb{Z}$, we have*

$$Ch_n^{(k)} = \sum_{l=0}^{n} 2^l \sum_{l_1+\cdots+l_{k-1}=l} \binom{n}{l} \binom{l}{l_1, l_2 \ldots, l_{k-1}} \frac{B_{l_1}^{(l_1)}}{l_1+1} \frac{B_{l_2}^{(l_2)}}{l_1+l_2+1} \cdots \frac{B_{l_{k-1}}^{(l_{k-1})}}{l_1+\cdots+l_{k-1}+1} Ch_{n-l}.$$

In particular,

$$Ch_n^{(2)} = \sum_{l=0}^{n} 2^l \binom{n}{l} \frac{B_l^{(l)}}{l+1} Ch_{n-l}.$$

Note that

$$\begin{aligned}
\mathrm{Ei}_k\left(\log(1+2t)\right) =& t(2+t)\sum_{n=0}^{\infty} Ch_n^{(k)}\frac{t^n}{n!} \\
=&\sum_{n=0}^{\infty} 2Ch_n^{(k)}\frac{t^{n+1}}{n!}+\sum_{n=0}^{\infty} Ch_n^{(k)}\frac{t^{n+2}}{n!} \\
=&2Ch_0^{(k)}t+\sum_{n=2}^{\infty}\left(2nCh_{n-1}^{(k)}+n(n-1)Ch_{n-2}^{(k)}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.10}$$

Hence, by (2.2) and (2.10), we obtain the following theorem.

Theorem 2.8. *For each $n \geq 2$ and each $k \in \mathbb{Z}$, we have*

$$2Ch_{n-1}^{(k)}+(n-1)Ch_{n-2}^{(k)}=\frac{2^n}{n}\sum_{m=1}^{n}\frac{S_1(n,m)}{m^{k-1}}.$$

By replacing t to $\frac{1}{2}\left(e^t-1\right)$ in (2.1), we have

$$\begin{aligned}
&\frac{2}{\left(e^t-1\right)\left(2+\frac{1}{2}\left(e^t-1\right)\right)}\mathrm{Ei}_k(t) \\
=&\left(\frac{t}{e^t-1}\right)\left(\frac{2}{2+\frac{1}{2}\left(e^t-1\right)}\right)\frac{1}{t}\sum_{n=1}^{\infty}\frac{t^n}{(n-1)!n^k} \\
=&\left(\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}Ch_n\frac{\left(\frac{1}{2}\left(e^t-1\right)^n\right)^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{t^n}{n!(n+1)^k}\right) \\
=&\left(\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{Ch_m}{2^m}S_2(n,m)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{t^n}{n!(n+1)^k}\right) \\
=&\sum_{n=0}^{\infty}\left(\sum_{a=0}^{n}\sum_{r=0}^{a}\sum_{m=0}^{r}\binom{n}{a}\binom{a}{r}\binom{r}{m}\frac{S_2(r,m)}{(n-a+1)^k2^m}Ch_mB_{a-r}\right)\frac{t^n}{n!},
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty}Ch_n^{(k)}\frac{\left(e^t-1\right)^n}{2^n n!} \\
=&\sum_{n=0}^{\infty}\frac{Ch_n^{(k)}}{2^n}\frac{1}{n!}n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!} \\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{Ch_m^{(k)}}{2^m}S_2(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.12}$$

Thus, by (2.11) and (2.12), we obtain the following theorem.

Theorem 2.9. *For each nonnegative integer n and each integer k, we have*

$$\sum_{m=0}^{n}\frac{Ch_m^{(k)}}{2^m}S_2(n,m)=\sum_{a=0}^{n}\sum_{r=0}^{a}\sum_{m=0}^{r}\binom{n}{a}\binom{a}{r}\binom{r}{m}\frac{S_2(r,m)}{2^m(n-a+1)^k}Ch_mB_{a-r}.$$

