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SOME MORE RESULTS ON f-KENMOTSU 3-MANIFOLDS

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ABSTRACT. In the present paper, we characterize f-Kenmotsu 3-manifolds with respect to the semi-symmetric metric connection satisfying certain curvature conditions: $\bar{R} \cdot \bar{Q} = 0$, $\bar{Q} \cdot \bar{P} = 0$, $\bar{P} \cdot \phi = 0$. Moreover, we have discussed Ricci symmetries in an f-Kenmotsu 3-manifold with respect to the semi-symmetric metric connection. In the paper, we also provided an example of f-Kenmotsu 3-manifolds, where f is a smooth function.

1. Introduction

In 1972, Kenmotsu [16] introduced and studied a new class of almost contact metric manifolds, later known as Kenmotsu manifolds. Olszak and Rosca [21] studied an f-Kenmotsu manifold, an almost contact metric manifold which is normal and locally conformal almost cosymplectic. Further, they give a geometric interpretation of f-Kenmotsu manifolds and proved that a Ricci symmetric f-Kenmotsu manifold is an Einstein manifold. Some certain curves on a 3-dimensional f-Kenmotsu manifold have been studied by Calin et al. [4], Majhi and Biswas [17] and Mondal [18]. Recently, f-Kenmotsu manifolds have been studied by various authors such as Haseeb and Prasad [9], Hui et al. [14], Nagaraja and Venu [19], Yildiz [31] and many others. In 1924, the notion of the semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten [6]. Later in 1932, Hayden [10] introduced the idea of metric connection with a torsion on a Riemannian manifold. A semi-symmetric connection on a Riemannian manifold was systematically studied by Yano [30], which was further studied by Barman and De [2], Özgur et al. [22], Shaikh and Hui [28, 29] and many others.

The projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a $(2n+1)$-dimensional Riemannian manifold [13]. If there exists a one to one correspondence between each coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes, the projective curvature tensor is defined by [1, 7, 23]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{1.1}$$

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where $X, Y, Z \in \chi(M)$, $R(X,Y)Z$ and $S(Y,Z)$ are the curvature tensor and the Ricci tensor with respect to the Levi-Civita connection, respectively.

Let (M,g) be a Riemannian manifold with the Levi-Civita connection ∇. A linear connection $\overline{\nabla}$ on (M,g) is said to be semi-symmetric [30] if its torsion tensor $\bar{T}$ is given by

$$\bar{T}(X,Y) = \pi(Y)X - \pi(X)Y,$$

where π is a 1-form on M and associated with the vector field ρ by

$$\pi(X) = g(X,\rho)$$

for all vector fields X on M.

A semi-symmetric connection $\overline{\nabla}$ is called a semi-symmetric metric connection if it satisfies the condition

$$\overline{\nabla} g = 0.$$

On an almost contact metric manifold, a semi-symmetric metric connection is defined by replacing 1-form π by the contact 1-form η, i.e.,

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y,$$

where $g(X,\xi) = \eta(X)$ for all $X \in \chi(M)$.

A relation between a semi-symmetric metric connection $\overline{\nabla}$ and the Levi-Civita connection ∇ on M and is given by [30]

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi, \tag{1.2}$$

where $\eta(X) = g(X,\xi)$.

The paper is organized as follows: In section 2, we give a brief introduction of f-Kenmotsu manifolds and an example of f-Kenmotsu 3-manifold is also given in this section. In section 3, we present the curvature tensor and the Ricci tensor of an f-Kenmotsu 3-manifold with respect to the semi-symmetric metric connection. In section 4, we show that the Ricci symmetries in an f-Kenmotsu 3-manifold with respect to the semi-symmetric metric connection and the Levi-Civita connection are equivalent if and only if the manifold is an η-Einstein manifold. Sections 5 and 6 are devoted to study f-Kenmotsu 3-manifolds satisfying the curvature conditions $\bar{R} \cdot \bar{Q} = 0$ and $\bar{Q} \cdot \bar{P} = 0$, respectively. In section 7, we study ϕ-projectively semisymmetric f-Kenmotsu 3-manifolds with respect to the semi-symmetric metric connection. The last section 8 deals with the study of f-Kenmotsu 3-manifolds with respect to the semi-symmetric metric connection admitting cyclic η-recurrent Ricci tensor.

2. Preliminaries

Let M be a (2n+1)-dimensional differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g) which satisfies the following equations [3]:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \tag{2.1}$$

$$\eta(X) = g(X,\xi), \quad \eta \circ \phi = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.3}$$

for all vector fields $X, Y \in \chi(M)$, where $\chi(M)$ is a set of all smooth vector fields on M, η is a 1-form, ξ is a vector field, g is a metric tensor field and ϕ is a vector field of type $(1,1)$. We say that (M, ϕ, ξ, η, g) is an f-Kenmotsu manifold if the Levi-Civita connection of g satisfies

$$(\nabla_X \phi)Y = f[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{2.4}$$

where $f \in C^\infty(M)$ such that $df \wedge \eta = 0$ [16]. If $f = \beta$ is nonzero constant, then the manifold is said to be β-Kenmotsu manifold [5, 15, 26, 27]. 1-Kenmotsu manifold is a Kenmotsu manifold [16]. If $f = 0$, then the manifold is cosymplectic. An f-Kenmotsu manifold is said to be regular if $f^2 + f' \neq 0$, where $f' = \xi f$.

