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A note on derangement polynomials and degenerate derangement polynomials

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Abstract

In 1708, Pierre Rémonde de Motmort introduced the problem of counting derangement for the first time. A derangement is a permutation that has no fixed points. Recently, many researchers have studied the derangement polynomials and T. Kim introduced the degenerate derangement polynomials and investigated some identities of those polynomials. In [6], Jang-Kim-Kim-Lee introduced some identities involving derangement polynomials and numbers and moments of gamma random variables. In this paper, we study some identities and properties of the derangement polynomials and degenerate derangement polynomials and investigate the zeros of derangement polynomials. Moreover, we investigate the numerical pattern of the roots of the polynomials $D_{n,\lambda}(x)$ varying the degree of polynomials from 1 to 40.

1 Introduction

The derangement number D_n is the number of fixed point free permutations on an $n(n \geq 1)$ element set. Then the few derangement numbers $D_n(n \geq 0)$ are $1, 0, 1, 2, 9, 44, 2654, 1854, 14833, \cdots$.

For $n \geq 0$, the derangement numbers are given by

$$D_n = \sum_{k=0}^{n} \binom{n}{k} (n-k)!(-1)^k = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}, \quad \text{(see [1, 11–13])}. \tag{1.1}$$

The derangement numbers are defined by the generating function to be

$$\frac{1}{1-t} e^{-t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}, \quad \text{(see [2–4, 11, 12, 17])}. \tag{1.2}$$

From (1.2), we obtain the recurrence relation:

$$(-1)^n = D_n - nD_{n-1}, \quad (n \geq 1), \quad \text{(see [5, 8, 13, 14, 18])}.$$

The derangement polynomials, denoted by $D_n(x)$, are defined by the generating function, to be

$$\frac{e^{-t}}{1-t} e^{xt} = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}, \quad \text{(see [14])}. \tag{1.3}$$

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From (1.3), we have

$$\begin{aligned}\sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!} &= \frac{e^{-t}}{1-t}e^{xt} = \frac{1}{1-t}e^{(x-1)t} \\ &= \sum_{l=0}^{\infty} t^l \sum_{m=0}^{\infty}(x-1)^m \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\Big(n!\sum_{m=0}^{n}\frac{(x-1)^m}{m!}\Big)\frac{t^n}{n!}.\end{aligned} \tag{1.4}$$

By (1.3) and (1.4), we have

$$D_n(x) = n!\sum_{m=0}^{n}\frac{(x-1)^m}{m!}. \tag{1.5}$$

As is well known, the Stirling numbers of the first kind are given by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \tag{1.6}$$

where $(x)_n$ are defined by

$$(x)_n = \begin{cases} 1, & n=0, \\ \Big(x+1-n\Big)(x)_{n-1}, & n \geq 1. \end{cases}$$

The Stirling number of the second kind are given by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad \text{(see [7, 9, 19])}. \tag{1.7}$$

The Euler polynomials are defined by the generating function to be

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad \text{(see [9, 18])}. \tag{1.8}$$

The Bell polynomials are defined by the generating function to be

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}, \quad \text{(see [16])}. \tag{1.9}$$

For nonzero $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda^1(t) = e_\lambda(x) = (1+\lambda t)^{\frac{1}{\lambda}}, \quad \text{(see [7, 11])}. \tag{1.10}$$

Note that $\lim_{\lambda\to 0} e_\lambda^x(t) = \lim_{\lambda\to 0}(1+\lambda t)^{\frac{x}{\lambda}} = e^{xt}$.

From (1.10), we have

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \tag{1.11}$$

where

$$(x)_{n,\lambda} = \begin{cases} 0, & \text{if } \ n=0, \\ (x-(n-1)\lambda)(x)_{n-1,\lambda}, & \text{if } \ n \geq 1. \end{cases}$$

Recently, Kim-Kim introduced the degenerate derangement polynomials which were defined by the generating function to be

$$\frac{1}{1-t}e_{\lambda}^{x-1}(t)=\sum_{n=0}^{\infty}D_{n,\lambda}(x)\frac{t^n}{n!},\quad (x,t\neq 1),\qquad \text{(see [11])}, \tag{1.12}$$

where $x=0$, $D_{n,\lambda}=D_{n,\lambda}(0)$ are called the degenerate derangement numbers.

