Proceedings of the Jangjeon Mathematical Society
24 (2021), No. 3, pp. 403 - 416

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2021.24.3.403

A logarithmic inequality involving prime numbers

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Abstract: Assume that N is a sufficiently large positive number. In this paper we show that for a small constant $\varepsilon > 0$, the logarithmic inequality

$$\big|p_1 \log p_1 + p_2 \log p_2 + p_3 \log p_3 - N\big| < \varepsilon$$

has a solution in prime numbers $p_1,\, p_2,\, p_3$.

Keywords: Diophantine inequality, Logarithmic inequality, Prime numbers.

2020 Math. Subject Classification: 11J25 · 11P55 · 11L07

1 Introduction and statements of the result

One of the most remarkable diophantine inequality with prime numbers is the ternary Piatetski-Shapiro inequality. The first solution is due to Tolev [13]. In 1992 he considered the the diophantine inequality

$$|p_1^c + p_2^c + p_3^c - N| < \varepsilon\,, \tag{1}$$

where N is a sufficiently large positive number, $p_1,\, p_2,\, p_3$ are prime numbers, $c > 1$ is not an integer and $\varepsilon > 0$ is a small constant. Overcoming all difficulties Tolev [13] showed that (1) has a solution for

$$1 < c < \frac{15}{14}\,.$$

Afterwards the result of Tolev was improved by Cai [3] to

$$1 < c < \frac{13}{12}\,,$$

by Cai [4] and Kumchev and Nedeva [9] to

$$1 < c < \frac{12}{11}\,,$$

by Cao and Zhai [6] to

$$1 < c < \frac{237}{214},$$

by Kumchev [10] to

$$1 < c < \frac{61}{55},$$

by Baker and Weingartner [2] to

$$1 < c < \frac{10}{9},$$

by Cai [5] to

$$1 < c < \frac{43}{36},$$

by Baker [1] to

$$1 < c < \frac{6}{5}$$

and this is the best result up to now.

Inspired by these profound investigations in this paper we introduce new diophantine inequality with prime numbers.

Consider the logarithmic inequality

$$\big|p_1 \log p_1 + p_2 \log p_2 + p_3 \log p_3 - N\big| < \varepsilon, \tag{2}$$

where N is a sufficiently large positive number and $\varepsilon > 0$ is a small constant. Having the methods of the aforementioned number theorists we expect that (2) can be solved in primes $p_1,\, p_2,\, p_3$. Thus we make the first step and prove the following theorem.

Theorem 1. *Let N is a sufficiently large positive number. Let X is a solution of the equality*

$$N = 2X \log(2X/3).$$

Then the logarithmic inequality

$$\big|p_1 \log p_1 + p_2 \log p_2 + p_3 \log p_3 - N\big| < X^{-\frac{1}{25}} \log^8 X$$

is solvable in prime numbers $p_1,\, p_2,\, p_3$.

As usual the corresponding binary problem is out of reach of the current state of the mathematics. In other words we have the following challenge.

Conjecture 1. *Let N is a sufficiently large positive number and $\varepsilon > 0$ is a small constant. Then the logarithmic inequality*

$$\big|p_1 \log p_1 + p_2 \log p_2 - N\big| < \varepsilon$$

is solvable in prime numbers $p_1,\, p_2$.

We believe that the future development of analytic number theory will lead to the solution of this binary logarithmic conjecture.

2 Notations

For positive A and B we write $A \asymp B$ instead of $A \ll B \ll A$. As usual $\mu(n)$ is Möbius' function, $\tau(n)$ denotes the number of positive divisors of n and $\Lambda(n)$ is von Mangoldt's function. Moreover $e(y) = e^{2\pi \imath y}$. We denote by $[y]$ the integer part of y. The letter p with or without subscript will always denote prime number. Let N be an sufficiently large positive number. Let X is a solution of the equality

$$N = 2X \log(2X/3).$$

Denote

$$\varepsilon = X^{-\frac{1}{25}} \log^8 X\,; \tag{3}$$

$$\tau = X^{-\frac{23}{25}}\,; \tag{4}$$

$$K = X^{\frac{1}{25}} \log^{-6} X\,; \tag{5}$$

$$S(\alpha) = \sum_{X/2<p\leq X} e(\alpha p \log p) \log p\,; \tag{6}$$

$$I(\alpha) = \int\limits_{X/2}^{X} e(\alpha y \log y)\, dy\,. \tag{7}$$

3 Lemmas

Lemma 1. *Let $k \in \mathbb{N}$. There exists a function $\psi(y)$ which is k times continuously differentiable and such that*