In an f-Kenmotsu manifold, from (2.4) we have

$$\nabla_X \xi = f[X - \eta(X)\xi]. \tag{2.5}$$

Using (2.5), we have

$$(\nabla_X \eta)Y = f[g(X,Y) - \eta(X)\eta(Y)]. \tag{2.6}$$

The condition $df \wedge \eta = 0$ holds if dim $M \geq 5$. This does not hold in general if dim $M = 3$ [21]. In a 3-dimensional Riemannian manifold, we have

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &\quad - S(X,Z)Y - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{2.7}$$

In an f-Kenmotsu 3-manifold M, we have [20]

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} + 2f^2 + 2f')[g(Y,Z)X - g(X,Z)Y] \\ &\quad - (\frac{r}{2} + 3f^2 + 3f')[g(Y,Z)\eta(X)\xi \\ &\quad - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y], \end{aligned} \tag{2.8}$$

$$S(X,Y) = (\frac{r}{2} + f^2 + f')g(X,Y) - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\eta(Y), \tag{2.9}$$

$$QX = (\frac{r}{2} + f^2 + f')X - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\xi, \tag{2.10}$$

for all vector fields $X, Y, Z \in \chi(M)$, where R, S, Q and r are the Riemannian curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature, respectively on M. Also from (2.8), (2.9) and (2.10), we get

$$R(X,Y)\xi = -(f^2 + f')[\eta(Y)X - \eta(X)Y], \tag{2.11}$$

$$R(\xi, X)Y = -(f^2 + f')[g(X,Y)\xi - \eta(Y)X], \tag{2.12}$$

$$\eta(R(X,Y)Z) = -(f^2 + f')[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.13}$$

$$S(X, \xi) = -2(f^2 + f')\eta(X), \tag{2.14}$$

$$Q\xi = -2(f^2 + f')\xi \tag{2.15}$$

for all $X, Y, Z \in \chi(M)$.

Definition 2.1. *An f-Kenmotsu 3-manifold M is said to be an η-Einstein manifold if its non vanishing Ricci tensor is of the form*

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a, b are smooth functions on M. If $b = 0$ (resp., $a = 0$), then M is said to be an Einstein (resp., special type of $\eta-$Einstein) manifold.

An example of an f-Kenmotsu manifold when f is a smooth function. We consider the 3-dimensional manifold $M = \{(u,v,w) \in R^3\}$, where (u,v,w) are the standard coordinates in R^3. Let e_1, e_2 and e_3 be the vector fields on M given by

$$e_1 = e^{-w}\frac{\partial}{\partial u},\ e_2 = e^{-w}\frac{\partial}{\partial v},\ e_3 = e^{-w}\frac{\partial}{\partial w} = \xi,$$

which are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1,\ g(e_1,e_2) = g(e_2,e_3) = g(e_3,e_1) = 0.$$

Let η be the 1-form on M defined as $\eta(U) = g(U,e_3) = g(U,\xi)$ for all $U \in \chi(M)$, and let ϕ be the $(1,1)$ tensor field on M defined as

$$\phi e_1 = e_2,\ \phi e_2 = -e_1,\ \phi e_3 = 0.$$

By applying the linearity of ϕ and g, we have

$$\eta(\xi) = g(\xi,\xi) = 1,\ \phi^2 U = -U + \eta(U)\xi,\ \eta(\phi U) = 0,$$

$$g(U,\xi) = \eta(U),\ g(\phi U,\phi V) = g(U,V) - \eta(U)\eta(V)$$

for all $U, V \in \chi(M)$. Then we have

$$[e_1,e_2] = 0,\ [e_3,e_1] = -e^{-w}e_1,\ [e_2,e_3] = e^{-w}e_2.$$

The Riemannian connection ∇ of the metric tensor g is given by

$$2g(\nabla_U V,W) = Ug(V,W) + Vg(W,U) - Wg(U,V) - g(U,[V,W]) + g(V,[W,U]) + g(W,[U,V]),$$

which is known as Koszul's formula [11, 12]. Using Koszul's formula, we can easily calculate

$$\nabla_{e_1}e_1 = -e^{-w}e_3,\ \nabla_{e_1}e_2 = 0,\ \nabla_{e_1}e_3 = e^{-w}e_1,\ \nabla_{e_2}e_1 = 0,$$

$$\nabla_{e_2}e_2 = -e^{-w}e_3,\ \nabla_{e_2}e_3 = e^{-w}e_2,\ \nabla_{e_3}e_1 = 0,\ \nabla_{e_3}e_2 = 0,\ \nabla_{e_3}e_3 = 0.$$

It can be easily verified that the manifold satisfies

$$\nabla_U\xi = f[U - \eta(U)\xi] \quad \text{and} \quad (\nabla_U\phi)V = f[g(\phi U,V)\xi - \eta(V)\phi U] \text{ for } \xi = e_3, \text{ where } f = e^{-w}.$$

Hence we conclude that M is an f-Kenmotsu 3-manifold. Also we have $f^2 + f' \neq 0$. Hence M is a regular f-Kenmotsu 3-manifold.

