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HAMMING DISTANCE BETWEEN THE STRINGS GENERATED BY ADJACENCY MATRIX OF GENERALISED COMPLEMENTS OF A GRAPH AND THEIR SUM

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ABSTRACT. Let $A(G)$ be the adjacency matrix of a graph G. Let $s(v)$ denote the row entries of $A(G)$ corresponding to the vertex v of G. The Hamming distance between the strings $s(u)$ and $s(v)$ is the number of positions in which $s(u)$ and $s(v)$ differ. In this paper, we study the Hamming distance between the strings generated by the adjacency matrix of generalised complements of a graph. We also compute sum of Hamming distances between all pairs of strings generated by the adjacency matrix of G_k^P and $G_{k(i)}^P$.

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1. Introduction

Let $Z_2 = \{0, 1\}$ and $(Z_2, +)$ be the additive group, where + denotes addition modulo 2. For any positive integer n, $Z_2^n = Z_2 \times Z_2 \times \ldots \times Z_2$ (n factors)$= \{(x_1, x_2, ..., x_n)|x_1, x_2, ..., x_n \in Z_2\}$. Thus every element of Z_2^n is an $n-$tuple $(x_1, x_2, \ldots, x_n)$ written as $x = x_1x_2...x_n$, where every x_i is either 0 or 1, called a string or word. The number of $1's$ in $x = x_1x_2...x_n$ is called the weight of x, denoted by $w(x)$. Let $x = x_1x_2...x_n$ and $y = y_1y_2...y_n$ be the elements of Z_2^n. The sum $x+y$ is computed by adding the corresponding components of x and y under addition modulo 2. In other words, $x_i+y_i = 0$ if $x_i = y_i$ and $x_i + y_i = 1$ if $x_i \neq y_i, i = 1, 2, ..., n$. The Hamming distance $H_d(x, y)$ between the strings $x = x_1x_2...x_n$ and $y = y_1y_2...y_n$ is the number of $i's$ such that $x_i \neq y_i, 1 \leq i \leq n$. Thus $H_d(x, y) =$ Number of positions in which x and y differ $= w(x + y)$.
Example 1. Let $x = 01001$ and $y = 11010$. Therefore $x + y = 10011$. Hence $H_d(x, y) = w(x + y) = 3$.

2. Preliminaries

A graph G with vertex set $V(G)$ is called a Hamming graph [2, 3] if each vertex $v \in V(G)$ can be labeled by a string $s(v)$ of a fixed length such that $H_d(s(u), s(v)) = d_G(u, v)$ for all $u, v \in V(G)$, where $d_G(u, v)$ is the length of shortest path joining u and v in G.

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The degree of a vertex v, denoted by $deg_G(v)$ is the number of neighbours of v also denoted by $n(v)$. The vertices which are adjacent to both u and v are called common neighbours of u and v. The vertices which are neither adjacent to u nor adjacent to v are called non common neighbours of u and v. We denote two vertices u adjacent to v by $u \sim v$ and u not adjacent to v by $u \nsim v$.

Sum of Hamming distances between all pairs of strings generated by the adjacency matrix of a graph G is denoted by $HA(G)$ is given by,

$$HA(G) = \sum_{1 \leq i < j \leq n} H_d(s(v_i), s(v_j)).$$

$HA(G)$ is a graph invariant.

Definition 2.1. [5] *For a graph $G(V, E)$, let $P = \{V_1, V_2, \ldots, V_k\}$ be the partition of vertex set $V(G)$ of order $k \geq 2$. For each set V_r in P, remove the edges of graph G inside V_r and add the edges of the complement of G joining the vertices of V_r. The graph thus obtained is called the $k(i)-$complement of graph G with respect to partition P denoted by $G^P_{k(i)}$.*

Definition 2.2. [4] *Let $G(V, E)$ be any graph. Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of V of order $k \geq 2$. For all V_i and V_j in $P, i \neq j$, remove the edges between V_i an V_j in G and add the edges between V_i and V_j which are not in G. The graph thus obtained is called $k-$complement of G with respect to P denoted by G^P_k.*

Throughout the paper, $H_{d_{k(i)}}(s(u), s(v))$ and $H_{d_k}(s(u), s(v))$ denote Hamming distance between the strings $s(u)$ and $s(v)$ generated by adjacency matrix of $G^P_{k(i)}$ and G^P_k respectively.

3. Hamming distance between the strings in $k(i)-$complement and $k-$ complement of a graph.

Theorem 3.1. *Let $G(V, E)$ be any graph of order n and $S \subseteq V$. Let $P = \{S, V - S\}$ be the partition of vertex set $V(G)$.*

(1) *Suppose the vertices u and v have $k_1(k_2)$ common neighbours, $l_1(l_2)$ non- common neighbours respectively in $\langle S \rangle$ ($\langle V - S \rangle$).*
   (a) *If $u \sim v$ in G and $u, v \in S$, then $H_{d_{2(i)}}(s(u), s(v)) = n - (k_1 + k_2) - (l_1 + l_2) - 2$.*
   (b) *If $u \nsim v$ in G and $u, v \in S$, then $H_{d_{2(i)}}(s(u), s(v)) = n - (k_1 + k_2) - (l_1 + l_2)$.*

(2) *Suppose $r_1 = n(u), r_2 = n(v)$ and $r_3 = n(u), r_4 = n(v)$ in $\langle S \rangle$ and $\langle V - S \rangle$ respectively.*
   (a) *If $u \sim v$ in G and $u \in S, v \in V - S$, then $H_{d_{2(i)}}(s(u), s(v)) = n - (r_1 + r_2 + r_3 + r_4)$.*
   (b) *If $u \nsim v$ in G and $u \in S, v \in V - S$, then $H_{d_{2(i)}}(s(u), s(v)) = n - (r_1 + r_2 + r_3 + r_4) - 2$.*

Proof. Let G be any graph of order n and $S \subseteq V$. Let $P = \{S, V - S\}$ be the partition of vertex set V.