$$\begin{aligned} &\psi(y) = 1 && \textit{for} \quad |y| \leq 3\varepsilon/4\,;\\ &0 \leq \psi(y) < 1 && \textit{for} \quad 3\varepsilon/4 < |y| < \varepsilon\,;\\ &\psi(y) = 0 && \textit{for} \quad |y| \geq \varepsilon\,. \end{aligned}$$

and its Fourier transform

$$\Psi(x) = \int\limits_{-\infty}^{\infty} \psi(y) e(-xy) dy$$

satisfies the inequality

$$|\Psi(x)| \leq \min\left(\frac{7\varepsilon}{4}, \frac{1}{\pi|x|}, \frac{1}{\pi|x|}\left(\frac{k}{2\pi|x|\varepsilon/8}\right)^k\right).$$

Proof. See ([11]). □

Lemma 2. *Let* $|f^{(m)}(u)| \asymp YX^{1-m}$ *for* $1 < X < u \leq 2X$ *and* $m = 1, 2, 3, \ldots$ *Then*

$$\Big|\sum_{X<n\leq 2X} e(f(n))\Big| \ll Y^{\varkappa}X^{\lambda} + Y^{-1},$$

where $(\varkappa, \lambda)$ *is any exponent pair.*

Proof. See ([7], Ch. 3). □

Lemma 3. *For any complex numbers* $a(n)$ *we have*

$$\Big|\sum_{a<n\leq b} a(n)\Big|^2 \leq \left(1 + \frac{b-a}{Q}\right)\sum_{|q|\leq Q}\left(1 - \frac{|q|}{Q}\right)\sum_{a<n,\, n+q\leq b} a(n+q)\overline{a(n)},$$

where Q *is any positive integer.*

Proof. See ([8], Lemma 8.17). □

Lemma 4. *Assume that* $F(x)$*,* $G(x)$ *are real functions defined in* $[a, b]$*,* $|G(x)| \leq H$ *for* $a \leq x \leq b$ *and* $G(x)/F'(x)$ *is a monotonous function. Set*

$$I = \int\limits_a^b G(x)e(F(x))dx\,.$$

If $F'(x) \geq h > 0$ *for all* $x \in [a, b]$ *or if* $F'(x) \leq -h < 0$ *for all* $x \in [a, b]$ *then*

$$|I| \ll H/h\,.$$

If $F''(x) \geq h > 0$ *for all* $x \in [a, b]$ *or if* $F''(x) \leq -h < 0$ *for all* $x \in [a, b]$ *then*

$$|I| \ll H/\sqrt{h}\,.$$

Proof. See ([12], p. 71). □

Lemma 5. *We have*

(i) $$\sum_{n\leq X} \tau^2(n) \ll X\log^3 X,$$

(ii) $$\sum_{n\leq X} \Lambda^2(n) \ll X\log X.$$

Lemma 6. *If* $|\alpha| \leq \tau$ *then*

$$S(\alpha) = I(\alpha) + \mathcal{O}\Big(Xe^{-(\log X)^{1/5}}\Big)\,.$$

Proof. This lemma is very similar to result of Tolev [13]. Inspecting the arguments presented in ([13], Lemma 14), (with $T = X^{\frac{1}{2}}$) the reader will easily see that the proof of Lemma 6 can be obtained by the same manner. □

Lemma 7. *We have*

$$\int_{-\infty}^{\infty} I^3(\alpha)e(-N\alpha)\Psi(\alpha)\,d\alpha \gg \varepsilon\frac{X^2}{\log X}.$$

Proof. Denoting the above integral with Θ, using (7), the definition of $\psi(y)$ and the inverse Fourier transformation formula we obtain

$$\begin{aligned}\Theta &= \int_{X/2}^{X}\int_{X/2}^{X}\int_{X/2}^{X}\int_{-\infty}^{\infty} e\big((y_1\log y_1+y_2\log y_2+y_3\log y_3-N)\alpha\big)\Psi(\alpha)\,d\alpha\,dy_1\,dy_2\,dy_3\\ &= \int_{X/2}^{X}\int_{X/2}^{X}\int_{X/2}^{X} \psi(y_1\log y_1+y_2\log y_2+y_3\log y_3-N)\,dy_1\,dy_2\,dy_3\\ &\geq \underset{|y_1\log y_1+y_2\log y_2+y_3\log y_3-N|<3\varepsilon/4}{\int_{X/2}^{X}\int_{X/2}^{X}\int_{X/2}^{X}} dy_1\,dy_2\,dy_3 \geq \int_{\lambda X}^{\mu X}\int_{\lambda X}^{\mu X}\left(\int_{\Delta} dy_3\right)dy_1\,dy_2\,, \qquad (8)\end{aligned}$$

