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FACE MAGIC LABELING OF DOUBLE DUPLICATION OF GRAPHS

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Abstract. In this paper, the face magic labeling of the double duplication $DD_{VV}(T_n)$ for $n \geq 2$, $DD_{VV}(C_n)$ for $n \geq 4$ and $DD_{EV}(C_n)$ for $n \geq 4$ of types (1,0,1), (1,1,0) and (0,1,1) are studied.

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1. Introduction

We consider finite, simple and undirected graphs. The graph labeling was first defined by Rosa in 1967. The applications of graph labeling are in various fields such as chemical graph theory, coding theory, formal languages, automata theory and so on. The face magic labeling was first investigated by Baca [2] for some planar graphs. He also studied the face magic labeling for fan graphs, planar pyramid graphs and ladder graphs, [3]. Amarajothi et al. [1] have proved the existence of a face magic labeling of vertex and edge duplication of graphs.

Let $G(V, E, F)$ be a finite planar graph where V, E, F denote the sets of vertices, edges and interior faces, respectively, with $|V| = p$, $|E| = q$ and $|F| = f$. A labeling of type (a, b, c) of G is the one which assigns labels from the set $\{1, 2, 3, \ldots, ap + bq + cf\}$ to the vertices, edges and faces of G such that each vertex receives label a, each edge receives label b and each face receives label c and each label is used exactly once. The values of a, b and c are restricted to {0,1}. Labelings of type (1,0,0), (0,1,0) and (0,0,1) are called a vertex labeling, edge labeling and face labeling, respectively. The weight $wt(f)$ of a face f under a labeling is the sum of labels of that face together with labels of vertices and edges forming that face. A labeling is said to be magic if for every positive integer s, all s sided faces have the same weight and we allow different weights for different s.

Definition 1.1. [10] *Duplication of a vertex v_k by a new edge $e = v'v''$ in a graph G produces a new graph G' such that $N_{G'}(v') = \{v_k, v''\}$ and $N_{G''}(v'') = \{v_k, v'\}$.*

Definition 1.2. [11] *Duplication of an edge $e = v_iv_{i+1}$ by a vertex v' in a graph G produces a new graph G' such that $N_{G'}(v') = \{v_i, v_{i+1}\}$.*

The following definitions are taken from [9] which are needed in our study:

Definition 1.3. *The double duplication of a vertex by an edge of a graph is defined as a duplication of a vertex v_k by an edge $e = \{v'_kv''_k\}$ in a graph*

G produces a graph G' in which $N_{G'}(v'_k) = \{v_k, v''_k\}$ and $N_{G'}(v''_k) = \{v_k, v'_k\}$. Again duplication of vertices v_k, v'_k and v''_k by edges $e' = \{u_k w_k\}$, $e'' = \{u'_k w'_k\}$ and $e''' = \{u''_k w''_k\}$, respectively, in G' produces a new graph G'' such that $N_{G'}(u_k) = \{w_k, v_k\}$, $N_{G'}(w_k) = \{u_k, v_k\}$, $N_{G'}(u'_k) = \{w'_k, v'_k\}$, $N_{G'}(w'_k) = \{u'_k, v'_k\}$, $N_{G'}(u''_k) = \{w''_k, v''_k\}$ and $N_{G'}(w''_k) = \{u''_k, v''_k\}$. It is usually denoted by $DD_{VV}(G)$.

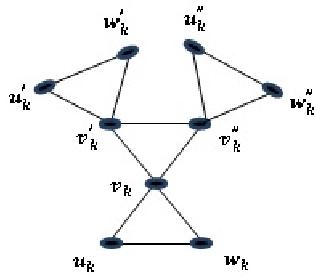

FIGURE 1. $DD_{VV}(G)$

Definition 1.4. *The double duplication of an edge by a vertex followed by vertex by edge of a graph is defined as, a duplication of an edge $e_k = \{v_k v_{k+1}\}$ by a vertex v'_k in a graph G produces a graph G' in which $N(v'_k) = \{v_k, v_{k+1}\}$. Again duplication of v_k, v_{k+1} and v'_k by the edges $e' = \{u_k w_k\}$, $e'' = \{u_{k+1} w_{k+1}\}$ and $e''' = \{u'_k w'_k\}$ in G' produces a new graph G'' such that $N(u_k) = \{v_k, w_k\}$, $N(w_k) = \{u_k, v_k\}$, $N(u_{k+1}) = \{v_{k+1}, w_{k+1}\}$, $N(w_{k+1}) = \{u_{k+1}, v_{k+1}\}$, $N(u'_k) = \{v'_k, w'_k\}$ and $N(w'_k) = \{u'_k, v'_k\}$. It is denoted by $DD_{EV}(G)$.*