(1) Let the vertices u, v of G be have k_1 common neighbours, l_1 non common neighbours in $\langle S\rangle$ and k_2 common neighbours, l_2 non common neighbours in $\langle V-S\rangle$.

(a) If $u \sim v$ in G and $u, v \in S$, then $u \nsim v$ in $G^P_{2(i)}$. We observe that u and v have (k_2+l_1) common neighbours and (k_1+l_2) non common neighbours. Remaining $n-(k_2+l_1)-(k_1+l_2)-2$ vertices other than u and v are adjacent to either u or v. Therefore strings of u and v from $A(G^P_{2(i)})$ will be of the form
$s(u) = x_1x_2x_3\ldots x_{k_2+l_1+1}x_{k_2+l_1+2}x_{k_2+l_1+3}\cdots x_{k_2+l_1+k_1+l_2+2}$
$x_{k_2+l_1+k_1+l_2+3}\cdots x_n$ and
$s(v) = y_1y_2y_3\cdots y_{k_2+l_1+1}y_{k_2+l_1+2}y_{k_2+l_1+3}\cdots y_{k_2+l_1+k_1+l_2+2}$
$y_{k_2+l_1+k_1+l_2+3}\cdots y_n$,
where $x_1 = 0, x_2 = 0, y_1 = y_2 = 0, x_i = y_i = 1$ for $i = 3, 4, \ldots, k_2+l_1+2, x_i = y_i = 0$ for $i = k_2+l_1+3, k_2+l_1+4, \ldots, k_2+l_1+k_1+l_2+2$ and $x_i \neq y_i$ for $i = k_2+l_1+k_1+l_2+3, k_2+l_1+k_1+l_2+4, \ldots, n$. Thus $s(u)$ and $s(v)$ differ at $n-(k_2+l_2)-(k_1+l_2)-2$ places.
Hence $H_{d_{2(i)}}(s(u), s(v)) = n-(k_1+k_2)-(l_1+l_2)-2$.

(b) If $u \nsim v$ in G and $u, v \in S$, then $u \sim v$ in $G^P_{2(i)}$. The vertices u and v have (k_2+l_1) common neighbours and (k_1+l_2) non common neighbours. Remaining $n-(k_2+l_1)-(k_1+l_2)-2$ vertices are adjacent to either u or v. Therefore strings of u and v from $A(G^P_{2(i)})$ will be of the form
$s(u) = x_1x_2x_3\ldots x_{k_2+l_1+1}x_{k_2+l_1+2}x_{k_2+l_1+3}\cdots x_{k_2+l_1+k_1+l_2+2}$
$x_{k_2+l_1+k_1+l_2+2}\cdots x_n$.
and $s(v) = y_1y_2y_3\ldots y_{k_2+l_1+1}y_{k_2+l_1+2}y_{k_2+l_1+3}\cdots y_{k_2+l_1+k_1+l_2+2}$
$y_{k_2+l_1+k_1+l_2+3}\cdots y_n$.
Where $x_1 = 0, x_2 = 1, y_1 = 1, y_2 = 0, x_i = y_i = 1$ for $i = 3, 4, \ldots, k_2+l_1+2, x_i = y_i = 0$ for $i = k_2+l_1+3, k_2+l_1+4, \ldots, k_2+l_1+k_1+l_2+2$ and $x_i \neq y_i$ for $i = k_2+l_1+k_1+l_2+3, k_2+l_1+k_1+l_2+4, \ldots, n$. Thus $s(u)$ and $s(v)$ differ at $n-(k_2+l_2)-(k_1+l_2)-2+2$ places. Hence $H_{d_{2(i)}}(s(u), s(v)) = n-(k_1+k_2)-(l_1+l_2)$.

(2) Let $r_1 = n(u), r_2 = n(v)$ and $r_3 = n(u), r_4 = n(v)$ in $\langle S\rangle$ and $\langle V-S\rangle$ respectively.