where λ and μ are real numbers such that

$$\frac{1}{2}<\frac{2}{3}<\lambda<\mu<\frac{5}{7}<1$$

and

$$\Delta=\big[X/2,X\big]\cap\big[y_3',y_3''\big]=\big[y_3',y_3''\big],$$

where the interval $\big[y_3',y_3''\big]$ is a solution of the inequality

$$N-3\varepsilon/4-y_1\log y_1-y_2\log y_2<y_3\log y_3<N+3\varepsilon/4-y_1\log y_1-y_2\log y_2.$$

Let y be an implicit function of t defined by

$$y\log y=t, \qquad (9)$$

where

$$t\asymp X\log X \qquad (10)$$

and therefore

$$y\asymp X. \qquad (11)$$

The first derivative of y is

$$y'=\frac{1}{1+\log y}. \qquad (12)$$

By (11) and (12) we conclude

$$y'\asymp\frac{1}{\log X}. \qquad (13)$$

Thus by the mean-value theorem we get

$$\Theta\gg\varepsilon\int_{\lambda X}^{\mu X}\int_{\lambda X}^{\mu X} y'\big(\xi_{y_1,y_2}\big)\,dy_1\,dy_2\,, \qquad (14)$$

where

$$\xi_{y_1,y_2} \asymp X \log X.$$

From (13) and (14) it follows that

$$\Theta \gg \varepsilon \frac{X^2}{\log X}.$$

The lemma is proved. □

Lemma 8. *We have*

$$\text{(i)} \qquad \int\limits_{-\tau}^{\tau} |S(\alpha)|^2 \, d\alpha \ll X \log^2 X,$$

$$\text{(ii)} \qquad \int\limits_{-\tau}^{\tau} |I(\alpha)|^2 \, d\alpha \ll X,$$

$$\text{(iii)} \qquad \int\limits_{n}^{n+1} |S(\alpha)|^2 \, d\alpha \ll X \log^2 X.$$

Proof. We only prove (i). The cases (ii) and (iii) are analogous.

Having in mind (6) we write

$$\begin{aligned}
\int\limits_{-\tau}^{\tau} |S(\alpha)|^2 \, d\alpha &= \sum_{X/2<p_1,p_2\le X} \log p_1 \log p_2 \int\limits_{-\tau}^{\tau} e((p_1 \log p_1 - p_2 \log p_2))\alpha) \, d\alpha \\
&\ll \sum_{X/2<p_1,p_2\le X} \log p_1 \log p_2 \min\left(\tau, \frac{1}{|p_1 \log p_1 - p_2 \log p_2|}\right) \\
&\ll \tau \sum_{\substack{X/2<p_1,p_2\le X \\ |p_1 \log p_1 - p_2 \log p_2|\le 1/\tau}} \log p_1 \log p_2 \\
&+ \sum_{\substack{X/2<p_1,p_2\le X \\ |p_1 \log p_1 - p_2 \log p_2|> 1/\tau}} \frac{\log p_1 \log p_2}{|p_1 \log p_1 - p_2 \log p_2|} \\
&\ll U\tau \log^2 X + V \log^2 X, \qquad (15)
\end{aligned}$$

where

$$U = \sum_{\substack{X/2<n_1,n_2\le X \\ |n_1 \log n_1 - n_2 \log n_2|\le 1/\tau}} 1\,,$$

$$V = \sum_{\substack{X/2<n_1,n_2\le X \\ |n_1 \log n_1 - n_2 \log n_2|> 1/\tau}} \frac{1}{|n_1 \log n_1 - n_2 \log n_2|}\,.$$