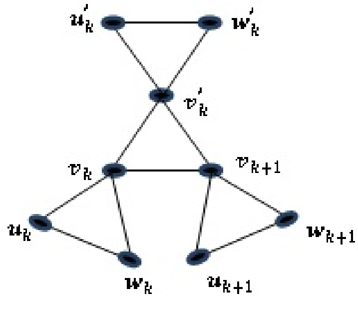

FIGURE 2. $DD_{EV}(G)$

2. Main results

In this section, the face magic labeling of types (1,0,1), (1,1,0) and (0,1,1) were proved for double duplication of graphs:

Theorem 2.1. *Let T_n be a tree of order n. Then the graph $DD_{VV}(T_n)$ for $n \geq 2$ is face magic.*

Proof. Let $G(V, E, F)$ be an arbitrary tree of order n with vertex set $V = \{u_k : 1 \leq k \leq n\}$ and edge set $E = \{e_k : 1 \leq k \leq n-1\}$. Let $G'(V', E', F')$ be the graph obtained by double duplication of vertex by edge in G with

$V' = \{v_k, w_k, x_k, y_k, p_k, q_k, r_k, s_k : 1 \leq k \leq n\} \cup V,$

$E' = \{u_kv_k, u_kw_k, v_kw_k, u_kx_k, u_ky_k, x_ky_k, p_kx_k, q_kx_k, p_kq_k, r_ky_k, s_ky_k, r_ks_k : 1 \leq k \leq n\} \cup E,$

$F' = \{f_k : u_kx_ky_k : 1 \leq k \leq n\} \cup \{f'_k : u_kv_kw_k : 1 \leq k \leq n\} \cup \{f''_k : x_kp_kq_k : 1 \leq k \leq n\} \cup \{f'''_k : y_kr_ks_k : 1 \leq k \leq n\}.$

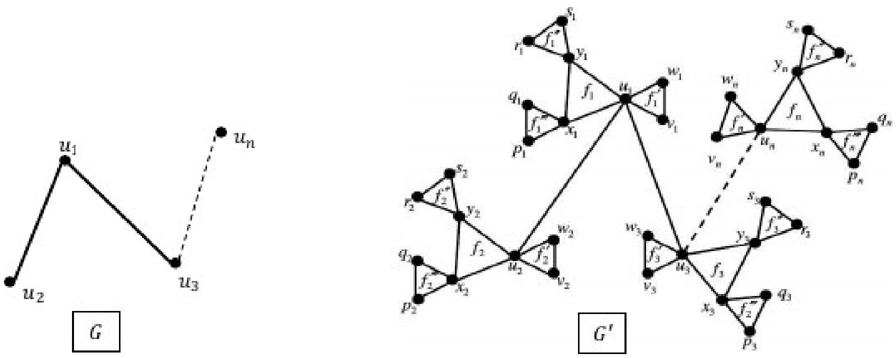

FIGURE 3. G is a tree T_n with $n \geq 3$ and G' is $DD_{VV}(T_n)$ with $n \geq 3$

Type 1: (1,0,1)
Let $1 \leq k \leq n$. A function $\lambda_1 : V' \cup F' \to \{1, 2, 3, \cdots, 13n\}$ is given by

$\lambda_1(u_k) = k,$ $\lambda_1(v_k) = 6n + k,$ $\lambda_1(w_k) = 6n + 1 - k,$
$\lambda_1(x_k) = 4n + 1 - k,$ $\lambda_1(y_k) = 8n + 1 - k,$ $\lambda_1(p_k) = 8n + k,$
$\lambda_1(q_k) = 2n + k,$ $\lambda_1(r_k) = 4n + k,$ $\lambda_1(s_k) = 2n + 1 - k,$
$\lambda_1(f_k) = 12n + k,$ $\lambda_1(f'_k) = 12n + 1 - k,$ $\lambda_1(f''_k) = 10n + 1 - k,$
$\lambda_1(f'''_k) = 10n + k.$

Therefore the weight of each 3-sided face is $w(f_k) = 24n + 2$.