(a) If $u \sim v$ in G and $u \in S, v \in V-S$, then $u \sim v$ in $G^P_{2(i)}$. The vertices u and v have (r_2+r_3) common neighbours and (r_1+r_4) non common neighbours. Remaining $n-(r_2+r_3)-(r_1+r_4)-2$ vertices are adjacent to either u or v. Therefore strings of u and v from $A(G^P_{2(i)})$ will be of the form
$s(u) = x_1x_2x_3\ldots x_{r_2+r_3+1}x_{r_2+r_3+2}x_{r_2+r_3+3}\ldots x_{r_2+r_3+r_1+r_4+1}$
$x_{r_2+r_3+r_1+r_4+2}x_{r_2+r_3+r_1+r_4+3}\ldots x_n$ and
$s(v) = y_1y_2y_3\cdots y_{r_2+r_3+1}y_{r_2+r_3+2}y_{r_2+r_3+3}\cdots y_{r_2+r_3+r_1+r_4+1}$
$y_{r_2+r_3+r_1+r_4+2}y_{r_2+r_3+r_1+r_4+3}\cdots y_n$, where $x_1 = 0, x_2 = 1, y_1 = 1, y_2 = 0, x_i = y_i = 1$ for $i = 3, 4, \ldots, (r_2+r_3+2)$ and $x_i = y_i = 0$ for $i = (r_2+r_3+3), (r_2+r_3+4), \ldots, (r_2+r_3+r_1+r_4+2), x_i \neq y_i$

for $i = (r_2 + r_3 + r_1 + r_4 + 3), \ldots, n$. Thus $s(u)$ and $s(v)$ differ at $n - (r_2 + r_3) - (r_1 + r_4) - 2 + 2$ places.
Hence $H_{d_{2(i)}}(s(u), s(v)) = n - (r_1 + r_2 + r_3 + r_4)$.

(b) If $u \nsim v$ in G and $u \in S, v \in V - S$, then $u \nsim v$ in $G^P_{2(i)}$. The vertices u and v have (r_2+r_3) common neighbours and (r_1+r_4) non common neighbours. Remaining $n-(r_2+r_3)-(r_1+r_4)-2$ vertices are adjacent to either u or v. Therefore strings of u and v from $A(G^P_{2(i)})$ will be of the form
$s(u) = x_1x_2x_3 \ldots x_{r_2+r_3+1}x_{r_2+r_3+2}x_{r_2+r_3+3} \ldots x_{r_2+r_3+r_1+r_4+1}$
$x_{r_2+r_3+r_1+r_4+2}x_{r_2+r_3+r_1+r_4+3} \ldots x_n$ and
$s(v) = y_1y_2y_3 \ldots y_{r_2+r_3+1}y_{r_2+r_3+2}y_{r_2+r_3+3} \ldots y_{r_2+r_3+r_1+r_4+1}$
$y_{r_2+r_3+r_1+r_4+2}y_{r_2+r_3+r_1+r_4+3} \ldots y_n$.
Where $x_1 = 0, x_2 = 0, y_1 = 0, y_2 = 0, x_i = y_i = 1$ for $i = 3, 4, \ldots, (r_2 + r_3 + 2)$ and $x_i = y_i = 0$ for $i = (r_2 + r_3 + 3), (r_2 + r_3+4), \ldots, (r_2+r_3+r_1+r_4+2), x_i \neq y_i$ for $i = (r_2+r_3+r_1+r_4+3), \ldots, n$. Thus $s(u)$ and $s(v)$ differ at $n-(r_2+r_3)-(r_1+r_4)-2$ places.
Hence $H_{d_{2(i)}}(s(u), s(v)) = n - (r_1 + r_2 + r_3 + r_4) - 2$.

□

Corollary 3.2. *Let $G(V, E)$ be any graph of order n and $S \subseteq V$. Let $P = \{S, V - S\}$ be the partition of vertex set $V(G)$.*

(1) *Suppose the vertices u, v of G have k_1 common neighbours, l_1 non common neighbours in $\langle S\rangle$ and k_2 common neighbours, l_2 non common neighbours in $\langle V - S\rangle$.*
  (a) *If $u \sim v$ in G and $u, v \in S$, then $H_{d_2}(s(u), s(v)) = n - (k_1 + k_2) - (l_1 + l_2)$.*
  (b) *If $u \nsim v$ in G and $u, v \in S$, then $H_{d_2}(s(u), s(v)) = n - (k_1 + k_2) - (l_1 + l_2) - 2$.*

(2) *Suppose $r_1 = n(u), r_2 = n(v)$ in $\langle S\rangle$ and $r_3 = n(u), r_4 = n(v)$ in $\langle V - S\rangle$.*
  (a) *If $u \sim v$ in G and $u \in S, v \in V - S$, then $H_{d_2}(s(u), s(v)) = n - (r_1 + r_2 + r_3 + r_4) - 2$.*
  (b) *If $u \nsim v$ in G and $u \in S, v \in V - S$, then $H_{d_2}(s(u), s(v)) = n - (r_1 + r_2 + r_3 + r_4)$.*

Theorem 3.3. *Let $G(V, E)$ be any graph of order n and $S \subseteq V$. Let $P = \{S, V - S\}$ be the partition of vertex set $V(G)$.*

(1) *Let the vertices u, v of G have k_1 common neighbours, l_1 non common neighbours in $\langle S\rangle$ and k_2 common neighbours, l_2 non common neighbours in $\langle V - S\rangle$.*
  (a) *If $u \sim v$ in G and u, $v \in S$, then $deg_{G^P_{2(i)}}(u) + deg_{G^P_{2(i)}}(v) = n + (k_2 + l_1) - (k_1 + l_2) - 2$.*
  (b) *If $u \nsim v$ in G and $u, v \in S$, then $deg_{G^P_{2(i)}}(u) + deg_{G^P_{2(i)}}(v) = n + (k_2 + l_1) - (k_1 + l_2)$.*

(2) *Let $r_1 = n(u), r_2 = n(v)$ in $\langle S\rangle$ and $r_3 = n(u), r_4 = n(v)$ in $\langle V - S\rangle$.*

(a) *If* $u \sim v$ *in* G *and* $u \in S, v \in V - S$, *then* $deg_{G^P_{2(i)}}(u) + deg_{G^P_{2(i)}}(v) = n + r_2 + r_3 - (r_1 + r_4)$.