From (1.12), we have

$$\begin{aligned}\sum_{n=0}^{\infty}D_{n,\lambda}(x)\frac{t^n}{n!}&=\frac{1}{1-t}e_{\lambda}^{x-1}(t)\\&=\sum_{l=0}^{\infty}t^l\sum_{m=0}^{\infty}(x-1)_{m,\lambda}\frac{t^m}{m!}\qquad \text{(see [11])}.\\&=\sum_{n=0}^{\infty}\Big(n!\sum_{m=0}^{n}\frac{(x-1)_{m,\lambda}}{m!}\Big)\frac{t^n}{n!},\end{aligned} \tag{1.13}$$

By (1.13), we get

$$D_{n,\lambda}(x)=n!\sum_{m=0}^{n}\frac{(x-1)_{m,\lambda}}{m!},\quad \text{(see [11])}. \tag{1.14}$$

A continuous random variable X whose density function is given by

$$f(x)=\begin{cases}\lambda e^{-\lambda x}\frac{(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } x\geq 0,\\ 0, & \text{if } x<0,\end{cases}\qquad (\alpha,\lambda>0),\quad \text{(see [15])}, \tag{1.15}$$

is said to be the gamma random variable with parameter α,λ which is denoted by $X\sim\Gamma(\alpha,\lambda)$.

The n-th moment of $X\sim\Gamma(\alpha,\lambda)$ is given by

$$E(X^n)=\frac{\Gamma(\alpha+n)}{\lambda^n\Gamma(\alpha)},\quad \text{(see [6, 15])}. \tag{1.16}$$

The aim of this paper is to study the derangement polynomials and the degenerate derangement polynomials, their connections with degenerate cosine-derangement polynomials and degenerate sine-derangement polynomials. As their application, the degenerate derangement number represents the moment of an exponential random variable with parameter 1. Finally, we investigate the distribution of zeros for the derangement polynomials.

2 The derangement polynomials and the degenerate derangement polynomials

In this section, we will study some identities of the derangement and the degenerate derangement polynomials.

From (1.3), assuming that $|e^t-1|<1$, we have

$$\begin{aligned}\frac{1}{1-t}e^{-t}e^{xt}&=\frac{1}{1-t}e^{-t}(e^t-1+1)^x\\&=\left(\sum_{l=0}^{\infty}D_l\frac{t^l}{l!}\right)\left(\sum_{k=0}^{\infty}(x)_k\frac{1}{k!}(e^t-1)^k\right)\\&=\left(\sum_{l=0}^{\infty}D_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\sum_{k=0}^{m}(x)_kS_2(m,k)\frac{t^k}{m!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}(x)_kD_{n-m}S_2(m,k)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.1}$$

Therefore, by (2.1), we obtain the following theorem.

Theorem 1 *For $n \geq 0$, we have*

$$D_n(x) = \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{n}{m} (x)_k D_{n-m} S_2(m,k).$$

Now, we observe that

$$\begin{aligned}
&\int_{-\pi}^{\pi} \frac{1}{1-e^{\mathbf{i}\theta}} e^{-e^{\mathbf{i}\theta}} \sin n\theta \; d\theta \\
&= \int_{-\pi}^{\pi} \sum_{k=0}^{\infty} D_k \frac{e^{\mathbf{i}\theta k}}{k!} \sin n\theta \; d\theta \\
&= \sum_{k=0}^{\infty} D_k \frac{1}{k!} \int_{-\pi}^{\pi} e^{\mathbf{i}\theta k} \sin n\theta \; d\theta \\
&= \sum_{k=0}^{\infty} D_k \frac{1}{k!} \left(\int_{-\pi}^{\pi} \cos k\theta \sin n\theta \; d\theta + \mathbf{i} \int_{-\pi}^{\pi} \sin k\theta \sin n\theta \; d\theta \right)
\end{aligned}$$

- Case I: if $n \neq k$

  $$\int_{-\pi}^{\pi} \cos k\theta \sin n\theta \; d\theta = 0$$

  and

  $$\int_{-\pi}^{\pi} \sin k\theta \sin n\theta \; d\theta = 0$$

- Case II: if $n = k$

  $$\int_{-\pi}^{\pi} \cos k\theta \sin n\theta \; d\theta = 0$$

  and

  $$\int_{-\pi}^{\pi} \sin k\theta \sin n\theta \; d\theta = \pi.$$

Hence, we get

$$\int_{-\pi}^{\pi} \frac{1}{1-e^{\mathbf{i}\theta}} e^{-e^{\mathbf{i}\theta}} \sin n\theta \; d\theta = D_n \frac{\pi}{n!} \mathbf{i}. \tag{2.2}$$

Therefore, by (2.2), we obtain the witt's type formula of derangement number.