3. **Curvature tensor of f-Kenmotsu 3-manifolds with respect to the semi-symmetric metric connection**

Let $\bar{R}$ and R be the Riemannian curvature tensors with respect to the semi-symmetric metric connection $\bar{\nabla}$ and Levi-Civita connection ∇, respectively of an f-Kenmotsu 3-manifold M. Then $\bar{R}$ and R are related by [25]

$$
\begin{aligned}
(3.1)\qquad \overline{R}(X,Y)Z &= R(X,Y)Z-(f+1)[\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X]\\
&\quad -(f+1)[g(X,Z)\eta(Y)-g(Y,Z)\eta(X)]\xi\\
&\quad +(2f+1)[g(X,Z)Y-g(Y,Z)X].
\end{aligned}
$$

Contracting X in (3.1), we have

$$(3.2)\qquad \overline{S}(Y,Z)=S(Y,Z)+(f+1)\eta(Y)\eta(Z)-(3f+1)g(Y,Z),$$

where $\overline{S}$ and S are the Ricci tensors of the connections $\overline{\nabla}$ and ∇, respectively on M. The equation (3.2) yields

$$(3.3)\qquad \overline{Q}Y=QY+(f+1)\eta(Y)\xi-(3f+1)Y,$$

where $\overline{Q}$ and Q are the Ricci operators of the connections $\overline{\nabla}$ and ∇, respectively on M.

Lemma 3.1. *Let M be an f-Kenmotsu 3-manifold with respect to the semi-symmetric metric connection. Then, we have*

$$(3.4)\qquad \bar{R}(\xi,X)Y=-(f^2+f'+f)(g(X,Y)\xi-\eta(Y)X),$$

$$(3.5)\qquad \bar{S}(X,\xi)=-2(f^2+f'+f)\eta(X),$$

$$(3.6)\qquad \bar{Q}\xi=-2(f^2+f'+f)\xi,$$

$$(3.7)\qquad \bar{\nabla}_X\xi=(1+f)(X-\eta(X)\xi),$$

$$(3.8)\qquad (\bar{\nabla}_X\eta)Y=(1+f)(g(X,Y)-\eta(X)\eta(Y))$$

for all $X,Y\in\chi(M)$.

Proof. Putting $X=\xi$ in (3.1) and using (2.1), (2.2), (2.14), we get (3.4). From the equations (2.1), (2.2) (2.14) and (3.2), (3.5) follows. Taking $Z=\xi$ in (3.3) then using (2.1) and (2.15), we get (3.6). Taking $Y=\xi$ in (1.2) and making use of (2.1), (2.2) and (2.5), we obtain (3.7). (3.8) follows from (2.2), (2.6) and (3.7). □

4. **Ricci symmetry in an f-Kenmotsu 3-manifold with respect to the connections $\bar{\nabla}$ and ∇**

Assuming that the manifold is Ricci symmetric with respect to the semi-symmetric metric connection $\bar{\nabla}$, therefore we have

$$(4.1)\qquad (\bar{R}(X,Y)\cdot\bar{S})(U,V)=-\bar{S}(\bar{R}(X,Y)U,V)-\bar{S}(U,\bar{R}(X,Y)V)$$

for all $X,Y,U,V\in\chi(M)$. In view of (3.1), (4.1) takes the form

$$
\begin{aligned}
(4.2)\qquad (\bar{R}(X,Y)\cdot\bar{S})(U,V) &= -\bar{S}(R(X,Y)U,V)-\bar{S}(U,R(X,Y)V)\\
&\quad +(f+1)[\eta(X)\eta(U)\bar{S}(Y,V)-\eta(Y)\eta(U)\bar{S}(X,V)\\
&\quad +g(X,U)\eta(Y)\bar{S}(\xi,V)-g(Y,U)\eta(X)\bar{S}(\xi,V)\\
&\quad +\eta(X)\eta(V)\bar{S}(U,Y)-\eta(Y)\eta(V)\bar{S}(U,X)
\end{aligned}
$$

$$+g(X,V)\eta(Y)\bar{S}(U,\xi) - g(Y,V)\eta(X)\bar{S}(U,\xi)]$$
$$-(2f+1)[g(X,U)\bar{S}(Y,V) - g(Y,U)\bar{S}(X,V)$$
$$+g(X,V)\bar{S}(U,Y) - g(Y,V)\bar{S}(U,X)].$$