(b) *If* $u \nsim v$ *in* G *and* $u \in S, v \in V - S$ *in* $\langle S \rangle$, *then* $deg_{G^P_{2(i)}}(u) + deg_{G^P_{2(i)}}(v) = n + r_2 + r_3 - (r_1 + r_4) - 2$.

Proof. Let $G(V, E)$ be any graph of order n and $S \in V$. Let $P = \{S, V - S\}$ be the partition of vertex set $V(G)$.

(1) Let the vertices u, v of G have k_1 common neighbours, l_1 non common neighbours in $\langle S \rangle$ and k_2 common neighbours, l_2 non common neighbours in $\langle V - S \rangle$.

(a) If $u \sim v$ in G and $u, v \in S$, then $u \nsim v$ in $G^P_{2(i)}$. The vertices u and v have $(k_2 + l_1)$ common neighbours and $(k_1 + l_2)$ non common neighbours. Remaining $n - (k_2 + l_1) - (k_1 + l_2) - 2$ vertices are adjacent to either u or v. Therefore $deg_{G^P_{2(i)}}(u) + deg_{G^P_{2(i)}}(v) = (k_2 + l_1) + (k_2 + l_1) + n - (k_2 + l_1) - (k_1 + l_2) - 2 = n + (k_2 + l_1) - (k_1 + l_2) - 2$.

(b) If $u \nsim v$ in G and $u, v \in S$, then $u \sim v$ in $G^P_{2(i)}$. The vertices u and v have $(k_2 + l_1)$ common neighbours and $(k_1 + l_2)$ non common neighbours. Remaining $n - (k_2 + l_1) - (k_1 + l_2) - 2$ vertices are adjacent to either u or v. Therefore $deg_{G^P_{2(i)}}(u) + deg_{G^P_{2(i)}}(v) = (K_2 + l_1) + (k_2 + l_1) + n - (k_2 + l_1) - (k_1 + l_2) - 2 + 2 = n + (k_2 + l_1) - (k_1 + l_2)$.

(2) Let $r_1 = n(u), r_2 = n(v)$ in $\langle S \rangle$ and $r_3 = n(u), r_4 = n(v)$ in $\langle V - S \rangle$.

(a) If $u \sim v$ in G and $u \in S, v \in V - S$, then $u \sim v$ in $G^P_{2(i)}$. The vertices u and v have $(r_2 + r_3)$ common neighbours and $(r_1 + r_4)$ non common neighbours. Remaining $n - (r_2 + r_3) - (r_1 + r_4) - 2$ vertices are adjacent to either u or v. Therefore $deg_{G^P_{2(i)}}(u) + deg_{G^P_{2(i)}}(v) = (r_2 + r_3) + (r_2 + r_3) + n - (r_2 + r_3) - (r_1 + r_4) - 2 + 2 = n + r_2 + r_3 - (r_1 + r_4)$.

(b) If $u \nsim v$ in G and $u \in S, v \in V - S$, then $u \nsim v$ in $G^P_{2(i)}$. The vertices u and v have $(r_2 + r_3)$ common neighbours and $(r_1 + r_4)$ non common neighbours. Remaining $n - (r_2 + r_3) - (r_1 + r_4) - 2$ vertices are adjacent to either u or v. Therefore $deg_{G^P_{2(i)}}(u) + deg_{G^P_{2(i)}}(v) = (r_2 + r_3) + (r_2 + r_3) + n - (r_2 + r_3) - (r_1 + r_4) - 2 = n + r_2 + r_3 - (r_1 + r_4) - 2$.

□

Corollary 3.4. *Let* $G(V, E)$ *be any graph of order* n *and* $S \subseteq V$. *Let* $P = \{S, V - S\}$ *be the partition of vertex set* $V(G)$.

(1) *Suppose the vertices* u, v *of* G *have* k_1 *common neighbours,* l_1 *non common neighbours in* $\langle S \rangle$ *and* k_2 *common neighbours,* l_2 *non common neighbours in* $\langle V - S \rangle$.

(a) *If* $u \sim v$ *in* G *and* $u, v \in S$, *then* $deg_{G^P_2}(u) + deg_{G^P_2}(v) = n + (k_1 + l_2) - (k_2 + l_1)$.

(b) *If* $u \nsim v$ *in* G *and* $u, v \in S$, *then* $deg_{G^P_2}(u) + deg_{G^P_2}(v) = n + (k_1 + l_2) - (k_2 + l_1) - 2$.

(2) *Suppose* $r_1 = n(u), r_2 = n(v)$ *in* $\langle S \rangle$ *and* $r_3 = n(u), r_4 = n(v)$ *in* $\langle V - S \rangle$.
   (a) *If* $u \sim v$ *in* G *and* $u \in S, v \in V - S$, *then* $deg_{G_2^P}(u) + deg_{G_2^P}(v) = n + r_1 + r_4 - (r_2 + r_3) - 2$.
   (b) *If* $u \nsim v$ *in* G *and* $u \in S, v \in V - S$, *then* $deg_{G_2^P}(u) + deg_{G_2^P}(v) = n + r_1 + r_4 - (r_2 + r_3) - 2$.