Theorem 2 (Witt's type formula of derangement number) *For $n \geq 0$, we have*

$$D_n = \frac{n!}{\pi} \mathbf{Im} \left(\int_{-\pi}^{\pi} \frac{1}{1-e^{\mathbf{i}\theta}} e^{-e^{\mathbf{i}\theta}} \sin n\theta \; d\theta \right).$$

Now, we investigate the convolution formula of derangement polynomials.

$$\begin{aligned}&\left(\frac{1}{1-t}\right)e^{(x-1)t}\left(\frac{1}{1-t}\right)e^{(y-1)t}\\&=\left(\sum_{l=0}^{\infty}D_l(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}D_m(y)\frac{t^m}{m!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}D_{n-m}(x)D_m(y)\right)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\Big(D(x)*D(y)\Big)\frac{t^n}{n!}.\end{aligned}$$

So, we define the convolution formula of derangement polynomials as follows:

$$(f*g)(n)=\sum_{k=0}^{n}\binom{n}{k}f(k)g(n-k).$$

The higher-order derangement numbers $D_n^{(r)}$ are defined by the generating function,

$$\left(\frac{1}{1-t}\right)^r e^{-t}=\sum_{n=0}^{\infty}D_n^{(r)}\frac{t^n}{n!},\quad \text{(see [8])}.$$

Now, we observe that

$$\begin{aligned}\left(\frac{1}{1-t}\right)^2 e^{-t}e^{-t}&=\left(\frac{1}{1-t}e^{-t}\right)\left(\frac{1}{1-t}e^{-t}\right)\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}D_{n-m}D_m\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(D*D\right)(n)\frac{t^n}{n!}.\end{aligned}\tag{2.3}$$

On the other hand,

$$\begin{aligned}\left(\frac{1}{1-t}\right)^2 e^{-t}e^{-t}&=\left(\sum_{m=0}^{\infty}D_m^{(2)}\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\frac{(-1)^l t^l}{l!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}(-1)^{n-m}D_m^{(2)}\right)\frac{t^n}{n!}.\end{aligned}\tag{2.4}$$

Therefore, by (2.3) and (2.4), we obtain the following theorem.

Theorem 3 *For $n,m\geq 0$ with $n\geq m$, we have*

$$\begin{aligned}(D*D)(n)&=\sum_{m=0}^{n}\binom{n}{m}D_{n-m}D_m\\&=\sum_{m=0}^{n}\binom{n}{m}(-1)^{n-m}D_m^{(2)}.\end{aligned}$$

The degenerate sine and cosine functions are defined by

$$\cos_\lambda t=\frac{e_\lambda^{\mathbf{i}}(t)+e_\lambda^{-\mathbf{i}}(t)}{2},\quad \sin_\lambda t=\frac{e_\lambda^{\mathbf{i}}(t)-e_\lambda^{-\mathbf{i}}(t)}{2\mathbf{i}}\quad \text{(see [7])}.$$

From (1.12), we have

$$\frac{1}{1-t}e_\lambda^{-1}(t)e_\lambda^{x+\mathbf{i}y}(t)=\sum_{n=0}^{\infty}D_{n,\lambda}(x+\mathbf{i}y)\frac{t^n}{n!},\quad \text{(see [6, 10])} \tag{2.5}$$

$$\frac{1}{1-t}e_\lambda^{-1}(t)e_\lambda^{x-\mathbf{i}y}(t)=\sum_{n=0}^{\infty}D_{n,\lambda}(x-\mathbf{i}y)\frac{t^n}{n!}, \tag{2.6}$$

$$\cos_\lambda^{(y)}(t)=\frac{e_\lambda^{\mathbf{i}y}(t)+e_\lambda^{-\mathbf{i}y}(t)}{2},\quad \sin_\lambda^{(y)}(t)=\frac{e_\lambda^{\mathbf{i}y}(t)-e_\lambda^{-\mathbf{i}y}(t)}{2\mathbf{i}},\quad \text{(see [8])}. \tag{2.7}$$