By using (3.2) and (3.5) in (4.2), we have

$$(\bar{R}(X,Y)\cdot\bar{S})(U,V) = (R(X,Y)\cdot S)(U,V) + (3f+1)[R(X,Y,U,V) + R(X,Y,V,U)]$$
$$-(1+f)[\eta(R(X,Y)U)\eta(V) + \eta(R(X,Y)V)\eta(U) - \eta(X)\eta(U)S(Y,V)$$
$$+\eta(Y)\eta(U)S(X,V) - \eta(X)\eta(V)S(Y,U) + \eta(Y)\eta(V)S(X,U)]$$
$$-(1+f)(2f^2+2f'+f)[\eta(Y)\eta(V)g(X,U) - \eta(X)\eta(V)g(Y,U)$$
$$+\eta(Y)\eta(U)g(X,V) - \eta(X)\eta(U)g(Y,V)$$

which by using (2.13) and the fact that $R(X,Y,U,V) + R(X,Y,V,U) = 0$ reduces to

$$(4.3)\qquad (\bar{R}(X,Y)\cdot\bar{S})(U,V) = (R(X,Y)\cdot S)(U,V)$$
$$-(1+f)[(f^2+f')(g(X,U)\eta(Y) - g(Y,U)\eta(X))\eta(V)$$
$$+(f^2+f')(g(X,V)\eta(Y) - g(Y,V)\eta(X))\eta(U)$$
$$-\eta(X)\eta(U)S(Y,V) + \eta(Y)\eta(U)S(X,V)$$
$$-\eta(X)\eta(V)S(Y,U) + \eta(Y)\eta(V)S(X,U)]$$
$$-(1+f)(2f^2+2f'+f)[\eta(Y)\eta(V)g(X,U)$$
$$-\eta(X)\eta(V)g(Y,U) + \eta(Y)\eta(U)g(X,V) - \eta(X)\eta(U)g(Y,V)].$$

Suppose that $(\bar{R}(X,Y)\cdot\bar{S})(U,V) = (R(X,Y)\cdot S)(U,V)$, then from (4.3), it follows that

$$(4.4)\qquad (1+f)[(f^2+f')(g(X,U)\eta(Y) - g(Y,U)\eta(X))\eta(V)$$
$$+(f^2+f')(g(X,V)\eta(Y) - g(Y,V)\eta(X))\eta(U)$$
$$-\eta(X)\eta(U)S(Y,V) + \eta(Y)\eta(U)S(X,V)$$
$$-\eta(X)\eta(V)S(Y,U) + \eta(Y)\eta(V)S(X,U)]$$
$$+(1+f)(2f^2+2f'+f)[\eta(Y)\eta(V)g(X,U)$$
$$-\eta(X)\eta(V)g(Y,U) + \eta(Y)\eta(U)g(X,V) - \eta(X)\eta(U)g(Y,V)] = 0.$$

Taking $X = U = \xi$ in (4.4) then using (2.1), (2.2) and (2.14), we obtain

$$(4.5)\qquad S(Y,V) = -(3f^2+3f'+f)g(Y,V) + (f^2+f'+f)\eta(Y)\eta(V).$$

If $S(Y,V) = -(3f^2+3f'+f)g(Y,V) + (f^2+f'+f)\eta(Y)\eta(V)$, then from (4.3), it follows that

$$(\bar{R}(X,Y)\cdot\bar{S})(U,V) = (R(X,Y)\cdot S)(U,V).$$

Thus we have the following theorem:

Theorem 4.1. *Ricci symmetries in an f-Kenmotsu 3-manifold with respect to the semi-symmetric metric connection and the Levi-Civita connection are equivalent if and only if the manifold is an η-Einstein manifold of the form* (4.5).

5. f-Kenmotsu 3-manifolds with respect to the semi-symmetric metric connection satisfying $\bar{R}\cdot\bar{Q}=0$

Let an f-Kenmotsu 3-manifold with respect to the semi-symmetric metric connection satisfies $\bar{R}\cdot\bar{Q}=0$. Then we have

$$(\bar{R}(X,Y)\cdot\bar{Q})Z=\bar{R}(X,Y)\bar{Q}Z-\bar{Q}(\bar{R}(X,Y)Z)=0 \tag{5.1}$$

for all $X,Y,Z\in\chi(M)$. With the help of (3.1) and (3.3), we find

$$\begin{aligned}\bar{R}(X,Y)\bar{Q}Z=R(X,Y)\bar{Q}Z&-(f+1)(\eta(X)\eta(\bar{Q}Z)Y-\eta(Y)\eta(\bar{Q}Z)X\\&+g(X,\bar{Q}Z)\eta(Y)\xi-g(Y,\bar{Q}Z)\eta(X)\xi)\\&+(2f+1)(g(X,\bar{Q}Z)Y-g(Y,\bar{Q}Z)X),\end{aligned} \tag{5.2}$$

$$\begin{aligned}\bar{Q}(\bar{R}(X,Y)Z)=\bar{Q}(R(X,Y)Z)&-(f+1)(\eta(X)\eta(Z)\bar{Q}Y-\eta(Y)\eta(Z)\bar{Q}X\\&+g(X,Z)\eta(Y)\bar{Q}\xi-g(Y,Z)\eta(X)\bar{Q}\xi)\\&+(2f+1)(g(X,Z)\bar{Q}Y-g(Y,Z)\bar{Q}X).\end{aligned} \tag{5.3}$$