Theorem 3.5. *For every vertices* u *and* v *in graph* G,

$$|H_{d_k}(s(u), s(v)) - H_{d_{k(i)}}(s(u), s(v))| = 2.$$

Proof. We consider two cases here.

Case 1: Let $u \sim v$ in G_k^P. Then $u \nsim v$ in $G_{k(i)}^P$. We know that $\overline{G_k^P} \cong G_{k(i)}^P$ and $\overline{G_{k(i)}^P} \cong G_k^P$ [5]. If u and v have k_1 common and l_1 non common neighbours in G_k^P, then u and v have l_1 common and k_1 non common neighbours in $G_{k(i)}^P$. So $H_{d_k}(s(u), s(v)) = n - k_1 - l_1$ and $H_{d_{k(i)}}(s(u), s(v)) = n - l_1 - k_1 - 2$. Therefore

$$H_{d_k}(s(u), s(v)) - H_{d_{k(i)}}(s(u), s(v)) = 2. \tag{1}$$

Case 2: Let $u \nsim v$ in G_k^P. Then $u \sim v$ in $G_{k(i)}^P$. If u and v have k_2 common and l_2 non common neighbours in G_k^P, then u and v have l_2 common and k_2 non common neighbours in $G_{k(i)}^P$. So $H_{d_k}(s(u), s(v)) = n - k_2 - l_2 - 2$ and $H_{d_{k(i)}}(s(u), s(v)) = n - l_2 - k_2$. Therefore

$$H_{d_{k(i)}}(s(u), s(v)) - H_{d_k}(s(u), s(v)) = 2. \tag{2}$$

Hence from equations (1) and (2), we get

$$|H_{d_k}(s(u), s(v)) - H_{d_{k(i)}}(s(u), s(v))| = 2.$$

□

Theorem 3.6. *For a graph* G *on* n *vertices,*

$$HA(G_{k(i)}^P) = HA(G_k^P) + n(n-1) - 4q.$$

Where q *denotes the number of edges of* G_k^P.

Proof. We have $\overline{G_k^P} \cong G_{k(i)}^P$ and $\overline{G_{k(i)}^P} \cong G_k^P$ [5].

Also $HA(\overline{G}) = HA(G) + n(n-1) - 4q$ [1].

Therefore $HA(G_{k(i)}^P) = HA(G_k^P) + n(n-1) - 4q$. □

4. Sum of Hamming distances of 2 and 2(i) complement of some graphs

Theorem 4.1. *For a complete graph* K_n, $HA((K_n)_2^P) = (n - m)(mn - 4m) + n(n-1)$.

Proof. Let $P = \{V_1, V_2\}$ be a partition of vertex set of K_n, where $V_1 = \{v_1, v_2, \ldots, v_m\}$ and $V_2 = \{v_m, v_{m+1}, \ldots, v_n\}$.
We observe that $(K_n)_2^P = K_m \cup K_{n-m}$.
Let us consider following cases.

Case 1: Let $u, v \in V_1$.

Then by Theorem (4) of [1],

$$\sum_{u,v\in V_1} H_{d_2}(s(u),s(v)) = m(m-1).$$

Case 2: Let $u,v \in V_2$. By Theorem (4) of [1],

$$\sum_{u,v\in V_2} H_{d_2}(s(u),s(v)) = (n-m)(n-m-1).$$

Case 3: Let $u \in V_1$ and $v \in V_2$. In $(K_n)_2^P$, $u \nsim v$. There are $m(n-m)$ pairs of vertices with zero common and zero non common neighbours which have Hamming distance $n-2$.

Therefore

$$\begin{aligned} H_A((K_n)_2^P) = &\sum_{u,v\in V_1} H_{d_2}(s(u),s(v)) + \sum_{u,v\in V_2} H_{d_2}(s(u),s(v)) \\ &+ \sum_{u\in V_1, v\in V_2} H_{d_2}(s(u),s(v)) \\ =&m(m-1)+(n-m)(n-m-1)+m(n-m)(n-2) \\ =&(n-m)(mn-4m)+n(n-1). \end{aligned}$$

□

Corollary 4.2. $HA((K_n)_{2(i)}^P) = mn(n-m)$.

Theorem 4.3. *If $P = \{V_1, V_2\}$ is a partition of cycle graph C_n with $\langle V_1\rangle = P_m$ and $\langle V_2\rangle = P_{n-m}$, then $HA((C_n)_2^P) = (n-m)(mn-8m)+2(n^2-2n+8)$.*

Proof. Let $V_1 = \{v_1, v_2, \ldots, v_m\}$ and $V_2 = \{v_{m+1}, v_{m+2}, \ldots, v_n\}$ be the partition of C_n.

Case 1: Let $u,v \in V_1$. Assume that $u \sim v$ in C_n. Then $u \sim v$ in $(C_n)_2^P$. There are $m-1$ pairs of vertices in C_n with distance 1. Out of $m-1$ pairs of vertices, 2 pairs (v_1,v_2) and (v_{m-1},v_m) have $n-m-1$ common neighbours and $m-3$ non common neighbours in $(C_n)_2^P$. Hence $H_{d_2}(s(u),s(v)) = n-(n-m-1)-(m-3) = 4$.