Type 2: (1,1,0)
Let $1 \leq k \leq n$. A function $\lambda_2 : V' \cup E' \to \{1, 2, 3, \cdots, 22n - 1\}$ is given by

$\lambda_2(u_k) = k,$ $\lambda_2(v_k) = 6n + k,$ $\lambda_2(w_k) = 6n + 1 - k,$
$\lambda_2(x_k) = 4n + 1 - k,$ $\lambda_2(y_k) = 8n + 1 - k,$ $\lambda_2(p_k) = 8n + k,$
$\lambda_2(q_k) = 2n + k,$ $\lambda_2(r_k) = 4n + k,$ $\lambda_2(s_k) = 2n + 1 - k,$
$\lambda_2(u_kv_k) = 14n + k,$ $\lambda_2(u_kw_k) = 20n + 1 - k,$ $\lambda_2(v_kw_k) = 12n + 1 - k,$
$\lambda_2(u_kx_k) = 14n + 1 - k,$ $\lambda_2(u_ky_k) = 12n + k,$ $\lambda_2(x_ky_k) = 20n + k,$

$\lambda_2(x_kp_k) = 10n+1-k,$ $\lambda_2(x_kq_k) = 16n+k,$ $\lambda_2(p_kq_k) = 18n+1-k,$
$\lambda_2(y_kr_k) = 16n+1-k,$ $\lambda_2(y_ks_k) = 10n+k,$ $\lambda_2(r_ks_k) = 18n+k.$

For $1 \le k \le n-1$, $\lambda_2(e_k) = 22n-k$. Therefore the weight of each 3-sided face is $w(f_k) = 58n+3$.

Type 3: (0,1,1)

Let $1 \le k \le n$. A function $\lambda_3 : E' \cup F' \to \{1,2,3,\cdots,16n+2\}$ is given by

$\lambda_3(u_kv_k) = 8n-3k,$ $\lambda_3(u_kw_k) = 2n+2+3k,$ $\lambda_3(v_kw_k) = 9n+3+3k,$
$\lambda_3(u_kx_k) = 7n+3+k,$ $\lambda_3(u_ky_k) = 2n-k,$ $\lambda_3(x_ky_k) = 10n-k,$
$\lambda_3(x_kp_k) = n+1+3k,$ $\lambda_3(x_kq_k) = 8n+2-3k,$ $\lambda_3(p_kq_k) = 9n+1+3k,$
$\lambda_3(y_kr_k) = n+2+3k,$ $\lambda_3(y_ks_k) = 8n+1-3k,$ $\lambda_3(r_ks_k) = 9n+2+3k,$
$\lambda_3(f_k) = 15n+3+k,$ $\lambda_3(f'_k) = 16n-3k,$ $\lambda_3(f''_k) = 16n+2-3k,$
$\lambda_3(f'''_k) = 16n+1-3k.$

For $1 \le k \le n-1$, $\lambda_3(e_k) = k$. Therefore the weight of each 3-sided face is $w(f_k) = 36n-2$. □

Corollary 2.2. *Let P_n be a path graph with $n \ge 2$. The graph $DD_{VV}(P_n)$ is face magic for the types (1,0,1), (1,1,0) and (0,1,1) with the same constant as that of the tree T_n.*

Theorem 2.3. *For $n \ge 4$, the graph $DD_{VV}(C_n)$ is face magic.*

Proof. Let $G(V,E,F)$ be a cycle of order $n \ge 4$ with vertex set $V = \{u_k : 1 \le k \le n\}$ and edge set $E = \{u_ku_{k+1} : 1 \le k \le n-1\} \cup \{u_1u_n\}$. Let $G'(V',E',F')$ be the graph obtained by double duplication of vertex by edge in G with

$V' = \{v_k, w_k, x_k, y_k, p_k, q_k, r_k, s_k : 1 \le k \le n\} \cup V,$

$E' = \{u_kv_k, u_kw_k, v_kw_k, u_kx_k, u_ky_k, x_ky_k, p_kx_k, q_kx_k, p_kq_k, r_ky_k, s_ky_k, r_ks_k : 1 \le k \le n\} \cup E,$

$F' = \{f_k : u_kx_ky_k : 1 \le k \le n\} \cup \{f'_k : u_kv_kw_k : 1 \le k \le n\} \cup \{f''_k : x_kp_kq_k : 1 \le k \le n\} \cup \{f'''_k : y_kr_ks_k : 1 \le k \le n\} \cup \{f_{n+1} : u_1u_2u_3\ldots u_k\}.$