Also from (2.8) and (3.3), we find

$$\begin{aligned}R(X,Y)\bar{Q}Z=&(\frac{r}{2}+2f^2+2f')(\bar{S}(Y,Z)X-\bar{S}(X,Z)Y)\\&-(\frac{r}{2}+3f^2+3f')[\bar{S}(Y,Z)\eta(X)\xi-\bar{S}(X,Z)\eta(Y)\xi\\&-2(f^2+f'+f)(\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y)],\end{aligned} \tag{5.4}$$

$$\begin{aligned}\bar{Q}(R(X,Y)Z)=&(\frac{r}{2}+2f^2+2f')(g(Y,Z)\bar{Q}X-g(X,Z)\bar{Q}Y)\\&-(\frac{r}{2}+3f^2+3f')[g(Y,Z)\eta(X)\bar{Q}\xi-g(X,Z)\eta(Y)\bar{Q}\xi\\&+\eta(Y)\eta(Z)\bar{Q}X-\eta(X)\eta(Z)\bar{Q}Y].\end{aligned} \tag{5.5}$$

By making use of (5.2)-(5.5) and $\eta(\bar{Q}Z)=-2(f^2+f'+f)\eta(Z)$, (5.1) becomes

$$\begin{aligned}&(\frac{r}{2}+2f^2+2f'-2f-1)(\bar{S}(Y,Z)X-\bar{S}(X,Z)Y-g(Y,Z)\bar{Q}X+g(X,Z)\bar{Q}Y)\\&+(\frac{r}{2}+3f^2+3f'-f-1)[\bar{S}(X,Z)\eta(Y)\xi-\bar{S}(Y,Z)\eta(X)\xi-\eta(X)\eta(Z)\bar{Q}Y+\eta(Y)\eta(Z)\bar{Q}X\\&+2(f^2+f'+f)(\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi)]=0.\end{aligned} \tag{5.6}$$

Taking the inner product of (5.6) with ξ, we obtain

$$\bar{S}(Y,Z)\eta(X)-\bar{S}(X,Z)\eta(Y)+2(f^2+f'+f)(g(Y,Z)\eta(X)-g(X,Z)\eta(Y))=0, \tag{5.7}$$

where $(f^2+f'+f)\neq 0$. Putting $X=\xi$ in (5.7) and using (2.1), (2.2) and (3.5), it follows that

$$\bar{S}(Y,Z)=-2(f^2+f'+f)g(Y,Z). \tag{5.8}$$

In view of (3.2), (5.8) takes the form

$$S(Y,Z)=(f+1-2f^2-2f')g(Y,Z)-(1+f)\eta(Y)\eta(Z). \tag{5.9}$$

Thus we have the following theorem:

Theorem 5.1. *If an f-Kenmotsu 3-manifold with respect to the semi- symmetric metric connection satisfies $\bar{R}\cdot\bar{Q}=0$, then the manifold is an $\eta-$Einstein manifold of the form* (5.9).

6. f-Kenmotsu 3-manifolds with respect to the semi-symmetric metric connection satisfying $\bar{Q}\cdot\bar{P}=0$

Let an f-Kenmotsu 3-manifold with respect to the semi-symmetric metric connection satisfies $\bar{Q}\cdot\bar{P}=0$, where $\bar{P}$ is the projective curvature tensor with respect to the semi-symmetric metric connection. Then we have

$$\bar{Q}(\bar{P}(X,Y)Z)-\bar{P}(\bar{Q}X,Y)Z-\bar{P}(X,\bar{Q}Y)Z-\bar{P}(X,Y)\bar{Q}Z=0 \tag{6.1}$$

for all $X,Y,Z\in\chi(M)$. In view of (1.1), (6.1) takes the form

$$\begin{aligned}\bar{Q}(\bar{R}(X,Y)Z)-\bar{R}(\bar{Q}X,Y)Z-\bar{R}(X,\bar{Q}Y)Z\\ -\bar{R}(X,Y)\bar{Q}Z+\bar{S}(Y,\bar{Q}Z)X-\bar{S}(X,\bar{Q}Z)Y=0.\end{aligned} \tag{6.2}$$

Taking the inner product of (6.2) with ξ, we have

$$\begin{aligned}\eta(\bar{Q}(\bar{R}(X,Y)Z))-\eta(\bar{R}(\bar{Q}X,Y)Z)-\eta(\bar{R}(X,\bar{Q}Y)Z)\\ -\eta(\bar{R}(X,Y)\bar{Q}Z)+\bar{S}(Y,\bar{Q}Z)\eta(X)-\bar{S}(X,\bar{Q}Z)\eta(Y)=0.\end{aligned} \tag{6.3}$$