On the one hand by the mean-value theorem we get

$$U \ll \sum_{\substack{X/2<n_1\le X \quad X/2<n_2\le X \\ n_1\log n_1-1/\tau\le n_2\log n_2\le n_1\log n_1+1/\tau}} 1 \ll \frac{1}{\tau} \sum_{X/2<n_\le X} y'(\xi),$$

where y is implicit function defined by the equation (9) and ξ satisfies (10). Bearing in mind (12), (13) and the last inequality we find

$$U \ll \frac{1}{\tau} \cdot \frac{X}{\log X} \,. \tag{16}$$

On the other hand

$$V \le \sum_{l} V_l \,, \tag{17}$$

where

$$V_l = \sum_{\substack{X/2<n_1,n_2\le X \\ l<|n_1\log n_1 - n_2\log n_2|\le 2l}} \frac{1}{|n_1\log n_1 - n_2\log n_2|}$$

and l takes the values $2^k/\tau,\ k=0,1,2,...$, with $l \le X\log X$.

Using (9) – (13) and the mean-value theorem we conclude

$$\begin{aligned} V_l &\ll \frac{1}{l} \sum_{\substack{X/2<n_1\le X \quad X/2<n_2\le X \\ n_1\log n_1 + l < n_2\log n_2 \le n_1\log n_1 + 2l}} \sum 1 \\ &+ \frac{1}{l} \sum_{\substack{X/2<n_1\le X \quad X/2<n_2\le X \\ n_1\log n_1 - 2l < n_2\log n_2 \le n_1\log n_1 - l}} \sum 1 \\ &\ll \sum_{X/2<n\le X} y'(\xi) \\ &\ll \frac{X}{\log X} \,. \end{aligned} \tag{18}$$

The proof follows from (15) – (18). □

Lemma 9. *Assume that* $\tau \le |\alpha| \le K$. *Then*

$$|S(\alpha)| \ll X^{24/25}\log^3 X. \tag{19}$$

Proof. Without loss of generality we may assume that $\tau \le \alpha \le K$.

From (6) we have

$$S(\alpha) = \sum_{X/2<n\le X} \Lambda(n) e(\alpha n\log n) + \mathcal{O}(X^{1/2}). \tag{20}$$

On the other hand

$$\begin{aligned} &\sum_{X/2<n\le X} \Lambda(n) e(\alpha n \log n) \\ &= \sum_{X/2<n\le X} \Lambda(n) e\Big(\alpha n\Big(\ \log(n+1) + \mathcal{O}\big(1/n\big)\Big)\Big) \\ &= e\big(\mathcal{O}(|\alpha|)\big) \sum_{X/2<n\le X} \Lambda(n) e\big(\alpha n\log(n+1)\big) \\ &\ll |S_1(\alpha)|, \end{aligned} \tag{21}$$

where

$$S_1(\alpha) = \sum_{X/2<n\le X} \Lambda(n) e\big(\alpha n \log(n+1)\big). \tag{22}$$

We denote

$$f(d,l) = \alpha dl \log(dl+1). \tag{23}$$

Using (22), (23) and Vaughan's identity (see [14]) we get

$$S_1(\alpha) = U_1 - U_2 - U_3 - U_4, \tag{24}$$

where

$$U_1 = \sum_{d\le X^{1/3}} \mu(d) \sum_{X/2d<l\le X/d} (\log l) e(f(d,l)), \tag{25}$$

$$U_2 = \sum_{d\le X^{1/3}} c(d) \sum_{X/2d<l\le X/d} e(f(d,l)), \tag{26}$$

$$U_3 = \sum_{X^{1/3}<d\le X^{2/3}} c(d) \sum_{X/2d<l\le X/d} e(f(d,l)), \tag{27}$$

$$U_4 = \sum\sum_{\substack{X/2<dl\le X \\ d>X^{1/3},\, l>X^{1/3}}} a(d)\Lambda(l) e(f(d,l)), \tag{28}$$

and where

$$|c(d)| \le \log d, \quad |a(d)| \le \tau(d). \tag{29}$$

Estimation of U_1 and U_2

Consider first U_2 defined by (26). Bearing in mind (23) we find

$$\frac{\partial^n f(d,l)}{\partial l^n} = (-1)^n \left[\frac{\alpha d^n (n-2)!}{(dl+1)^{n-1}} + \frac{\alpha d^n (n-1)!}{(dl+1)^n}\right], \quad \text{for } n\ge 2. \tag{30}$$

By (30) and the restriction and the restriction

$$X/2 < dl \le X \tag{31}$$

we deduce

$$\left|\frac{\partial^n f(d,l)}{\partial l^n}\right| \asymp \alpha d (Xd^{-1})^{1-n}, \quad \text{for } n\ge 2. \tag{32}$$