Remaining $m-3$ pairs of vertices have $n-m$ common and $m-4$ non common neighbours in $(C_n)_2^P$. Thus they have Hamming distance 4. Therefore

$$\sum_{d_{C_n}=1} H_{d_2}(s(u),s(v)) = 4m-4.$$

Let $d_{C_n}(u,v) = 2$, where $u \nsim v$ in C_n and $(C_n)_2^P$. There are $m-2$ pairs of vertices which are at distance 2. Out of which 2 pairs (v_1,v_3) and (v_{m-2},v_m) have $n-m$ common neighbours and $m-4$ non common neighbours in $(C_n)_2^P$. So $H_{d_2}(s(u),s(v)) = n-(n-m)-(m-4)-2 = 2$.

Remaining $m-4$ pairs of vertices have $n-m+1$ common neighbours and $m-5$ non common neighbours. Hence $H_{d_2}(s(u),s(v)) = n-(n-m+1)-(m-5)-2 = 2$.

Therefore

$$\sum_{d_{C_n}=2} H_{d_2}(s(u),s(v)) = 2m-4.$$

Let $d_{C_n}(u,v) > 2$. We observe that $u \nsim v$ in $(C_n)_2^P$. There are $\binom{m}{2}-2m+3$ pairs of vertices which are at distance greater than 2. Out of these vertices, $2(m-4)$ pairs of vertices have Hamming distance 4 as they have $n-m-1$ common and $m-5$ non common neighbours and one pair (v_1, v_m) have $n-m-2$ common and $m-4$ non common neighbours which results in $H_{d_2}(u,v) = 4$. Remaining $\binom{m}{2} - 4m + 10$ pairs of vertices have $n-m$ common and $m-6$ non common neighbours. So $H_{d_2}(u,v) = 4$. Hence

$$\sum_{d_{C_n}>2} H_{d_2}(s(u), s(v)) = 2m(m-1) - 8m + 12.$$

Therefore

$$\begin{aligned}\sum_{u,v\in V_1} H_{d_2}(s(u), s(v)) &= \sum_{d_{C_n}=1} H_{d_2}(s(u), s(v)) + \sum_{d_{C_n}=2} H_{d_2}(s(u), s(v)) \\ &+ \sum_{d_{C_n}>2} H_{d_2}(s(u), s(v)) \\ &= 4m - 4 + 2m - 4 + 2m(m-1) - 8m + 12 \\ &= 2m^2 - 4m + 4.\end{aligned}$$

Case 2: Let $u, v \in V_2$. Then

$$\sum_{u,v\in V_2} H_{d_2}(s(u), s(v)) = 2(n-m)^2 - 4(n-m) + 4.$$

Case 3: Let $u \in V_1$ and $v \in V_2$ such that $u \sim v$ in C_n. Then $u \nsim v$ in $(C_n)_2^P$. There are 2 pairs of vertices (v_1, v_n) and (v_m, v_{m+1}) of distance 1 in C_n. Since these two pairs of vertices have 2 common and zero non common neighbours in $(C_n)_2^P$, Hamming distance is $n-4$.

Let $u \nsim v$ in C_n. Then $u \sim v$ in $(C_n)_2^P$. There are $m(n-m)-2$ such pairs of vertices. Among these pairs, (v_1, v_{m+1}) and (v_m, v_n) have 2 common and 2 non common neighbours. Hence these two pairs have Hamming distance $n-4$. The four pairs of vertices $(v_m, v_{m+2}), (v_1, v_{n-1}), (v_n, v_2)$ and (v_{m+1}, v_{m-1}) have Hamming distance $n-2$ as they have 2 common and zero non common neighbours. There are $2n-12$ pairs of vertices having three common and one non common neighbours, which are of the form $(v_1, v_j), j = m+2, m+3, \ldots, n-2, (v_m, v_j), j = m+3, m+4, \ldots, n-1, (v_i, v_{m+1}), i = 2, 3, \ldots, m-2$ and $(v_i, v_{n-m}), i = 3, 4, \ldots, m-1$ with Hamming distance $n-4$. Remaining $m(n-m) - 2n + 4$ pairs of vertices have 4 common and zero non common neighbours and thus have Hamming distance $n-4$. So,

$$\sum_{u\in V_1, v\in V_2} H_{d_2}(s(u), s(v)) = (m(n-m) + 4n - 24)(n-4) + 4(n-2).$$

Therefore,

$$
\begin{aligned}
H_A((C_n)_2^P) = &\sum_{u,v\in V_1} H_{d_2}(s(u),s(v)) + \sum_{u,v\in V_2} H_{d_2}(s(u),s(v)) + \\
&\sum_{u\in V_1, v\in V_2} H_{d_2}(s(u),s(v)) \\
=& 2m^2 - 4m + 4 + 2(n-m)^2 - 4(n-m) + 4 + (m(n-m) + 4n - 24) \\
&(n-4) + 4(n-2) \\
=& (n-m)(mn-8m) + 2(n^2 - 2n + 8).
\end{aligned}
$$

□

Corollary 4.4. *If $P = \{V_1, V_2\}$ is a partition of cycle graph C_n with $\langle V_1\rangle = P_m$ and $\langle V_2\rangle = P_{n-m}$, then $HA((C_n)_{2(i)}^P) = (n-m)(mn-12m) + 3n^2 - 9n + 32$.*