Note that

$$\lim_{\lambda\to 0}\cos_\lambda(t)=\cos t,\quad \lim_{\lambda\to 0}\sin_\lambda(t)=\sin t,$$

and

$$\lim_{\lambda\to 0}\cos_\lambda^{(y)}(t)=\cos yt,\quad \lim_{\lambda\to 0}\sin_\lambda^{(y)}(t)=\sin yt.$$

From (2.7), we have

$$\begin{aligned}
e_\lambda^{\mathbf{i}y}(t)=(1+\lambda t)^{\frac{\mathbf{i}y}{\lambda}}=e^{\frac{\mathbf{i}y}{\lambda}\log(1+\lambda t)}&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\lambda^{n-k}\mathbf{i}^k y^k S_1(n,k)\right)\frac{t^n}{n!},\\
\cos_\lambda^{(y)}(t)=\frac{e_\lambda^{\mathbf{i}y}(t)+e_\lambda^{-\mathbf{i}y}(t)}{2}&=\sum_{n=0}^{\infty}\left(\sum_{k=2n}^{\infty}\lambda^{n-2k}(-1)^k y^{2k}S_1(n,2k)\right)\frac{t^n}{n!},\\
\sin_\lambda^{(y)}(t)=\frac{e_\lambda^{\mathbf{i}y}(t)-e_\lambda^{-\mathbf{i}y}(t)}{2\mathbf{i}}&=\sum_{n=0}^{\infty}\left(\sum_{k=2n+1}^{\infty}\lambda^{n-2k-1}(-1)^k S_1(n,2k+1)\right)\frac{t^n}{n!}.
\end{aligned}$$

From (2.5) and (2.6), we have

$$\frac{1}{1-t}e_\lambda^{x-1}(t)\cos_\lambda^{(y)}(t)=\sum_{n=0}^{\infty}\left(\frac{D_{n,\lambda}(x+\mathbf{i}y)+D_{n,\lambda}(x-\mathbf{i}y)}{2}\right)\frac{t^n}{n!}, \tag{2.8}$$

and

$$\frac{1}{1-t}e_\lambda^{x-1}(t)\sin_\lambda^{(y)}(t)=\sum_{n=0}^{\infty}\left(\frac{D_{n,\lambda}(x+\mathbf{i}y)-D_{n,\lambda}(x-\mathbf{i}y)}{2\mathbf{i}}\right)\frac{t^n}{n!}. \tag{2.9}$$

By (2.8) and (2.9), we define the degenerate cosine-derangement polynomials and degenerate sine-derangement polynomials respectively by

$$\frac{1}{1-t}e_\lambda^{x-1}(t)\cos_\lambda^{(y)}(t)=\sum_{n=0}^{\infty}D_{n,\lambda}^{(c)}(x,y)\frac{t^n}{n!}, \tag{2.10}$$

and

$$\frac{1}{1-t}e_\lambda^{x-1}(t)\sin_\lambda^{(y)}(t)=\sum_{n=0}^{\infty}D_{n,\lambda}^{(s)}(x,y)\frac{t^n}{n!}. \tag{2.11}$$

From (2.10) and (2.11), we have

$$\begin{aligned}\sum_{n=0}^{\infty} D_{n,\lambda}^{(c)}(x,y)\frac{t^n}{n!} &= \frac{1}{1-t}e_\lambda^{x-1}(t)\cos_\lambda^y(t)\\ &= \left(\sum_{l=0}^{\infty} D_{l,\lambda}(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\sum_{k=0}^{\left[\frac{m}{2}\right]}\lambda^{m-2k}(-1)^k y^{2k}S_1(m,2k)\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{\left[\frac{m}{2}\right]}\binom{n}{m}\lambda^{m-2k}(-1)^k y^{2k}S_1(m,2k)D_{n-m,\lambda}(x)\right)\frac{t^n}{n!}\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} D_{n,\lambda}^{(s)}(x,y)\frac{t^n}{n!} &= \frac{1}{1-t}e_\lambda^{x-1}(t)\sin_\lambda^y(t)\\ &= \left(\sum_{l=0}^{\infty} D_{l,\lambda}(x)\frac{t^l}{l!}\right)\left(\sum_{m=1}^{\infty}\sum_{k=0}^{\left[\frac{m-1}{2}\right]}\lambda^{m-2k-1}(-1)^k y^{2k+1}S_1(m,2k+1)\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=1}^{n}\sum_{k=0}^{\left[\frac{m-1}{2}\right]}\binom{n}{m}\lambda^{m-2k-1}(-1)^k y^{2k+1}S_1(m,2k+1)D_{n-m,\lambda}(x)\right)\frac{t^n}{n!}\end{aligned} \tag{2.13}$$

Therefore, by (2.12) and (2.13), we obtain the following theorem.