Type 1: (1,0,1)

Let $1 \le k \le n$. A function $\delta_1 : V' \cup F' \to \{1,2,3,\cdots,13n+1\}$ is given by

$\delta_1(u_k) = k,$ $\delta_1(v_k) = 6n+k,$ $\delta_1(w_k) = 6n+1-k,$
$\delta_1(x_k) = 4n+1-k,$ $\delta_1(y_k) = 8n+1-k,$ $\delta_1(p_k) = 8n+k,$
$\delta_1(q_k) = 2n+k,$ $\delta_1(r_k) = 4n+k,$ $\delta_1(s_k) = 2n+1-k,$
$\delta_1(f_k) = 12n+k,$ $\delta_1(f'_k) = 12n+1-k,$ $\delta_1(f''_k) = 10n+1-k,$
$\delta_1(f'''_k) = 10n+k,$ $\delta_1(f_{k+1}) = 13k+1.$

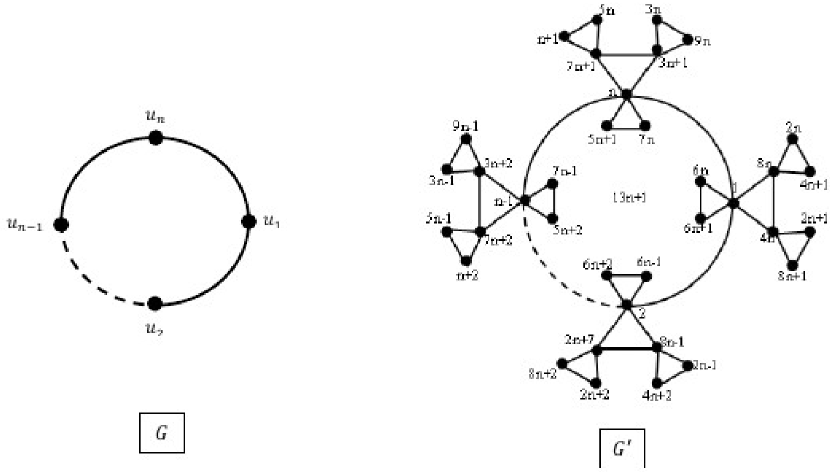

FIGURE 4. G is a cycle graph (C_n) with $n \geq 4$ and G' is $DD_{VV}(C_n)$ with $n \geq 4$

Therefore the weight of each 3-sided face is $w(f_k) = 24n + 2$. Hence the weight of n-sided face

$$\begin{aligned} w(f_k) &= \sum_{k=1}^{n} k + 13n + 1 \\ &= \frac{n(n+1)}{2} + 13n + 1 \\ &= \frac{n^2 + 27n + 2}{2}. \end{aligned}$$

Type 2: (1,1,0)
A function $\delta_2 : V' \cup E' \to \{1, 2, 3, \ldots, 22n\}$ is given by

$\delta_2(u_k) = k,$ $\delta_2(v_k) = 6n + k,$ $\delta_2(w_k) = 6n + 1 - k,$
$\delta_2(x_k) = 4n + 1 - k,$ $\delta_2(y_k) = 8n + 1 - k,$ $\delta_2(p_k) = 8n + k,$
$\delta_2(q_k) = 2n + k,$ $\delta_2(r_k) = 4n + k,$ $\delta_2(s_k) = 2n + 1 - k,$
$\delta_2(u_k v_k) = 14n + k,$ $\delta_2(u_k w_k) = 20n + 1 - k,$ $\delta_2(v_k w_k) = 12n + 1 - k,$
$\delta_2(u_k x_k) = 14n + 1 - k,$ $\delta_2(u_k y_k) = 12n + k,$ $\delta_2(x_k y_k) = 20n + k,$
$\delta_2(x_k p_k) = 10n + 1 - k,$ $\delta_2(x_k q_k) = 16n + k,$ $\delta_2(p_k q_k) = 18n + 1 - k,$
$\delta_2(y_k r_k) = 16n + 1 - k,$ $\delta_2(y_k s_k) = 10n + k,$ $\delta_2(r_k s_k) = 18n + k.$

