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FORGOTTEN INDEX OF GRAPHS WITH DELETED EDGES

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ABSTRACT. The forgotten index $F(G)$ of a graph G is a degree based topological index which is defined as the sum of the cubes of the degrees of its vertices which was introduced by Furtula and Gutman in 2015. It was used in earlier works in relation with the first Zagreb index but not named until 2015. In this work, we present the effect of deleting edges from a simple graph on forgotten index is studied. In particular, some statements for the change of forgotten index of path, cycle, complete, star, complete bipartite and tadpole graphs are obtained. Also the same effect is determined for regular graphs.

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1. INTRODUCTION

Throughout this paper, all the graphs are simple, finite and undirected. Let $G = G(V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The degree of a vertex $v \in V(G)$ is denoted by $d_G v$ or briefly by dv if there is no risk of confusion. A vertex having degree one is called a pendant vertex and an edge incident to such a vertex is called a pendant edge. Here we used P_n, C_n, S_n, K_n, K_r, s, and T_r, s to denote the path, cycle, star, complete, complete bipartite and tadpole graphs of order n, respectively.

2. FORGOTTEN INDEX

Topological indices are used to study the required characteristics of drug molecules and many other real life situations, see [5, 6, 7, 8, 9, 10, 11]. Among the different types of topological indices, degree based topological indices are the most studied type of topological indices which play a prominent role in chemical graph theory. One of the oldest degree-based topological indices is Zagreb index which was introduced by Gutman and Trinajstić in the year 1972, [3]. There are two fundamental types of Zagreb indices, namely, the first Zagreb index which is defined as

$$M_1(G) = \sum_{v \in V(G)} (dv)^2 = \sum_{uv \in E(G)} (du + dv)$$

and the second Zagreb index which is defined as

$$M_2(G) = \sum_{uv \in E(G)} dudv.$$

All the other Zagreb indices are defined as several variants of these two indices. Furtula and Gutman in 2015 named the topological graph index of the form

$$F(G) = \sum_{v \in V(G)} (dv)^3 = \sum_{uv \in E(G)} ((du)^2 + (dv)^2)$$

as the forgotten topological index or F-index, [2]. In [1], a recent problem called the inverse problem was settled for the forgotten index.

Edge/vertex deletion and addition is an important method in graph theory as it helps us to calculate some required property of a graph in terms of the same property of a smaller graph. In this work, we find the amount of change in the forgotten index of a given graph G when an edge e is deleted. This effect is also calculated for regular graphs and for deleting two successive edges.

3. Change in the forgotten index when an edge is deleted

In this section, we will determine the amount of change in the forgotten index when an edge is deleted from a simple graph. We have

Theorem 3.1. *Let G be a simple graph and let $G-e$ be the graph obtained by deleting an edge $e = uv$ from G. Then*

$$F(G) - F(G-e) \;=\; 3(d_G u(d_G u - 1) + d_G v(d_G v - 1)) + 2.$$

Proof. Let $e = uv$ be the edge in the graph G. Let the neighbouring vertices of u be $u_1, u_2, \cdots, u_{d_G u}$ of degrees $d_G u_1, d_G u_2, \cdots, d_G u$ and the neighbouring vertices of v be $v_1, v_2, \cdots, v_{d_G v}$ of degrees $d_G v_1, d_G v_2, \cdots, d_G v$, respectively. Let us delete the existing edge e between the vertices u and $v \in G$ of degrees $d_G u$ and $d_G v$, respectively.

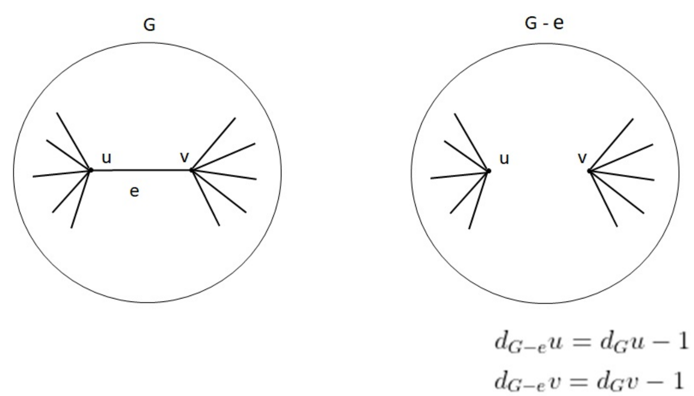

Figure 1. G and $G - e$

We know that the forgotten index of a graph G can be stated as

$$\begin{aligned} F(G) &= \sum_{w \in V(G)} (d_G w)^3 \\ &= (d_G u)^3 + (d_G v)^3 + \sum_{w \neq u,v} (d_G w)^3 \end{aligned}$$