Theorem 4.5. *Let $P = \{V_1, V_2\}$ be a partition of path graph P_n such that $V_1 = \langle P_m\rangle$ and $V_2 = \langle P_{n-m}\rangle$. Then $HA((P_n)_2^P) = (n-m)(mn-8m) + 2(n^2 - 2n + 7)$.*

Proof. Let $P = \{V_1, V_2\}$ be a partition of path graph P_n such that $V_1 = \{v_1, v_2, \ldots, v_m\}$ and $V_2 = \{v_{m+1}, v_{m+2}, \ldots, v_n\}$.
We consider following cases.
Case 1: Let $u, v \in V_1$, $u \sim v$ in P_n. They remain adjacent in $(P_n)_2^P$.
There are $m-1$ pairs of vertices of distance 1. Out of $m-1$ pairs, a pair of vertices (v_{m-1}, v_m) has $n-m-1$ common neighbours and $m-3$ non common neighbours. Thus Hamming distance of (v_{m-1}, v_m) is 4. Also (v_1, v_2) have $n-m$ common neighbours and $m-3$ non common neighbours. Hence Hamming distance of (v_1, v_2) is 3. Remaining $m-3$ pairs of vertices have $n-m$ common and $m-4$ non common neighbours. Hence, Hamming distance of these pairs of vertices is 4. Therefore

$$\sum_{d_{pn}=1} H_{d_2}(s(u),s(v)) = 4m - 5.$$

There are $m-2$ pairs of vertices which are at distance 2. These vertices are non adjacent in $(P_n)_2^P$. Out of $m-2$ pairs of vertices, (v_{m-2}, v_m) has $n-m$ common and $m-4$ non common neighbours and (v_1, v_3) has $n-m+1$ common and $m-4$ non common neighbours. So Hamming distance of (v_{m-2}, v_m) is 2 and that of (v_1, v_3) is 1. Remaining $m-4$ pairs of vertices have $n-m+1$ common and $m-5$ non common neighbours. Hence Hamming distance of such pairs of vertices is 2.

$$\sum_{d_{pn}=2} H_{d_2}(s(u),s(v)) = 2m - 5.$$

There are $\binom{m}{2} - 2m + 3$ pairs of vertices which are at distance greater than 2. Since $m-4$ pairs of vertices of the form $(v_1, v_i), i = 4, 5, \ldots, m-1$ have $n-m$ common and $m-5$ non common neighbours, Hamming distance of (v_1, v_i) is 3. Similarly, the Hamming distance of pairs of vertices of the form $(v_j, v_m), j = 2, 3, \ldots, m-3$ is 4, since (v_j, v_m) pairs have $n-m-1$ common and $m-5$ non common neighbours. Hamming distance of (v_1, v_m) is 4 as it has $n-m-1$ common and $m-4$ non common neighbours. Remaining $\binom{m}{2}-4m+10$ pairs of vertices have $n-m$ common and $m-6$ non common neighbours. Hence their Hamming distance is 4.

$$\sum_{d_{p_n}>2} H_{d_2}(s(u), s(v)) = 2m^2 - 11m + 15.$$

Therefore

$$\begin{aligned}\sum_{u,v\in V_1} H_{d_2}(s(u), s(v)) =& \sum_{d_{p_n}=1} H_{d_2}(s(u), s(v)) + \sum_{d_{p_n}=2} H_{d_2}(s(u), s(v)) \\ &+ \sum_{d_{p_n}>2} H_{d_2}(s(u), s(v)) \\ =& 2m^2 - 5m + 5.\end{aligned}$$

Case 2: If $u, v \in V_2$, then

$$\sum_{u,v\in V_2} H_{d_2}(s(u), s(v)) = 2(n-m)^2 - 5(n-m) + 5.$$

Case 3: Let $u \in V_1$ and $v \in V_2$. There are $m(n-m)$ pairs of vertices between V_1 and V_2. Among these pairs, consider (v_m, v_{m+1}) and (v_1, v_n). Now $v_m \sim v_{m+1}$ and $v_1 \nsim v_n$ in P_n. Also $v_m \nsim v_{m+1}$ and $v_1 \sim v_n$ in $(P_n)_2^P$. So both the pairs have 2 common and zero non common neighbours. Thus they have Hamming distance $n-4$ and $n-2$ respectively. Also (v_m, v_n), (v_1, v_{m+1}) have 2 common and one non common neighbours and thus have Hamming distance $n-3$. The pairs $(v_{m-1}, v_{m+1}), (v_m, v_{m+2})$ have Hamming distance $n-2$ as they have 2 common and zero non common neighbours. Further $n-4$ pairs of vertices which are of the form $(v_1, v_i), i = m+2, m+3, \ldots, n-m-1$ and $(v_j, v_n), j = 2, 3, \ldots, m-1$ have Hamming distance $n-4$ as they have 3 common and zero non common neighbours. There are $n-6$ pairs of vertices which are of the form $(v_i, v_{m+1}), i = 2, 3, \ldots, m-2$ and $(v_m, v_j), j = m+3, m+4, \ldots, n-m-1$, having 3 common and 1 non common neighbours. Hence they have Hamming distance $n-4$. Remaining $m(n-m)-2n+4$ pairs of vertices have 4 common and zero non common neighbours. Hence Hamming distance of these pairs of vertices is $n-4$.
Thus

$$\begin{aligned}\sum_{u\in V_1, v\in V_2} H_{d_2}(s(u), s(v)) =& (n-4) + (n-4)(n-3) + 2(n-3) + 3(n-2) \\ &+ (n-6)(n-4) \\ &+ (m(n-m) - 2n + 4)(n-4) \\ =& mn(n-m) + 4m^2 - 4mn + n + 4.\end{aligned}$$