For $1 \leq k \leq n - 1$, $\delta_2(u_k u_{k+1}) = 22n + 1 - k$ and $\delta_2(u_1 u_k) = 21n + 1$. Therefore the weight of each 3-sided face is $w(f_k) = 58n + 3$. Hence the weight of n-sided face

$$\begin{aligned} w(f_k) &= \sum_{k=1}^{n} k + \sum_{k=1}^{n-1} [\delta_2(u_k u_{k+1}) + \delta_2(u_1 u_k)] \\ &= 22n^2 + n. \end{aligned}$$

Type 3: (0,1,1)
Let $1 \leq k \leq n$. A function $\delta_3 : E' \cup F' \to \{1, 2, 3, \cdots, 17n+1\}$ is given by

$\delta_3(u_k v_k) = n + k,$ $\delta_3(u_k w_k) = 9n + 1 - k,$ $\delta_3(v_k w_k) = 9n + k,$
$\delta_3(u_k x_k) = 7n + k,$ $\delta_3(u_k y_k) = 3n + 1 - k,$ $\delta_3(x_k y_k) = 11n + 1 - k,$
$\delta_3(x_k p_k) = 7n + 1 - 2k,$ $\delta_3(x_k q_k) = 3n + 2k,$ $\delta_3(p_k q_k) = 11n + 2 + 2k,$
$\delta_3(y_k r_k) = n + 2 + 3k,$ $\delta_3(y_k s_k) = 3n - 1 + 2k,$ $\delta_3(r_k s_k) = 10n + 3 + 2k,$
$\delta_3(f_k) = 15n +_k,$ $\delta_3(f'_k) = 17n + 1 - k,$ $\delta_3(f''_k) = 15n + 2 - 2k,$
$\delta_3(f'''_k) = 15n + 1 - 2k,$ $\delta_3(f_{n+1}) = 17n + 1.$

For $1 \leq k \leq n-1$, $\delta_3(u_k u_{k+1}) = k$, and $\delta_3(u_1 u_n) = n$. Therefore the weight of each 3-sided face is $w(f_k) = 36n + 2$. Hence the weight of n-sided face is

$$\begin{aligned} w(f_k) &= \sum_{k=1}^{n-1} [\delta_3(u_k u_{k+1})] + \delta_3(u_1 u_k) + (17n+1) \\ &= \frac{n^2 + 35n + 2}{2}. \end{aligned}$$

□

Theorem 2.4. *For $n \geq 4$, the graph $DD_{EV}(C_n)$ is face magic.*

Proof. Let $G(V, E, F)$ be the graph of a cycle C_n with vertex set $V = \{u_k : 1 \leq k \leq n\}$ and edge set $E = \{u_k u_{k+1} : 1 \leq k \leq n-1\} \cup \{u_1 u_n\}$. Let $G'(V', E', F')$ be the graph obtained by double duplication of edge by vertex followed by vertex by edge in G with vertex set, edge set and face set as follows:
$V' = \{v_k, x_k, y_k, p_k, q_k : 1 \leq k \leq n\} \cup V,$

$E' = \{u_k v_k, u_k x_k, u_k y_k, x_k y_k, p_k v_k, q_k v_k, p_k q_k : 1 \leq k \leq n\} \cup \{v_k u_{k+1} : 1 \leq k \leq n-1\} \cup \{u_1 v_n\} \cup E,$

$F' = \{f_k : u_k v_k u_{k+1} : 1 \leq k \leq n\} \cup \{f'_k : u_k x_k y_k : 1 \leq k \leq n\} \cup \{f''_k : v_k p_k q_k : 1 \leq k \leq n\} \cup \{f_{n+1} : u_1 u_2 u_3 \ldots u_k\}.$

Type 1: (1,0,1)
Let $1 \leq k \leq n$. A function $\theta_1 : V' \cup F' \to \{1, 2, 3, \cdots, 9n+1\}$ is given by

$\theta_1(u_k) = 4n + k,$ $\theta_1(v_k) = 2n + 1 - k,$ $\theta_1(x_k) = 4n + 1 - k,$
$\theta_1(y_k) = k,$ $\theta_1(p_k) = 6n + 1 - k,$ $\theta_1(q_k) = 2n + k,$
$\theta_1(f_k) = 7n - k,$ $\theta_1(f'_k) = 9n + 1 - k,$ $\theta_1(f''_k) = 7n + k,$
$\theta_1(f_n) = 7k.$