and hence

$$\begin{aligned} F(G-e) &= (d_{G-e} u)^3 + (d_{G-e} v)^3 + \sum_{w \neq u,v} (d_{G-e} w)^3 \\ &= (d_G u - 1)^3 + (d_G v - 1)^3 + \sum_{w \neq u,v} (d_G w)^3 \end{aligned}$$

implying that

$$\begin{aligned} F(G) - F(G-e) &= 3(d_G u)^2 - 3d_G u + 1 + 3(d_G v)^2 - 3d_G v + 1 \\ &= 3(d_G u(d_G u - 1) + d_G v(d_G v - 1)) + 2. \end{aligned}$$

□

As a special case of this main result, we deduce the following for an r-regular graph:

Corollary 3.2. *Let G be an r-regular graph. Then,*

$$F(G) - F(G-e) = 6r^2 - 6r + 2.$$

If we delete a second edge from an r-regular graph G, we obtain

Theorem 3.3. *Let G be a simple graph and let e be an edge of G.*

(a) *Let f be an adjacent edge to e. Without lose of generality, we assume that f is adjacent to e at vertex u, see Fig. 2. Then*

$$F(G) - F(G-e) = 2(6r^2 - 9r + 5).$$

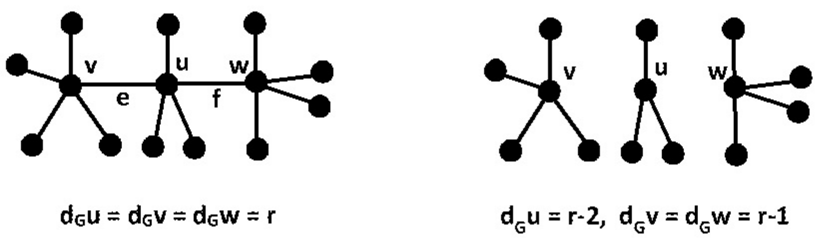

FIGURE 2. G and $G - e$

(b) *Let f be an edge of G which is not adjacent to e. Without lose of generality, we assume that f is adjacent to e at vertex u, see Fig. 3. Then*

$$F(G) - F(G-e) = 4(3r^2 - 3r + 1).$$

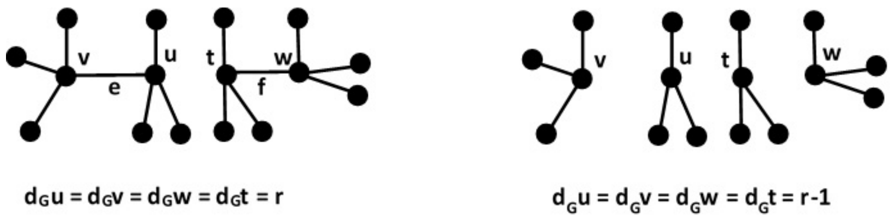

FIGURE 3. G and $G-e$

Proof. **(a)** If f is an adjacent edge to e, then

$$\begin{aligned} F(G)-F(G-\{e,f\}) &= 3r^3+\sum_{z\neq u,v,w}(d_G z)^3 \\ &\quad -[(r-1)^3+(r-2)^3+(r-1)^3+\sum_{z\neq u,v,w}(d_{G-\{e,f\}}z)^3] \\ &= 3r^3+\sum_{z\neq u,v,w}(d_G z)^3 \\ &\quad -[3r^3-12r^2+18r-10+\sum_{z\neq u,v,w}(d_G z)^3] \\ &= 2(6r^2-9r+5). \end{aligned}$$