Theorem 4 *For $n \in \mathbb{N}$, we have*

$$D_{n,\lambda}^{(c)} = \sum_{m=0}^{n}\sum_{k=0}^{\left[\frac{m}{2}\right]}\binom{n}{m}\lambda^{m-2k}(-1)^k y^{2k}S_1(m,2k)D_{n-m,\lambda}(x)$$

and

$$D_{n,\lambda}^{(s)} = \sum_{m=1}^{n}\sum_{k=0}^{\left[\frac{m-1}{2}\right]}\binom{n}{m}\lambda^{m-2k-1}(-1)^k y^{2k+1}S_1(m,2k+1)D_{n-m,\lambda}(x).$$

From (2.10) and (2.11), we have

$$\begin{aligned}\sum_{n=0}^{\infty} D_{n,\lambda}^{(c)}(x,y)\frac{t^n}{n!} &= \frac{1}{1-t}e_\lambda^{x-1}(t)\cos_\lambda^{(y)}(t)\\ &= \left(\sum_{l=0}^{\infty} l!\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\sum_{j=0}^{\left[\frac{k}{2}\right]}\binom{m}{k}\lambda^{k-2j}(-1)^j y^{2j}S_1(k,2j)(x)_{m-k,\lambda}\right)\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{j=0}^{\left[\frac{k}{2}\right]}\binom{n}{m}\binom{m}{k}\lambda^{k-2j}(-1)^j(n-m)!y^{2j}S_1(k,2j)(x)_{m-k,\lambda}\right)\frac{t^n}{n!}\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} D_{n,\lambda}^{(s)}(x,y)\frac{t^n}{n!} &= \frac{1}{1-t}e_{\lambda}^{x-1}(t)\sin_{\lambda}^{(y)}(t)\\ &= \left(\sum_{l=0}^{\infty} l!\frac{t^l}{l!}\right)\left(\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}\sum_{j=0}^{\left[\frac{k-1}{2}\right]}\binom{m}{k}\lambda^{k-2j-1}(-1)^j y^{2j+1}S_1(k,2j+1)(x)_{m-k,\lambda}\right)\frac{t^m}{m!}\right)\\ &= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\sum_{k=1}^{m}\sum_{j=0}^{\left[\frac{k-1}{2}\right]}\binom{n}{m}\binom{m}{k}\lambda^{k-2j-1}(-1)^j(n-m)!y^{2j+1}S_1(k,2j+1)(x)_{m-k,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.15}$$

Therefore, by (2.14) and (2.15), we obtain the following theorem.

Theorem 5 *For $n \in \mathbb{N}$, we have*

$$D_{n,\lambda}^{(c)}(x,y) = \sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{j=0}^{\left[\frac{k}{2}\right]}\binom{n}{m}\binom{m}{k}\lambda^{k-2j}(-1)^j(n-m)!y^{2j}S_1(k,2j)(x)_{m-k,\lambda}$$

and

$$D_{n,\lambda}^{(s)}(x,y) = \sum_{m=1}^{n}\sum_{k=1}^{m}\sum_{j=0}^{\left[\frac{k-1}{2}\right]}\binom{n}{m}\binom{m}{k}\lambda^{k-2j-1}(-1)^j(n-m)!y^{2j+1}S_1(k,2j+1)(x)_{m-k,\lambda}.$$

Let X be a gamma random variable with parameters 1,1 which is denoted by $X \sim \Gamma(1,1)$. Then we observe

$$\mathbf{E}\left[e^{Xt}\right] = \int_0^{\infty} e^{xt}e^{-x}dx = \int_0^{\infty} e^{-x(1-t)}dx = \frac{1}{1-t}. \tag{2.16}$$

From (2.16), we note that

$$\mathbf{E}\left[e^{Xt}\right]e_{\lambda}^{-1}(t) = \frac{1}{1-t}e_{\lambda}^{-1}(t) = \sum_{n=0}^{\infty} D_{n,\lambda}\frac{t^n}{n!}. \tag{2.17}$$

On the other hand,

$$\begin{aligned}\mathbf{E}\left[e^{Xt}\right]e_{\lambda}^{-1}(t) &= \left(\sum_{l=0}^{\infty}\mathbf{E}\left[X^l\right]\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(-1)_{m,\lambda}\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathbf{E}\left[X^l\right](-1)_{n-l,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.18}$$

Therefore, by (2.17) and (2.18), we obtain the following theorem.