Putting $Y=\xi$ in (6.3), we have

$$\begin{aligned}\eta(\bar{Q}(\bar{R}(X,\xi)Z))-\eta(\bar{R}(\bar{Q}X,\xi)Z)-\eta(\bar{R}(X,\bar{Q}\xi)Z)\\ -\eta(\bar{R}(X,\xi)\bar{Q}Z)+\bar{S}(\xi,\bar{Q}Z)\eta(X)-\bar{S}(X,\bar{Q}Z)=0.\end{aligned} \tag{6.4}$$

With the help of (3.3), (3.4) and (3.6), we find

$$\begin{cases}\eta(\bar{Q}(\bar{R}(X,\xi)Z))=\eta(\bar{R}(X,\bar{Q}\xi)Z)=2(f^2+f'+f)^2(\eta(X)\eta(Z)-g(X,Z)),\\ \eta(\bar{R}(\bar{Q}X,\xi)Z)=\eta(\bar{R}(X,\xi)\bar{Q}Z)=(f^2+f'+f)(\bar{S}(X,Z)+2(f^2+f'+f)\eta(X)\eta(Z)),\\ \bar{S}(\xi,\bar{Q}Z)=4(f^2+f'+f)^2\eta(Z).\end{cases} \tag{6.5}$$

Also from the equations (2.9), (3.2) and $\eta(\bar{Q}Z)=-2(f^2+f'+f)\eta(Z)$, we get

$$\begin{aligned}\bar{S}(X,\bar{Q}Z)=(\frac{r}{2}+f^2+f'-3f-1)\bar{S}(X,Z)\\ +2(f^2+f'+f)(\frac{r}{2}+3f^2+3f'-f-1)\eta(X)\eta(Z).\end{aligned} \tag{6.6}$$

By virtue of (6.5) and (6.6), (6.4) turns to

$$(\frac{r}{2}+3f^2+3f'-f-1)[\bar{S}(X,Z)+2(f^2+f'+f)\eta(X)\eta(Z)]=0. \tag{6.7}$$

Thus we have either $r=-6(f^2+f')+2(f+1)$, or

$$\bar{S}(X,Z)=-2(f^2+f'+f)\eta(X)\eta(Z). \tag{6.8}$$

In view of (3.2), (6.8) takes the form

$$S(X,Z)=(3f+1)g(Y,Z)-(2f^2+2f'+3f+1)\eta(X)\eta(Z). \tag{6.9}$$

Thus we have the following theorem:

Theorem 6.1. *If an f-Kenmotsu 3-manifold with respect to the semi- symmetric metric connection satisfies $\bar{Q}\cdot\bar{P}=0$, then either the scalar curvature is $-6(f^2+f')+2(f+1)$ or the manifold is an $\eta-$Einstein manifold of the form (6.9).*

7. ϕ-projectively semisymmetric f-Kenmotsu 3-manifolds with respect to the semi-symmetric metric connection

Definition 7.1. *An f-Kenmotsu 3-manifold with respect to the semi-symmetric metric connection is said to be ϕ-projectively semisymmetric if* [24]

$$\bar{P}(X,Y)\cdot\phi = 0$$

for all $X,Y \in \chi(M)$.

Let M be a ϕ-projectively semisymmetric f-Kenmotsu 3-manifold with respect to the semi-symmetric metric connection. Therefore $\bar{P}(X,Y)\cdot\phi = 0$ turns into

$$(\bar{P}(X,Y)\cdot\phi)Z = \bar{P}(X,Y)\phi Z - \phi\bar{P}(X,Y)Z = 0. \tag{7.1}$$

Taking $X = \xi$ in (7.1), we have

$$(\bar{P}(\xi,Y)\cdot\phi)Z = \bar{P}(\xi,Y)\phi Z - \phi\bar{P}(\xi,Y)Z = 0. \tag{7.2}$$

From (1.1), we find

$$\bar{P}(\xi,Y)\phi Z = -(f^2+f'+f)g(Y,\phi Z)\xi - \frac{1}{2}\bar{S}(Y,\phi Z)\xi, \tag{7.3}$$

$$\phi\bar{P}(\xi,Y)Z = 0. \tag{7.4}$$

By making use of (7.3) and (7.4) in (7.2), we obtain $\bar{S}(Y,\phi Z)\xi = -2(f^2+f'+f)g(Y,\phi Z)\xi$ which by taking the inner product with ξ gives

$$\bar{S}(Y,\phi Z) = -2(f^2+f'+f)g(Y,\phi Z). \tag{7.5}$$

By replacing Z by ϕZ in (7.5) and using (2.1), we get

$$\bar{S}(Y,Z) = -2(f^2+f'+f)g(Y,Z). \tag{7.6}$$

In view of (3.2), (7.6) takes the form

$$S(Y,Z) = (f+1-2f^2-2f')g(Y,Z) - (1+f)\eta(Y)\eta(Z). \tag{7.7}$$

Thus we have the following theorem:

Theorem 7.2. *A ϕ-projectively semisymmetric f-Kenmotsu 3-manifold with respect to the semi-symmetric metric connection is an $\eta-$Einstein manifold of the form* (7.7).