(b) If f is a non-adjacent edge to e, then

$$\begin{aligned} F(G)-F(G-\{e,f\}) &= 4r^3+\sum_{z\neq u,v,w,t}(d_G z)^3 \\ &\quad -[4(r-1)^3+\sum_{z\neq u,v,w,t}(d_{G-\{e,f\}}z)^3] \\ &= 4r^3+\sum_{z\neq u,v,w}(d_G z)^3 \\ &\quad -[4r^3+12r^2-12r+4]-\sum_{z\neq u,v,w,t}(d_G z)^3 \\ &= 4(3r^2-3r+1). \end{aligned}$$

□

This fact also follows from Corollary 3.2 as applying the edge deletion process twice to non-adjacent edges will reduce the effect of edge deletion twice.

Also, as another special case, if e is a pendant edge, then we deduce the following:

Corollary 3.4. *Let $G-e$ be the graph obtained by deleting the pendant edge $e=uv$ from G where v is a pendant vertex. Then,*

$$F(G)-F(G-e) = 3d_G u(d_G u-1)+2.$$

Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be the degree sequence of a graph G. Clearly, there are a_1 pendant edges in G. Let G^* be the graph obtained by deleting all the a_1 pendant edges from G. Then we have

Theorem 3.5. *If G^* is the graph obtained from G by deleting all pendant edges and if the degrees of the support vertices are $d_G u_1, d_G u_2, \cdots, d_G u_{a_1}$, then*

$$F(G^*) = F(G) - 3\sum_{i=1}^{a_1} d_G u_i(d_G u_i - 1) + 2a_1.$$

Proof. Effect of a pendant edge deletion was given in Corollary 3.4. Applying this result a_1 times, we get the result. □

Example 3.1. *Let G be as in Fig. 4. Deleting the tree pendant edges e_1, e_2, e_3 from G, we get the graph G^* in Fig. 5 and finally deleting e_4, we get the graph G^{**} in Fig. 6. The degree sequence of G is $\{1^{(3)}, 2^{(2)}, 3^{(1)}, 4^{(3)}\}$ and hence*

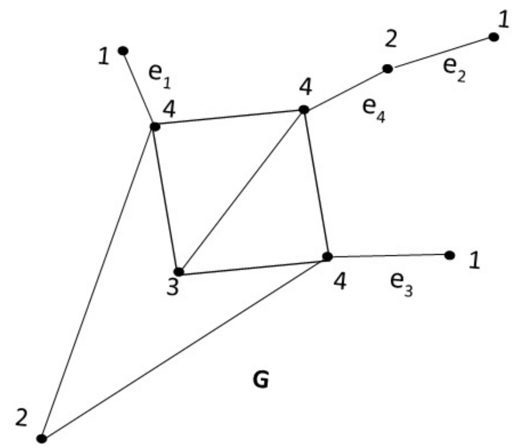

FIGURE 4. Graph G

$$\begin{aligned} F(G) &= 3 \cdot 1^3 + 2 \cdot 2^3 + 1 \cdot 3^3 + 3 \cdot 4^3 \\ &= 3 + 16 + 27 + 192 \\ &= 238. \end{aligned}$$

Similarly as degree sequence of G^* is $\{1^{(1)}, 2^{(1)}, 3^{(3)}, 4^{(1)}\}$, we have

$$\begin{aligned} F(G^*) &= 1 \cdot 1^3 + 1 \cdot 2^3 + 3 \cdot 3^3 + 1 \cdot 4^3 \\ &= 1 + 8 + 81 + 64 \\ &= 154 \end{aligned}$$