Theorem 6 *For $n \geq 0$, we have*

$$D_{n,\lambda} = \sum_{l=0}^{n}\binom{n}{l}\mathbf{E}\left[X^l\right](-1)_{n-l,\lambda}.$$

3 The zeros of degenerate derangement polynomials

It is well known that finding zeros of a polynomial is an extremely important work in applied mathematics. In general, the formula of the zeros of a polynomial is unknown. The distribution and structure of the zeros of Bernoulli polynomials had been studied by Woon [20] numerically. Hence, we will investigate the numerical pattern of the roots of the polynomials $D_{n,\lambda}(x)$. Using the mathematica , the polynomial $D_{n,\lambda}(x)$ can be determined explicitly. For example,

$$D_{1,\lambda}(x) = x - 1,$$
$$D_{2,\lambda}(x) = (x-1)^2 + (2-\lambda)(x-1),$$
$$D_{3,\lambda}(x) = (x-1)^3 + 3(1-\lambda)(x-1)^2 + (6 - 3\lambda + 2\lambda^2)(x-1),$$
$$D_{4,\lambda}(x) = (x-1)^4 + (4-6\lambda)(x-1)^3 + (12 - 12\lambda + 11\lambda^2)(x-1)^2 + (24 - 12\lambda + 8\lambda^2 - 6\lambda^3)(x-1).$$

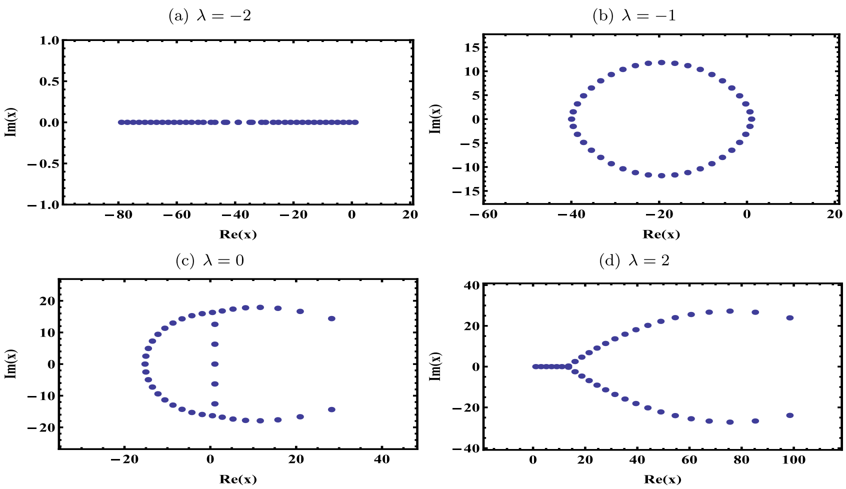

Fig. 1: The computed roots of $D^{L}_{40,\lambda}(x)$ with variable λ

From the definition of the polynomials $D_{n,\lambda}(x)$, one can show that the roots have the following properties:

- For any real number λ, the polynomials $D_{n,\lambda}(x)$ with $n = 1, 2$ have only real roots.
- For any real number λ and any positive integer n, all polynomials $D_{n,\lambda}(x)$ have a common root which is one.