From (32) and Lemma 2 with $(\varkappa,\lambda) = \big(\frac{1}{2},\frac{1}{2}\big)$ it follows

$$\begin{aligned}\sum_{X/2d<l\le X/d} e(f(d,l)) &\ll (\alpha d)^{1/2}\big(Xd^{-1}\big)^{1/2} + (\alpha d)^{-1} \\ &= \alpha^{1/2}X^{1/2} + \alpha^{-1}d^{-1}.\end{aligned} \tag{33}$$

Now (4), (5), (26), (29) and (33) give us

$$\begin{aligned}U_2 &\ll \big(\alpha^{1/2}X^{5/6} + \alpha^{-1/2}\big)\log^2 X \\ &\ll \big(K^{1/2}X^{5/6} + \tau^{-1/2}\big)\log^2 X \\ &\ll X^{23/25}\log^2 X.\end{aligned} \tag{34}$$

In order to estimate U_1 defined by (25) we apply Abel's transformation. Then arguing as in the estimation of U_2 we obtain

$$U_1 \ll X^{23/25} \log^2 X. \tag{35}$$

Estimation of U_3 and U_4

Consider first U_4 defined by (28). We have

$$U_4 \ll |U_5| \log X, \tag{36}$$

where

$$U_5 = \sum_{D<d\leq 2D} a(d) \sum_{\substack{L<l\leq 2L \\ X/2<dl\leq X}} \Lambda(l) e(f(d,l)) \tag{37}$$

and where

$$X^{1/3} \ll L \ll X^{1/2} \ll D \ll X^{2/3}, \quad DL \asymp X. \tag{38}$$

Using (29), (37), (38), Lemma 5 (i) and Cauchy's inequality we obtain

$$\begin{aligned} |U_5|^2 &\ll \sum_{D<d\leq 2D} \tau^2(d) \sum_{D<d\leq 2D} \left| \sum_{L_1<l\leq L_2} \Lambda(l) e(f(d,l)) \right|^2 \\ &\ll D(\log X)^3 \sum_{D<d\leq 2D} \left| \sum_{L_1<l\leq L_2} \Lambda(l) e(f(d,l)) \right|^2, \end{aligned} \tag{39}$$

where

$$L_1 = \max\left\{L, \frac{X}{2d}\right\}, \quad L_2 = \min\left\{2L, \frac{X}{d}\right\}. \tag{40}$$

Now from (38) – (40) and Lemma 3 with $Q \leq L$ and Lemma 5 (ii) we find

$$\begin{aligned} |U_5|^2 \ll D(\log X)^3 & \sum_{D<d\leq 2D} \frac{L}{Q} \sum_{|q|\leq Q} \left(1 - \frac{|q|}{Q}\right) \\ & \times \sum_{\substack{L_1<l\leq L_2 \\ L_1<l+q\leq L_2}} \Lambda(l+q)\Lambda(l) e(f(d,l+q) - f(d,l)) \\ & \ll \Bigg(\frac{LD}{Q} \sum_{0<|q|\leq Q} \sum_{\substack{L<l\leq 2L \\ L<l+q\leq 2L}} \Lambda(l+q)\Lambda(l) \left| \sum_{D_1<d\leq D_2} e\big(g_{l,q}(d)\big) \right| \\ & \qquad + \frac{(LD)^2}{Q} \log X \Bigg) \log^3 X, \end{aligned} \tag{41}$$

where

$$D_1 = \max\left\{D, \frac{X}{2l}, \frac{X}{2(l+q)}\right\}, \quad D_2 = \min\left\{2D, \frac{X}{l}, \frac{X}{l+q}\right\} \tag{42}$$

and

$$g(d) = g_{l,q}(d) = f(d,l+q) - f(d,l). \tag{43}$$

It is not hard to see that the sum over negative q in formula (41) is equal to the sum over positive q. Thus

$$|U_5|^2 \ll \left(\frac{LD}{Q}\sum_{1\leq q\leq Q}\sum_{L<l\leq 2L-q}\Lambda(l+q)\Lambda(l)\left|\sum_{D_1<d\leq D_2}e(g_{l,q}(d))\right| + \frac{(LD)^2}{Q}\log X\right)\log^3 X. \quad (44)$$