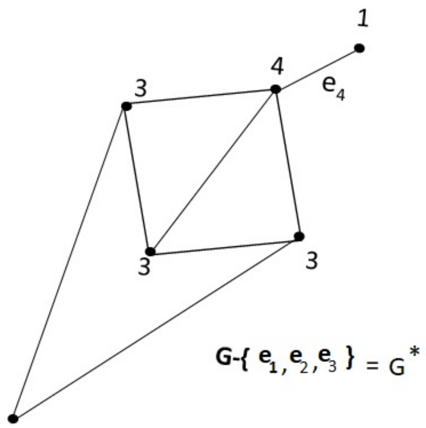

FIGURE 5. $G - \{e_1, e_2, e_3\} = G^*$

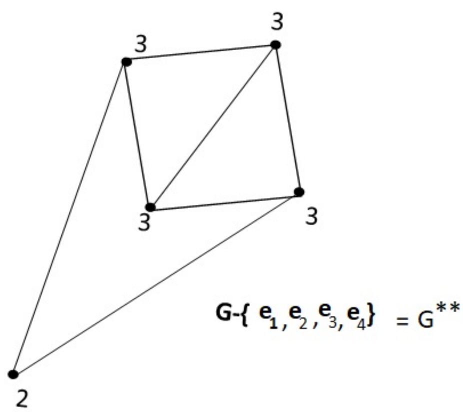

FIGURE 6. $G - \{e_1, e_2, e_3, e_4\} = G^{**}$

and as the degree sequence of G^{**} *is* $\left\{2^{(1)}, 3^{(4)}\right\}$*, we have*

$$\begin{aligned} F(G^{**}) &= 1 \cdot 2^3 + 4 \cdot 3^3 \\ &= 8 + 108 \\ &= 116. \end{aligned}$$

G *has three pendant edges with support vertices of degrees* $2, 4$ *and* 4*. So by Theorem 3.5, we get*

$$\begin{aligned} F(G^{*}) &= F(G) - [3 \cdot 2 \cdot 1 + 2 + 3 \cdot 4 \cdot 3 + 2 + 3 \cdot 4 \cdot 3 + 2] \\ &= 238 - 84 \\ &= 154. \end{aligned}$$

As G^* still has a pendant edge with support vertex of degree 4, by Theorem 3.5, we find

$$\begin{aligned} F(G^{**}) &= F(G^*) - (3 \cdot 4 \cdot 3 + 2) \\ &= 154 - 38 \\ &= 116 \end{aligned}$$

as we calculated.

By applying Theorem 3.5 successively, we can obtain a new graph G having no vertices of degree one and calculate its forgotten index of G.

We call a graph edge-homogeneous if all the edges have the same pairs of vertex degrees. In particular, if each edge of G has vertex degrees r and s, then G is called (r,s)-edge homogeneous. We then have

Theorem 3.6. *Let G be an (r,s)-edge homogeneous graph. If e is any edge of G, then*

$$F(G) - F(G-e) = 3(r(r-1) + s(s-1)) + 2.$$

Proof. Let $e = uv$ be an edge. Let us say $d_G u = r$ and $d_G v = s$. In $G-e$, $d_{G-e} u = r-1$ and $d_{G-e} = s-1$ implying the result.

We shall use some special types of vertices to establish our main result. We call a support vertex which do not lie on any cycle as a bud vertex. If a support vertex which lies on some cycle seperates the graph into two pieces, one is cyclic and the other is acyclic, then it will be called as a root vertex. In Fig. 7, there are two bud vertices v_1 and v_2 having degrees $d_G 1 = 2$ and $d_G 2 = 3$, and two root vertices u_1 and u_2 having degrees $e_G 1 = 3$ and $e_G 2 = 6$. □

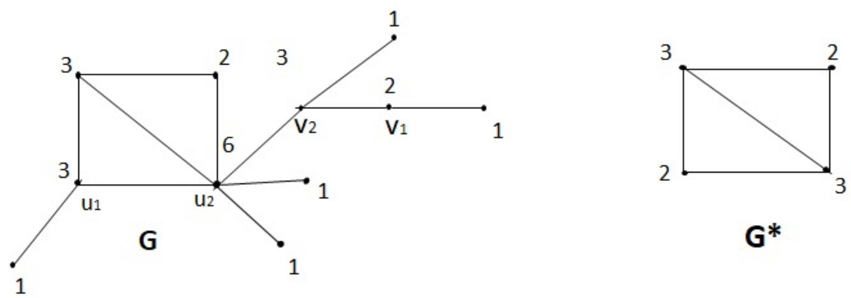

FIGURE 7. G and G^*

We shall denote the numbers of bud vertices and root vertices in a graph by k and t, respectively. For the graph in Fig. 7, $k = t = 2$. Our main result is as follows:

Theorem 3.7. *Let G be a simple graph. Let G^* be the graph obtained by deleting all the bud vertices. If G has k bud vertices $v_1, v_2, \cdots, v_k$ of degrees $d_G v_1, d_G v_2, \cdots, d_G v_k > 1$ and t root vertices $u_1, u_2, \cdots, u_t$ of vertices $e_G u_1, e_G u_2, \cdots, e_G u_t > 2$, then*

$$F(G) - F(G^*) = \sum_{i=1}^{k}(d_G i^3 + d_G i) + \sum_{j=1}^{t}(3l_j e_G j(e_G j - lj) + lj^3 + lj) - 2k$$

where l_j denotes the number of incident edges to u_j connecting u_j to a bud vertex.