From (2.2), we observe that

$$\begin{aligned}
&\sum_{n=0}^{\infty}Ch_n^{(k)}(x)\frac{t^n}{n!}\\
=&\frac{\mathrm{Ei}_k\left(\log(1+2t)\right)}{t(2+t)}(1+t)^x\\
=&\frac{(1+t)^x}{t\left((1+t)^2-1\right)}\left((1+t)-1\right)Ei_k\left(\log(1+2t)\right)\\
=&\frac{(1+t)^{x-1}\log(1+t)}{(1+t)-(1+t)^{-1}}\frac{1}{\log(1+t)}\left(\sum_{n=0}^{\infty}\sum_{m=1}^{n+1}\frac{2^{n+1}}{m^{k-1}}S_1(n+1,m)\frac{t^{n+1}}{(n+1)!}\right)\\
=&\frac{(1+t)^x\log(1+t)}{(1+t)-(1+t)^{-1}}\frac{t}{\log(1+t)}\frac{1}{1+t}\left(\sum_{n=0}^{\infty}\sum_{m=1}^{n+1}\frac{2^{n+1}}{m^{k-1}}\frac{S_1(n+1,m)}{n+1}\frac{t^n}{n!}\right)\\
=&\left(\sum_{n=0}^{\infty}d_n(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}b_n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(-1)^nt^n\right)\left(\sum_{n=0}^{\infty}\sum_{m=1}^{n+1}\frac{2^{n+1}}{m^{k-1}}\frac{S_1(n+1,m)}{n+1}\frac{t^n}{n!}\right)\\
=&\sum_{n=0}^{\infty}\left(\sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{l=0}^{r}\sum_{m=1}^{s-r+1}\binom{n}{s}\binom{s}{r}\binom{r}{l}\frac{(-1)^{n-s}2^{s-r+1}(n-s)!}{(s-r+1)m^{k-1}}S_1(s-r+1,m)b_{r-l}d_l(x)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.13}$$

3. Unipoly Changhee polynomials and numbers

Let $p(n)$ be a arithmetic function which is a real or complex valued function defined on the set of positive integers $\mathbb{N}$. In [9], Kim and Kim definded the *unipoly function* attached to $p(n)$ by

$$u_k(x|p)=\sum_{n=1}^{\infty}\frac{p(n)x^n}{n^k},\ (k\in\mathbb{Z}).\tag{3.1}$$

In particular, if we put $p(n)=1$ for each $n\in\mathbb{N}$, then

$$u_k(x|1)=\sum_{n=1}^{\infty}\frac{x^n}{n^k}=Li_k(x)$$

is ordinary polylogarithm function.

Note that for $k \geq 2$,

$$\begin{aligned}\frac{d}{dx}u_k(x|p) =& \sum_{n=1}^{\infty}\frac{p(n)}{n^k}nx^{n-1} \\ =& \frac{1}{x}\sum_{n=1}^{\infty}\frac{p(n)x^n}{n^{k-1}} \\ =& \frac{1}{x}u_{k-1}(x|p),\end{aligned} \tag{3.2}$$

and by (3.2), we get

$$u_k(x|p) = \int_0^x \frac{1}{t}\underbrace{\int_0^t \frac{1}{t}\cdots\int_0^t \frac{1}{t}}_{(k-2)-times} u_1(t|p)dtdt\cdots dt. \tag{3.3}$$

By using (3.1), we define the *unipoly Changhee polynomials* attached p as follows:

$$\frac{(1+t)^x}{t(2+t)}u_k\left(\log(1+2t)|\, p\right) = \sum_{n=0}^{\infty} Ch_{n,p}^{(k)}(x)\frac{t^n}{n!}. \tag{3.4}$$

When $x = 0$, $Ch_{n,p}^{(k)} = Ch_{n,p}^{(k)}(0)$ are called the *unipoly Changhee numbers* attached p.

If we put $p(n) = \frac{1}{\Gamma(n)}$, then

$$\begin{aligned}\sum_{n=0}^{\infty} Ch_{n,\frac{1}{\Gamma(n)}}^{(k)}(x)\frac{t^n}{n!} =& \frac{1}{t(2+t)}(1+t)^x u_k\left(\log(1+2t)\,\middle|\,\frac{1}{\Gamma}\right) \\ =& \frac{1}{t(2+t)}(1+t)^x\sum_{n=1}^{\infty}\frac{(\log(1+2t))^n}{n^k(n-1)!} \\ =& \frac{Ei_k\left(\log(1+2t)\right)}{t(2+t)}(1+t)^x \\ =& \sum_{n=0}^{\infty} Ch_n^{(k)}(x)\frac{t^n}{n!},\end{aligned} \tag{3.5}$$

and so we have the following theorem.