Consider the function $g(d)$. Taking into account (23), (30) and (43) we get

$$g^{(n)}(d) = (-1)^n\left[\frac{\alpha(l+q)^n(n-2)!}{(d(l+q)+1)^{n-1}} + \frac{\alpha l^n(n-1)!}{(dl+1)^n}\right], \quad \text{for } n\geq 2. \quad (45)$$

By (31) and (45) we conclude

$$|g^{(n)}(d)| \asymp \alpha L(XL^{-1})^{1-n}. \quad (46)$$

From (42), (46) and Lemma 2 with $(\varkappa,\lambda) = \left(\frac{11}{82},\frac{57}{82}\right)$ we obtain

$$\sum_{D_1<d\leq D_2}e(g(d)) \ll (\alpha L)^{11/82}\big(XL^{-1}\big)^{57/82} + (\alpha L)^{-1} = \alpha^{11/82}L^{-46/82}X^{57/82} + \alpha^{-1}L^{-1}. \quad (47)$$

Bearing in mind (44), (47), Lemma 5 (ii) and choosing $Q = L$ we find

$$\begin{aligned}|U_5|^2 &\ll \big(\alpha^{11/82}DLL^{36/82}X^{57/82} + \alpha^{-1}DL + D^2L\big)\log^4 X\\ &\ll \big(K^{11/82}L^{36/82}X^{139/82} + \tau^{-1}X + D^2L\big)\log^4 X. \quad (48)\end{aligned}$$

Now (4), (5), (38) and (48) give us

$$|U_5| \ll \big(K^{11/164}L^{36/164}X^{139/164} + \tau^{-1/2}X^{1/2} + XL^{-1/2}\big)\log^2 X \ll X^{24/25}\log^2 X. \quad (49)$$

From (36) and (49) it follows

$$U_4 \ll X^{24/25}\log^3 X. \quad (50)$$

Working as in the estimation of U_4 we obtain

$$U_3 \ll X^{24/25}\log^3 X. \quad (51)$$

Summarizing (20), (21), (24), (34), (35), (50) and (51) we establish the estimation (19).

The lemma is proved. □

4 Proof of the Theorem

Consider the sum

$$\Gamma = \sum_{\substack{X/2<p_1,p_2,p_3\leq X\\ |p_1\log p_1+p_2\log p_2+p_3\log p_3-N|<\varepsilon}}\log p_1\log p_2\log p_3.$$

The theorem will be proved if we show that $\Gamma \to \infty$ as $X \to \infty$.

According to the definition of $\psi(y)$ and the inverse Fourier transformation formula we have

$$\begin{aligned}
\Gamma \geq \Gamma_0 &= \sum_{X/2<p_1,p_2,p_3\leq X} \psi(p_1 \log p_1 + p_2 \log p_2 + p_3 \log p_3 - N) \log p_1 \log p_2 \log p_3 \\
&= \sum_{X/2<p_1,p_2,p_3\leq X} \log p_1 \log p_2 \log p_3 \\
&\times \int\limits_{-\infty}^{\infty} e((p_1 \log p_1 + p_2 \log p_2 + p_3 \log p_3 - N)\alpha)\Psi(\alpha)\, d\alpha \\
&= \int\limits_{-\infty}^{\infty} S^3(\alpha) e(-N\alpha)\Psi(\alpha)\, d\alpha. \qquad (52)
\end{aligned}$$

We decompose Γ_0 in three parts

$$\Gamma_0 = \Gamma_1 + \Gamma_2 + \Gamma_3. \qquad (53)$$

where

$$\Gamma_1 = \int\limits_{-\tau}^{\tau} S^3(\alpha)e(-N\alpha)\Psi(\alpha)\, d\alpha, \qquad (54)$$

$$\Gamma_2 = \int\limits_{\tau\leq|\alpha|\leq K} S^3(\alpha)e(-N\alpha)\Psi(\alpha)\, d\alpha, \qquad (55)$$

$$\Gamma_3 = \int\limits_{|\alpha|>K} S^3(\alpha)e(-N\alpha)\Psi(\alpha)\, d\alpha. \qquad (56)$$

Estimation of Γ_1

Denote the integrals

$$\Theta_\tau = \int\limits_{-\tau}^{\tau} I^3(\alpha)e(-N\alpha)\Psi(\alpha)\, d\alpha \qquad (57)$$

$$\Theta = \int\limits_{-\infty}^{\infty} I^3(\alpha)e(-N\alpha)\Psi(\alpha)\, d\alpha. \qquad (58)$$