Therefore

$$\begin{aligned}H_A((P_n)_2^P) =& \sum_{u,v\in V_1} H_{d_2}(s(u),s(v)) + \sum_{u,v\in V_2} H_{d_2}(s(u),s(v)) \\ &+ \sum_{u\in V_1, v\in V_2} H_{d_2}(s(u),s(v)) \\ =& 2m^2 - 5m + 5 + 2(n-m)^2 - 5(n-m) + 5 + mn(n-m) \\ &+ 4m^2 - 4mn + n + 4 \\ =& (n-m)(mn-8m) + 2(n^2 - 2n + 7).\end{aligned}$$

□

Corollary 4.6. *Let $P = \{V_1, V_2\}$ be a partition of path graph P_n such that $V_1 = \langle P_m\rangle$ and $V_2 = \langle P_{n-m}\rangle$. Then $HA((P_n)_{2(i)}^P) = (n-m)(mn-12m) + 3n^2 - 9n + 26$.*

Theorem 4.7. *Let $P = \{V_1, V_2\}$ be a partition of complete bipartite graph $K_{r,s}$ such that $\langle V_1\rangle = K_{a,b}$ and $\langle V_2\rangle = K_{r-a,s-b}$. Then $HA((K_{r,s})_2^P) = (r+s)[(r-s)(b-a) - (a-b)^2 + rs]$.*

Proof. Let $\langle V_1\rangle = K_{a,b}$ and $\langle V_2\rangle = K_{r-a,s-b}$.
Case 1: Let $u, v \in V_1$. Suppose $u \sim v$ in $K_{r,s}$. As there are ab pairs of vertices in $(K_{r,s})_2^P$ having no common as well as non common neighbours, the Hamming distance between u and v is $r+s$.

If $u \nsim v$ in $K_{r,s}$, then we see that $2-$ complement of $K_{r,s}$ will have $\binom{a}{2}$ pairs of vertices with $b+r-a$ common and $s-b+a-2$ non common neighbours and $\binom{b}{2}$ pairs of vertices with $a+s-b$ common and $r-a+b-2$ non common neighbours. Thus $H_{d_2}(s(u),s(v)) = 0$ for every (u,v) in $(K_{r,s})_2^P$. Hence

$$\begin{aligned}\sum_{u,v\in V_1} H_{d_2}(s(u),s(v)) =& ab(r+s) + \binom{a}{2}(0) + \binom{b}{2}(0) \\ =& ab(r+s).\end{aligned}$$

Case 2: Let $u, v \in V_2$. Then $\sum_{u,v\in V_2} H_{d_2}(s(u),s(v)) = (r-a)(s-b)(r+s)$.
Case 3: Let $u \in V_1$ and $v \in V_2$. Then there are $a(r-a) + b(s-a)$ pairs of vertices with Hamming distance $r+s$ as they have zero common and zero non common neighbours and remaining pairs of vertices have Hamming distance zero. So

$$\sum_{u\in V_1, v\in V_2} H_{d_2}(s(u),s(v)) = (r+s)(a(r-a) + b(s-a)).$$

Therefore

$$\begin{aligned}H_A((K_{r,s})_2^P) =& \sum_{u,v\in V_1} H_{d_2}(s(u),s(v)) + \sum_{u,v\in V_2} H_{d_2}(s(u),s(v)) \\ &+ \sum_{u\in V_1, v\in V_2} H_{d_2}(s(u),s(v)) \\ =& ab(r+s) + (r-a)(s-b)(r+s) + (r+s)(a(r-a)+b(s-a)) \\ =& (r+s)[(s-r)(b-a) - (a-b)^2 + rs].\end{aligned}$$

□

Corollary 4.8. *Let $P = \{V_1, V_2\}$ be a partition of complete bipartite graph $K_{r,s}$ such that $\langle V_1\rangle = K_{a,b}$ and $\langle V_2\rangle = K_{r-a,s-b}$. Then $HA((K_{r,s})_{2(i)}^P) = b(4r - r^2 + a(2r-8)) - r - a(-r^2+4r) + s^2(b-a+r+1) - s(2r-4(a-b) + (a-b)^2 - r^2 + 1) + r^2 - (a^2+b^2)(r-4)$.*

Corollary 4.9. *Let $P = \{V_1, V_2\}$ be a partition of complete bipartite graph $K_{r,s}$ with $\langle V_1\rangle = rK_1$ and $\langle V_2\rangle = sK_1$. Then $HA((K_{r,s})_2^P) = 0$.*

Proof. Since $(K_{r,s})_2^P$ is totally disconnected graph with respect to P, $HA((K_{r,s})_2^P) = 0$. □

Corollary 4.10. *Let $P = \{V_1, V_2\}$ be a partition of complete bipartite graph $K_{r,s}$ with $\langle V_1\rangle = rK_1$ and $\langle V_2\rangle = sK_1$. Then $HA((K_{r,s})_{2(i)}^P) = (r+s)(r+s-1)$.*

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