Proof. We will obtain G^* from G by deleting edges from outside to inside. Therefore we first delete the edges incident to bud vertices and then, the ones incident to root vertices. For each budvertex v_i of degree $d_G v_i > 1$, there is one incident edge which is the nearest incident edge of v_i to the cyclic part of G and the remaining $d_G v_i - 1$ incident edges lie on the acyclic side of v_i. First we delete the $d_G v_i - 1$ incident edges just mentioned which are of type $(1, d_G i), (1, d_G i - 1), (1, d_G i - 2), \cdots, (1, 2)$, respectively. By Corollary 3.4, for each bud vertex v_i,

$$\sum_{r=2}^{d_G v_i}(3r(r-1)+2) = d_G v_i((d_G v_i)^2 + 1) - 1$$

is reduced from $F(G)$. For k bud vertices, $F(G)$ reduced by

$$\sum_{i=1}^{k}((d_G i)^3 + d_G i) - 2k. \tag{1}$$

Secondly, we delete the edges lying between root vertices and bud vertices. If there are l_i edges at a root vertex u_i of degree $e_G i$ which do not belong to a cycle, then deleting them will decrease $F(G)$ by

$$\begin{aligned}\sum_{r=0}^{l_j-1}[3(e_G j - r)(e_G j - r - 1) + 2] &= \sum_{r=0}^{l_j-1}\left[3e_G j^2 - 3e_G j + 2 + (3 - 6e_G j)r + 3r^2\right] \\ &= (3e_G j^2 - 3e_G j + 2)l_j \\ &\quad + (3 - 6e_G j)\frac{(l_j - 1)l_j}{2} + \frac{3(l_j-1)l_j(2l_j - 1)}{6} \\ &= 3l_j e_G j(e_G j - l_j) + lj^3 + l_j.\end{aligned}$$

As there are t root vertices, $F(G)$ decreases by

$$\sum_{j=1}^{t}(3l_j e_G j(e_G j - l_j) + lj^3 + l_j). \tag{2}$$

By (1) and (2), the result is obtained. □

Example 3.2. *Consider the graph in Figure 7. Recall that there are $k = 2$ bud vertices v_1 and v_2 of degrees $d_G 1 = 2$ and $d_G 2 = 3$, respectively. Also there are t=2 root vertices u_1 and u_2 of degrees $e_G 1 = 3$ and $e_G 2 = 6$. Also $l_1 = 1$ and $l_2 = 3$. By Theorem 3.7, we have*

$$F(G) - F(G^*) = 248$$

which is correct as

$$F(G) = 318 \quad and \quad F(G^*) = 70.$$

4. Change of the forgotten index of some important graph classes

In this section, we calculate the effect of edge deletion on the F index of some frequently used graph classes.

4.1. **Path graph P_n.** Let P_n be a path graph of order n and $F(P_n) = 8n - 14$. Consider the edge deletion from P_n. There are two cases to consider:

a) One of the two pendant edges is deleted: In this case, the degree sequence of $P_n - e$ is $\{1^{(2)}, 2^{(n-3)}\}$. Hence

$$F(P_n - e) = 2 \cdot 1^3 + (n-3) \cdot 2^3 = 8n - 22.$$

b) Any one of the non-pendant edges is deleted: In this case, the degree sequence of $P_n - e$ is $\{1^{(4)}, 2^{(n-4)}\}$ and

$$F(P_n - e) = 4 \cdot 1^3 + (n-4) \cdot 2^3 = 8n - 28.$$

Corollary 4.1. *Considering the fact that $F(P_n) = 8n - 14$, we see that the forgotten index of P_n either reduces by 8 or by 14.*

4.2. **Cycle graph C_n.** Let C_n be a cycle graph of order n and $F(C_n) = 8n$. Consider the edge deletion from a cycle graph C_n. When an edge is deleted from C_n, we get a P_n. That is, $F(C_n - e) = F(P_n)$. We know that $F(P_n) = 8n - 14$. Therefore $F(C_n - e) = 8n - 14$.