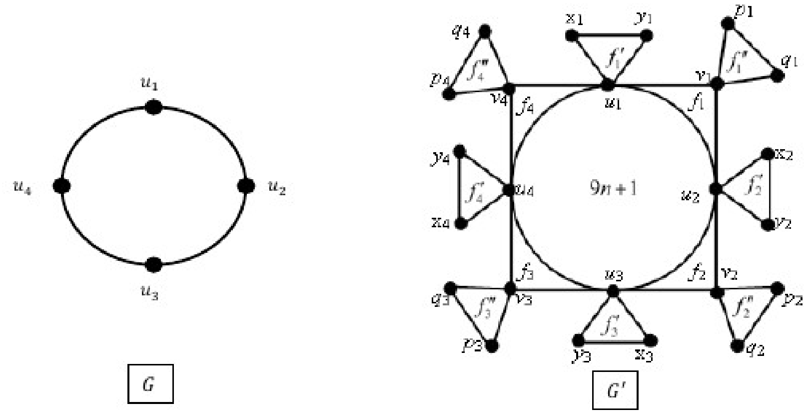

FIGURE 5. G is a cycle graph (C_4) and G' is $DD_{EV}(C_4)$

Therefore the weight of each 3-sided face is $w(f_k) = 17n + 2$. The weight of n-sided face would be

$$w(f_k) = \sum_{k=1}^{n}(4n + k) + 9n + 1$$
$$= \frac{9n^2 + 19n + 2}{2}.$$

Type 2: (1,1,0)
Let $1 \le k \le n$. A function $\theta_2 : V' \cup E' \to \{1, 2, 3, \cdots, 15n\}$ is given by

$\theta_2(u_k) = 4n + k$, $\theta_2(v_k) = 2n + 1 - k$, $\theta_2(x_k) = 4n + 1 - k$,
$\theta_2(y_k) = k$, $\theta_2(p_k) = 6n + 1 - k$, $\theta_2(q_k) = 2n + k$,
$\theta_2(u_k x_k) = 15n + 1 - k$, $\theta_2(u_k y_k) = 9n + k$, $\theta_2(x_k y_k) = 9n + 1 - k$,
$\theta_2(p_k q_k) = 7n + k$.

For $1 \le k \le n - 1$,

$\theta_2(u_k u_{k+1}) = 7n - k$, $\theta_2(v_k u_{k+1}) = 11n + 1 + k$, $\theta_2(u_k v_k) = 13n - k$,
$\theta_2(u_1 u_n) = 7n$, $\theta_2(u_1 v_n) = 11n + 1$, $\theta_2(u_n v_n) = 13n$.

Therefore the weight of each 3-sided face is $w(f_k) = 41n + 3$. Hence the weight of an n-sided face is $w(f_k) = 11n^2 + n$.

Type 3: (0,1,1)
A function $\theta_3 : E' \cup F' \to \{1, 2, 3, \cdots, 12n + 1\}$ is given by

$\theta_3(u_kx_k) = 4n + k,$ $\theta_3(u_ky_k) = 8n + 1 - k,$ $\theta_3(x_ky_k) = 2n + 1 - k,$
$\theta_3(p_kq_k) = 2n + k,$ $\theta_3(u_kv_k) = 6n + k,$ $\theta_3(v_kp_k) = 4n + 1 - k,$
$\theta_3(v_kq_k) = 8n + k,$ $\theta_3(f_k) = 12n + k,$ $\theta_3(f'_k) = 10n + 1 + k,$
$\theta_3(f''_k) = 10n + 2 - k.$

For $1 \leq k \leq n - 1$,

$\theta_3(u_ku_{k+1}) = k,$ $\theta_3(v_ku_{k+1}) = 6n + 1 - k,$
$\theta_3(u_1u_n) = n,$ $\theta_3(u_1v_n) = 5n + 1.$

Therefore the weight of each 3-sided face is $w(f_k) = 24n + 3$. Hence the weight of an n-sided face is $w(f_k) = \frac{n^2+19n+2}{2}$. □

Remark 2.5. *In Theorem 2.3 and Theorem 2.4, we have considered only interior faces in which G has exactly $n + 1$ interior faces and all the faces are triangular except one which is an n-sided face.*

Conclusion

In this paper, we have studied the face magic labelings for double duplication of graphs. In future, we planned to use the face magic labeling for encrypting and decrypting numbers.

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