For Γ_1 denoted by (54) we have

$$\Gamma_1 = (\Gamma_1 - \Theta_\tau) + (\Theta_\tau - \Theta) + \Theta. \qquad (59)$$

From (54), (57), Lemma 1, Lemma 6 and Lemma 8 (i), 8 (ii) we get

$$\begin{aligned}|\Gamma_1 - \Theta_\tau| &\ll \int_{-\tau}^{\tau} |S^3(\alpha) - I^3(\alpha)||\Psi(\alpha)|\,d\alpha\\ &\ll \varepsilon \int_{-\tau}^{\tau} |S(\alpha) - I(\alpha)|\big(|S(\alpha)|^2 + |I(\alpha)|^2\big)\,d\alpha\\ &\ll \varepsilon X e^{-(\log X)^{1/5}} \left(\int_{-\tau}^{\tau} |S(\alpha)|^2\,d\alpha + \int_{-\tau}^{\tau} |I(\alpha)|^2\,d\alpha\right)\\ &\ll \varepsilon X^2 e^{-(\log X)^{1/6}}. \end{aligned} \tag{60}$$

Using (4), (7), (57), (58), Lemma 1 and Lemma 4 we find

$$\begin{aligned}|\Theta_\tau - \Theta| &\ll \int_{\tau}^{\infty} |I(\alpha)|^3|\Psi(\alpha)|\,d\alpha \ll \frac{\varepsilon}{(1+\log X)^3}\int_{\tau}^{\infty}\frac{d\alpha}{\alpha^3}\\ &\ll \frac{\varepsilon}{\tau^2(1+\log X)^3} \ll \varepsilon\frac{X^2}{\log^2 X}. \end{aligned} \tag{61}$$

Bearing in mind (4), (58), (59), (60), (61) and Lemma 7 we conclude

$$\Gamma_1 \gg \varepsilon\frac{X^2}{\log X}. \tag{62}$$

Estimation of Γ_2

Now let us consider Γ_2 defined by (55). We have

$$\Gamma_2 \ll \int_{\tau}^{K} |S(\alpha)|^3|\Psi(\alpha)|\,d\alpha \ll \max_{\tau\le t\le K}|S(\alpha)|\int_{\tau}^{K}|S(\alpha)|^2|\Psi(\alpha)|\,d\alpha. \tag{63}$$

Using Lemma 1 and Lemma 8 (iii) we deduce

$$\begin{aligned}\int_{\tau}^{K} |S(\alpha)|^2|\Psi(\alpha)|\,d\alpha &\ll \varepsilon\int_{\tau}^{1/\varepsilon}|S(\alpha)|^2\,d\alpha + \int_{1/\varepsilon}^{K}|S(\alpha)|^2\,\frac{d\alpha}{\alpha}\\ &\ll \varepsilon\sum_{0\le n\le 1/\varepsilon}\int_{n}^{n+1}|S(\alpha)|^2\,d\alpha + \sum_{1/\varepsilon-1\le n\le K}\frac{1}{n}\int_{n}^{n+1}|S(\alpha)|^2\,d\alpha\\ &\ll X\log^3 X. \end{aligned} \tag{64}$$

From (63), (64) and Lemma 9 it follows

$$\Gamma_2 \ll X^{49/25}\log^6 X \ll \frac{\varepsilon X^2}{\log^2 X}. \tag{65}$$

Estimation of Γ_3
Using (56), Lemma 1 and choosing $k = [\log X]$ we find

$$\begin{aligned}\Gamma_3 &\ll \int\limits_K^\infty |S(\alpha)|^3 |\Psi(\alpha)|\, d\alpha \\ &\ll X^3 \int\limits_K^\infty \frac{1}{\alpha}\left(\frac{k}{2\pi\alpha\varepsilon/8}\right)^k d\alpha \\ &= X^3\left(\frac{4k}{\pi\varepsilon K}\right)^k \ll 1. \end{aligned} \tag{66}$$

The end of the proof
Bearing in mind (52), (53), (62), (65) and (66) we establish that

$$\Gamma \gg \varepsilon\frac{X^2}{\log X}. \tag{67}$$

Now (3) and (67) imply that $\Gamma \to \infty$ as $X \to \infty$.
The proof of the Theorem is complete.

Acknowledgments. The author thanks Professor Kaisa Matomäki and Professor Joni Teräväinen for their valuable remarks and useful discussions.

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