Firstly, we want to observe the impact of λ on the distribution of the roots of the polynomials. For the aims, we fix the degree of polynomials as $n = 40$. Since the explicit form of roots of $D_{n,\lambda}(x)$ are unknown for $n \geq 5$, we calculate the roots by using the Mathematical tool with 100 working precision. The absolute numerical error is bounded as following

$$\sum_{i=1}^{40} |D_{40,\lambda}(x_i)| < 10^{-62},$$

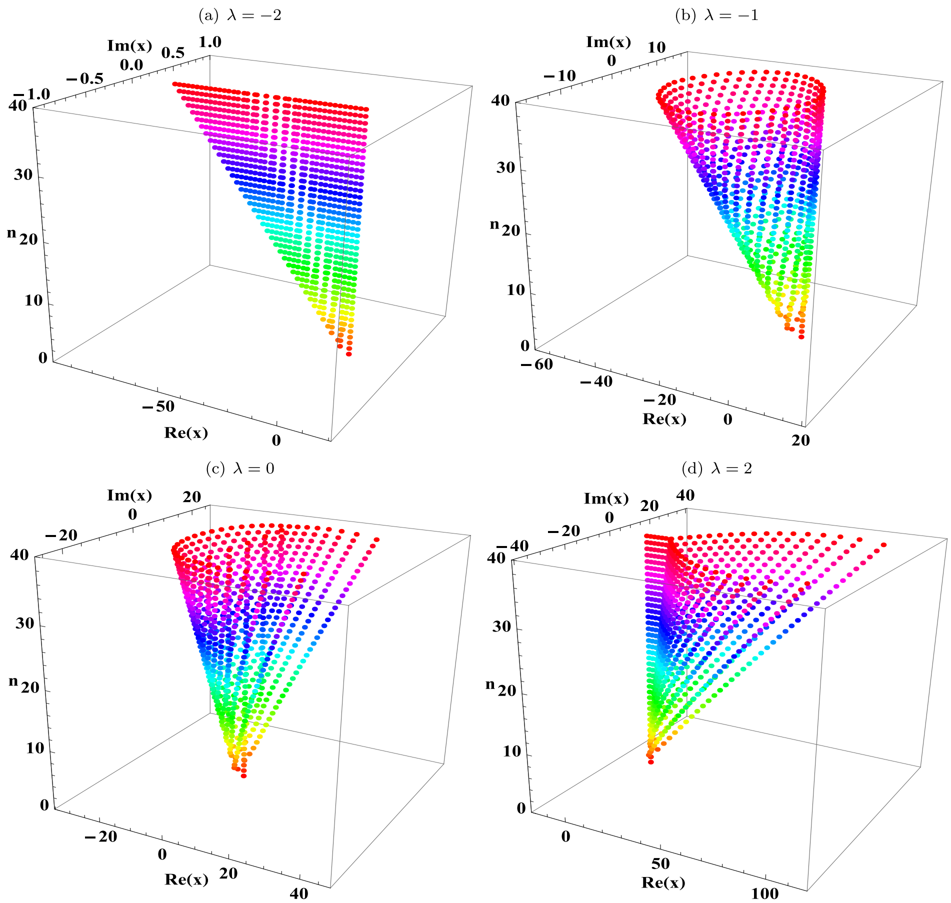

Fig. 2: The root distribution of $D_{n,\lambda}(x)$ with variable λ and different integer $n = 1, 2, \cdots, 40$.

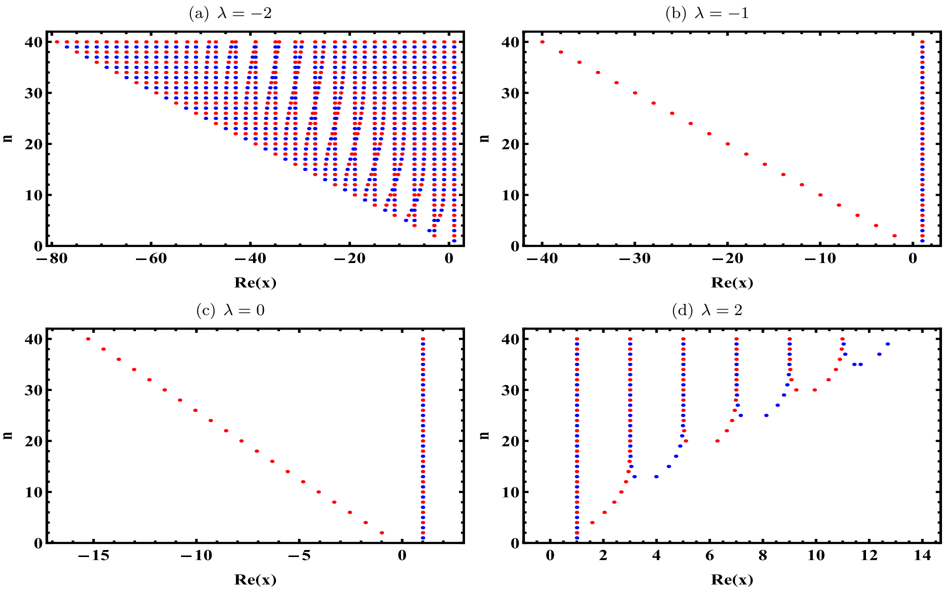

Fig. 3: Real zeros of $D_{n,\lambda}(x)$ for $\lambda = -2, -1, 0, 2$ and $1 \leq n \leq 40$.