8. f-Kenmotsu 3-manifolds with respect to the semi-symmetric metric connection admitting cyclic η-recurrent Ricci tensor

Definition 8.1. *An f-Kenmotsu 3-manifold with respect to the semi-symmetric metric connection is said to have cyclic η-recurrent Ricci tensor if its Ricci tensor $\overline{S}$ of type $(0,2)$ is non-zero and satisfies the following condition* [8]

$$(\overline{\nabla}_X\overline{S})(Y,Z) + (\overline{\nabla}_Y\overline{S})(Z,X) + (\overline{\nabla}_Z\overline{S})(X,Y) = \eta(X)\bar{S}(Y,Z) + \eta(Y)\bar{S}(Z,X) + \eta(Z)\bar{S}(X,Y) \tag{8.1}$$

for all $X,Y,Z \in \chi(M)$.

Let an f-Kenmotsu 3-manifold with respect to the semi-symmetric metric connection admits cyclic η-recurrent Ricci tensor, then (8.1) holds. By using (2.9) in (3.2), we have

$$\bar{S}(Y,Z) = (\frac{r}{2}+f^2+f'-3f-1)g(Y,Z) - (\frac{r}{2}+3f^2+3f'-f-1)\eta(Y)\eta(Z). \tag{8.2}$$

Taking the covariant derivative of (8.2) with respect to X, we find

$$\begin{aligned}(\overline{\nabla}_X\overline{S})(Y,Z) &= [\frac{dr(X)}{2}+(2f-3)(Xf)+(Xf')]g(Y,Z)\\ &-[\frac{dr(X)}{2}+(6f-1)(Xf)+3(Xf')]\eta(Y)\eta(Z)\\ &-(1+f)[\frac{r}{2}+3f^2+3f'-f-1](g(\phi X,\phi Y)\eta(Z)+g(\phi X,\phi Z)\eta(Y)).\end{aligned} \tag{8.3}$$

Similarly, we have

$$\begin{aligned}(\overline{\nabla}_Y\overline{S})(Z,X) &= [\frac{dr(Y)}{2}+(2f-3)(Yf)+(Yf')]g(Z,X)\\ &-[\frac{dr(Y)}{2}+(6f-1)(Yf)+3(Yf')]\eta(Z)\eta(X)\\ &-(1+f)[\frac{r}{2}+3f^2+3f'-f-1](g(\phi Y,\phi Z)\eta(X)+g(\phi Y,\phi X)\eta(Z))\end{aligned} \tag{8.4}$$

and

$$\begin{aligned}(\overline{\nabla}_Z\overline{S})(X,Y) &= [\frac{dr(Z)}{2}+(2f-3)(Zf)+(Zf')]g(X,Y)\\ &-[\frac{dr(Z)}{2}+(6f-1)(Zf)+3(Zf')]\eta(X)\eta(Y)\\ &-(1+f)[\frac{r}{2}+3f^2+3f'-f-1](g(\phi Z,\phi X)\eta(Y)+g(\phi Z,\phi Y)\eta(X)).\end{aligned} \tag{8.5}$$

By making use of (8.2)-(8.5) in (8.1) and taking $r=-6f^2+2f+2$, f being constant it follows that $g(Y,Z)\eta(X)+g(X,Z)\eta(Y)+g(X,Y)\eta(Z)=0$ which by putting $X=\xi$ gives

$$g(Y,Z) = -2\eta(Y)\eta(Z). \tag{8.6}$$

Now replacing Y by QY in (8.6) and using (2.10), we get $S(Y,Z)=4f^2\eta(Y)\eta(Z)$. Thus we have the following theorem:

Theorem 8.2. *If an f-Kenmotsu 3-manifold with respect to the semi- symmetric metric connection admits cyclic η-recurrent Ricci tensor, then the manifold is a special type of an $\eta-$Einstein manifold of the form $S(Y,Z)=4f^2\eta(Y)\eta(Z)$ if the scalar curvature is constant, provided f constant.*

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References

[1] Ahsan, Z., Tensors: Mathematics of Differential Geometry and Relativity, Prentice Hall of India Learning Pvt. Ltd., Delhi, 2015.

[2] Barman, A. and De, U. C., Projective curvature tensor of a semi-symmetric metric connection in a Kenmotsu manifold, Int. Electron. J. Geom., 6 (1) (2013), 159-169.

[3] Blair, D. E., Contact manifolds in Riemannian geometry, Lecture Note in Mathematics, 509, Springer-Verlag Berlin, 1976.

[4] Calin, C., Crasmareanu, M. and Munteanu, M., Slant curves in three-dimensional f-Kenmotsu manifolds, Journal of Mathematical Analysis and Applications, Vol. 394, Issue 1(2012), 400-407.

[5] Chakraborty, D., Mishra, V. N, and Hui, S. K., Ricci Solitons on three Dimensional $\beta-$Kenmotsu Manifolds with Respect to Schouten-Van Kampen Connection, Journal of Ultra Scientist of Physical Sciences, 30(1)(2018), 86-91.