Theorem 3.1. *For each nonnegative integer n and each integer k, if we put $p(n) = \frac{1}{\Gamma(n)}$, then*

$$Ch_{n,p}^{(k)}(x) = Ch_n^{(k)}(x).$$

By (3.4), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} Ch_{n,p}^{(k)}\frac{t^n}{n!}\\
=&\frac{1}{t(2+t)}\sum_{n=1}^{\infty}\frac{p(n)}{n^k}\left(\log(1+2t)\right)^n\\
=&\frac{1}{t(2+t)}\sum_{n=0}^{\infty}\frac{p(n+1)}{(n+1)^k}(n+1)!\sum_{l=n+1}^{\infty}S_1(l,n+1)\frac{(2t)^l}{l!}\\
=&\left(\frac{2}{2+t}\right)\left(\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\frac{p(n+1)(n+1)!}{(n+1)^k}S_1(n+l+1,n+1)\frac{(2t)^{n+l}}{(n+l+1)!}\right)\\
=&\left(\sum_{n=0}^{\infty}Ch_n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{p(m+1)(m+1)!2^n}{(m+1)^k}\frac{S_1(n+1,m+1)}{n+1}\frac{t^n}{n!}\right)\\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{r=0}^{m}\binom{n}{m}\frac{2^m p(r+1)(r+1)!}{(r+1)^k}\frac{S_1(m+1,r+1)}{m+1}Ch_{n-m}\right)\frac{t^n}{n!},
\end{aligned} \tag{3.6}$$

and thus, by (3.6), we obtain the following theorem.

Theorem 3.2. *For each nonnegative integer n and each integer k, we have*

$$Ch_{n,p}^{(k)}=\sum_{m=0}^{n}\sum_{r=0}^{m}\binom{n}{m}\frac{2^m p(r+1)(r+1)!}{(r+1)^k}\frac{S_1(m+1,r+1)}{m+1}Ch_{n-m}.$$

In particular,

$$Ch_{n,\frac{1}{\Gamma}}^{(k)}=\sum_{m=0}^{n}\sum_{r=0}^{m}\binom{n}{m}\frac{2^m S_1(m+1,r+1)}{(m+1)(r+1)^{k-1}}Ch_{n-m}.$$

By (3.4), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} Ch_{n,p}^{(k)} \frac{t^n}{n!} \\
=&\frac{1}{t(2+t)} u_k\left(\log(1+2t) \mid p\right) \\
=&\frac{1}{t(2+t)} \sum_{n=1}^{\infty} \frac{p(n)}{n^k}\left(\log(1+2t)\right)^n \\
=&\left(\frac{\log(1+2t)}{2t}\right)\left(\frac{2}{2+t}\right)\left(\sum_{n=0}^{\infty} \frac{p(n+1)}{(n+1)^k}\left(\log(1+2t)\right)^n\right) \\
=&\left(\sum_{n=0}^{\infty} D_n \frac{(2t)^n}{n!}\right)\left(\sum_{n=0}^{\infty} Ch_n \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{p(m+1)m!}{(m+1)^k} S_1(n,m) \frac{2^n t^n}{n!}\right) \\
=&\sum_{n=0}^{\infty}\left(\sum_{u=0}^{n}\sum_{r=0}^{u}\sum_{m=0}^{n-u} \binom{n}{u}\binom{u}{r} 2^{n-u+r} \frac{p(m+1)m!}{(m+1)^k} S_1(n-u,m) D_r Ch_{u-r}\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.7}$$

Therefore, by (3.7), we obtain the following theorem.

Theorem 3.3. *For each $n \in \mathbb{N} \cup \{0\}$ and each $k \in \mathbb{Z}$, we have*

$$Ch_{n,p}^{(k)} = \sum_{u=0}^{n}\sum_{r=0}^{u}\sum_{m=0}^{n-u} \binom{n}{u}\binom{u}{r} 2^{n-u+r} \frac{p(m+1)m!}{(m+1)^k} S_1(n-u,m) D_r Ch_{u-r}.$$

4. Conclusion

The Changhee polynomials were defined by Kim in [15], and these polynomials have been generalized and studied the properties of those polynomials by many researchers. In particular, Kim and Kim defined the type 2 Changhee polynomials by using p-adic integral on $\mathbb{Z}_p$ in [17], and derived some identities of those polynomials.

As another generalization of Changhee polynomials, we define the poly-Chagnhee polynomials and unpoly Changhee polynomials by using poly-exponential function and unpoly function, and found the properties of those polynomials. In addition, we show that these polynomials are expressed linear combinations of the Stirling numbers, Bernoulli numbers of the second kind, Daehee numbers, Changhee numbers and some special polynomials.

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