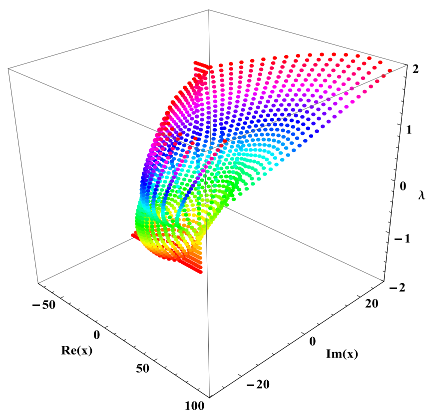

Fig. 4: Roots distribution structures vs $\lambda \in [-2, 2]$.

where x_i denotes the root of the polynomial. Hence, the results obtained from numerical computations are reliable. We compute the numerical roots of $D_{40,\lambda}(x)$ with four different parameters $\lambda = -2, -1, 0$ and 2 and the results are plotted in Fig. 1.

As observed in Fig. 1, the roots of the derangement polynomials have four patterns.

Secondly, we want to investigate the impact of the degree of polynomials on the distribution of roots of the polynomials. We compute the numerical roots of the polynomials increasing the degree of polynomials from 1 to 40 and presented in Fig. 2.

Thirdly, to investigate the real roots distribution structure of $D_{n,\lambda}(x)$, we compute the real roots and displayed them in Fig.3.

From the results of Fig.3, one can find a remarkably regular structure of the roots of the polynomial $D_{n,\lambda}(x)$. In order to find the structure of the roots , we count the number of real roots for $\lambda = -2, -1, 0$ and 2 and $n \in [1, 40]$ with in $x \in [-1000, 1000]$ and summarized as following:

- $\lambda = -2$: the number of real roots=n.
- $\lambda = -1, 0$: the number of real roots=$\begin{cases} 1 & n \text{ is odd} \\ 2 & n \text{ is even} \end{cases}$
- $\lambda = 2$: the number of real roots=$\begin{cases} 1, & n = 1, \cdots, 11, \\ 2, & n = 2, \cdots, 18, \\ 3, & n = 13, \cdots, 23, \\ 4, & n = 20, \cdots, 28, \\ 5, & n = 25, \cdots, 33, \\ 6, & n = 30, \cdots, 40, \\ 7, & n = 35, \cdots, 39. \end{cases}$

Finally, we compute the roots of the polynomials with a fixed $n = 40$ and varying the parameter $\lambda = \frac{k}{10}$, $k = -20, -19, \cdots, 20$. The numerical results are plotted in Fig. 4.

4 Conclusion

In 2018, Kim-Kim considered the derangement polynomials in (1.3) and the degenerate derangement polynomials in (1.14) and in [11], Jang-Kim-Kim-Lee introduced some identities involving derangement polynomials and numbers and moments of gamma random variables. In this paper, we obtained new identities of the derangement polynomials in Theorem 1 and Theorem 2. We also observed Witt's formula of the derangement numbers in Theorem 3. For the higher order derangement numbers, we obtained an interesting identity for those numbers in Theorem 4.

In particular, we defined the degenerate cosine-derangement polynomials and degenerate sine-derangement polynomials in (2.10) and (2.11), and obtained some identities for those polynomials in Theorem 5 and Theorem 6, and observed some relation identity between expectation of an exponential random variable with parameter 1 and degenerate derangement numbers in Theorem 7. Furthermore, we investigated the numerical pattern of the roots of the polynomials $D_{n,\lambda}(x)$ in Fig. 1, the numerical roots of the polynomials increasing the degree of polynomials from 1 to 40 in Fig. 2, the real roots distribution structure of the polynomials in Fig. 3, and computed the roots of the polynomials with $n = 40$ and varying the parameter $\lambda = \frac{k}{10}, k = -20, -19, \cdots, 20$ in Fig. 4.

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