[6] Friedmann, A. and Schouten, J. A., Uber die geometric derhalbsymmetrischen Ubertragung, Math., Zeitschr., 21 (1924), 211-223.

[7] Gautam, U. K., Haseeb, A. and Prasad, R., Some results on projective curvature tensor in Sasakian manifolds, Commun. Korean Math. Soc., 34(3) (2019), 881-896.

[8] Gray, A., Einstein-like manifolds which are not Einstein, Geom. Dedicata, 7 (1978), 259-280.

[9] Haseeb, A. and Prasad, R., On a class of three dimensional f-Kenmotsu manifolds, Bulletin of the Transilvania University of Brasov, Series III: Mathematics, Informatics, Physics, 11(60) (2018), 109-118.

[10] Hayden, H. A., Subspaces of space with torsion, Proc. London Math. Soc., 34 (1932), 27-50.

[11] Hui, S. K., Atceken, M., Pal, T. and Mishra, L. N., On Contact CR-submanifolds of $(LCS)_n$-manifolds, (Accepted in Thai J. Math.).

[12] Hui, S. K., Mishra, V. N., Pal, T. and Vandana, Some classes of invariant submanifolds of $(LCS)_n$-manifolds, Italian J. Pure and App. Math., 39 (2018), 359-372.

[13] Hui, S. K., Mishra, V. N. and Patra, A., Examples of gradient Ricci solitons on 4-dimensional Riemannian manifold, Modelling and Application and Theory, 1 (2016), 23-27.

[14] Hui, S. K., Yadav, S. K. and Patra, A., Almost conformal Ricci solitons on f-Kenmotsu manifolds, Khayyam J. Math., 5 (2019), 89-104.

[15] Janssens, D. and Vanhecke, L., Almost contact structures and curvature tensor, Kodai Math. J., 4 (1981), 1-27.

[16] Kenmotsu, K., A class of almost contact Riemannian manifolds, Tohoku Math. J., 24 (1972), 93-103.

[17] Majhi, P. and Biswas, A., Certain curves associated with f-Kenmotsu manifolds, Journal of Dynamical Systems and Geometric Theories, 18(1)(2020), 39-51.

[18] Mondal, A., Legendre curves on 3-dimensional f-Kenmotsu manifolds admitting Schouten-van Kampen connection, Facta Universitatis (NIS), Ser. Math. Inform., 35(2) (2020), 357-366.

[19] Nagaraja, H. G. and Venu, K., f-Kenmotsu manifolds metric as conformal Ricci soliton, Analele Univ. de Vest Timisora, Seria Matematica-Informatica, LV (2017), 119-127.

[20] Olszak, Z., Locally conformal almost cosymplectic manifolds, Colloq. Math., 57 (1989), 73-87.

[21] Olszak, Z. and Rosca R, Normal locally conformal almost cosymplectic manifolds, Publ. Math. Debrecen, 39 (1991), 315-323.

[22] Ozgur, C., Ahmad, M. and Haseeb, A., CR-submanifolds of an LP-Sasakian manifold with a semi-symmetric metric connection, Hacettepe Journal of Mathematics and Statistics, 39(2010), 489-496.

[23] Piscoran, L. I. and Mishra, V. N., Projective flatness of a new class of (α, β)-metrics, Georgian Mathematical Journal, 26(1)(2017), DOI: 10.1515/gmj-2017-0034.

[24] Prasad, R., Haseeb, A. and Gautam, U. K., On $\phi-$semisymmetric LP-Kenmotsu manifolds with a QSNM connection, Kragujevac J. Math., 45(2021), 815-827.

[25] Prasad, R., Haseeb, A. and Gautam, U. K., On 3-dimensional f-Kenmotsu manifolds with a certain connection (submitted).

[26] Shaikh, A. A. and Hui, S. K., On extended generalized ϕ-recurrent β-Kenmotsu manifolds, Publications De L Instititut Mathematique, Nouvelle Serie, tome 89(103) (2011), 77-88.

[27] Shaikh, A. A., Hui, S. K., On locally ϕ-symmetric $\beta-$Kenmotsu manifolds, Extracta Mathematicae, 24 (2009), 301-316.

[28] Shaikh, A.A. and Hui, S. K., On ϕ-symmetric LP-Sasakian manifolds admitting semi-symmetric metric connection, Novi Sad J. Math., 46(2016), 63-78.

[29] Shaikh, A. A. and Jana, S. K., Quarter symmetric metric connection on a $(k, \mu)-$ contact metric manifold, Commun. Fac. Sci. Univ. Ank. Series A1, 55(2006), 33-45.

[30] Yano, K., On semi-symmetric connection, Revue Roumaine de Math. Pure et Appliques, 15 (1970), 1570-1586.

[31] Yildiz, A., f-Kenmotsu manifold with the Schouten-van Kampen connection, Publications De L'Instititut Mathematique, Nouvelle Serie, tome 102(116) (2017), 93-105.

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