Corollary 4.2. *As $F(C_n) = 8n$, we see that when an edge is deleted from C_n, the forgotten index reduces by 14.*

4.3. **Star graph S_n.** Consider the star graph. A star graph is an edge-homogeneous graph like C_n. Hence, by Theorem 3.6, we have

$$\begin{aligned} F(S_n - e) &= F(S_n) - 3((n-1)(n-2) + 1 \cdot 0) + 2 \\ &= n^3 - 6n^2 + 13n - 10. \end{aligned}$$

Corollary 4.3. *As $F(S_n) = n^3 - 3n^2 + 4n - 2$, we see that when an edge is deleted from S_n, the forgoten index reduces by $3n^2 - 9n + 8$.*

4.4. **Complete graph K_n.** For a complete graph K_n which is also an edge-homogeneous graph, there are $\frac{n(n-1)}{2}$ times $(n-1, n-1)$-edges, so that

$$\begin{aligned} F(K_n - e) &= F(K_n) - 3((n-1)(n-2) + (n-1)(n-2)) + 2 \\ &= n^4 - 3n^3 - 3n^2 + 17n - 14. \end{aligned}$$

by Theorem 3.6.

Corollary 4.4. *As $F(K_n) = n^4 - 3n^3 + 3n^2 - n$, we see that when an edge is deleted from K_n, the forgotten index reduces by $6n^2 - 18n + 14$.*

4.5. **Complete bipartite graph $K_{r,s}$.** For a complete bipartite graph $K_{r,s}$, as all edges are the same, we have

$$F(K_{r,s}-e)=rs(r^2+s^2)-3(r^2+s^2)+3(r+s)-2.$$

by Theorem 3.6. Therefore as $F(K_{r,s})=rs(r^2+s^2)$, we see that

Corollary 4.5. *When an edge is deleted from $K_{r,s}$, the forgotten index reduces by* $3(r^2+s^2)-3(r+s)+2$.

4.6. **Tadpole graph $T_{r,s}$.** Let $T_{r,s}$ be a tadpole graph. Then $F(T_{r,s})=8r+8s+12$. Consider the edge deletion from a tadpole graph $T_{r,s}$. For a tadpole graph $T_{r,s}$, there are three cases to consider:

a) If a non-pendant edge e between two vertices of degrees 2 and 3 is deleted, then in this case, the degree sequence of $T_{r,s}-e$ is $\left\{1^{(2)},2^{(r+s-2)}\right\}$. Hence

$$F(T_{r,s}-e)=2\cdot 1^3+(r+s-2)\cdot 2^3=8r+8s-14.$$

b) If a non-pendant edge e between two vertices of degrees 2 is deleted, then in this case, the degree sequence of $T_{r,s}-e$ is $\left\{1^{(3)},3^{(1)},2^{(r+s-4)}.\right\}$. Hence

$$F(T_{r,s}-e)=3\cdot 1^3+1\cdot 3^3+(r+s-4)\cdot 2^3=8r+8s-2.$$

c) If the unique pendant edge e is deleted, then in this case, the degree sequence of $T_{r,s}-e$ is $\left\{1^{(1)},3^1,2^{(r+s-3)}\right\}$. Hence

$$F(T_{r,s}-e)=1\cdot 1^3+1\cdot 3^3+(r+s-3)\cdot 2^3=8r+8s+4.$$

As $F(T_{r,s})=8r+8s+12$, we see that

Corollary 4.6. *When an edge is deleted from $T_{r,s}$, the forgotten index reduces by* 26*,* 14 *or* 8*.*

5. Summary and conclusions

In this work, the effect of deleting an edge from a simple graph on forgotten index is studied. This effect differs for deleting different edges. These effects are formulized and all possible differences are determined for six most frequently used graph classes P_n, C_n, S_n, K_n, $K_{r,s}$ and $T_{r,s}$.

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