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FRECHET DIFFERENTIABILITY FOR A VISCOUS CAHN-HILLIARD EQUATION AND ITS APPLICATION TO A BILINEAR OPTIMAL CONTROL PROBLEM

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Abstract. We consider a viscous Cahn-Hilliard equation. The objective is to study the Fréchet differentiability of a nonlinear solution map from a bilinear control input to the solution of the equation. We use this to formulate a bilinear optimal control problem. We show the existence of an optimal control and find its necessary optimality condition.

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1. Introduction

Let Ω be a connected bounded open subset of $\mathbf{R}^n (n \leq 3)$ with a sufficiently smooth boundary Γ. We set $Q = (0,T) \times \Omega$ for $T > 0$. We consider the following viscous Cahn-Hilliard equation

$$(1) \qquad (1-\nu)y' + \Delta\Big(\Delta y + f(y) - \nu y'\Big) = 0 \quad \text{in } \ Q$$

together with the initial value $y(0,x) = y_0$ and appropriate boundary conditions, where $' = \frac{\partial}{\partial t}$, $\nu \in [0,1]$ is a constant and $f(y)$ is given by a potential function $F(y)$ for which $F'(y) = -f(y)$. The solution y of Eq.(1) means the concentration of one of the two phases in a phase transition. In the case $\nu = 0$, Eq.(1) reduces to the Cahn-Hilliard model for spinodal decomposition, which is a description of the process by which phase separation occurs in a binary alloy after the temperature is reduced suddenly below a critical value [5]; In case $\nu = 1$, Eq.(1) reduces to the Allen-Cahn model for grain-boundary migration, which is the process by which the interface between two differently aligned crystal lattice in a solid evolve with time. Novic and Cohen [17] proposed Eq.(1) in the case of $\nu \in (0,1)$ to interpret a chemical model that interpolates between the cases $\nu = 0$ and $\nu = 1$ in Eq.(1), as well as to include certain viscous effects neglected in [5]. Many researchers have contributed to the study of well-posedness of Eq.(1), as well as the existence and properties of global attractors in some Sobolev spaces. To mention just a few, we can refer to Elliott and Stuart [9], Carvalho and Dlotko [6], Grinfeld and Novic-Cohen [10] and references therein.

Zhao and Liu [21] studied quadratic cost optimal control problems using the frame work of Lions [16] on a one dimensional version of Eq.(1) with $f(y) = -y^3 + y$. They studied the well-posedness of the equation and proved the existence of an optimal control for the equation with quadratic cost. In our previous study [13], we studied quadratic cost optimal control problems with the equation considered in [21] in general dimensions of $n \leq 3$. We verified the well-posedness of the equation in the Hadamard sense and expanded the optimal control theory due to Lions [16] with emphasis on deriving necessary optimality conditions of optimal controls by showing the Gâteaux differentiability of the solution map from an external forcing control variable to the weak solution of the equation. As the study of the same direction, we can refer to [8].

The extension of optimal control theory to quasilinear equations is not easy because of the difficulties in showing differentiability of a solution map. Some researches have been devoted to the study of optimal control in specific quasilinear equations. For instance, we can refer to Hwang and Nakagiri [11, 14] and Hwang [12]. In this paper, we consider the following controlled viscous Cahn-Hilliard equation with the Dirichlet boundary conditions:

$$
\begin{cases} (1-\nu)y' + \Delta\Big(\Delta y - y^3 + y - \nu y'\Big) = vy + f \quad \text{in} \quad Q, \\ y = \Delta y = 0 \qquad \text{on} \quad \Sigma, \\ y(0,x) = y_0(x) \qquad \text{in} \quad \Omega, \end{cases} \tag{2}
$$

where $\Sigma = (0,T) \times \Gamma$, $\nu \in (0,1)$, v is a bilinear control input variable, f is a forcing term and $-y^3 + y$ is given as a negative of the derivative of the classical double well potential function $F(y) = \frac{1}{4}(1-y^2)^2$.

In most cases, the Gâteaux differentiability may be enough to solve a quadratic cost optimal control problem. However, the Fréchet differentiability of a solution map is more desirable for studying the problem with more general cost function like non-quadratic or non-convex functions,. In this paper, we show the Fréchet differentiability of the nonlinear solution map of Eq.(2): $v \to y$, that is, from the bilinear control input variables to the solutions of Eq.(2). Based on the result, we construct and solve a bilinear optimal control problem on Eq.(2). For the study, we refer to the linear results of a bilinear optimal control problems where the state equation is a linear partial differential equations, such as the Kirchhoff plate or the damped wave equations (see [3], [4], [15] and references therein).

To apply the variational approach due to the previous studies of [3], [4], [15] and [16] to our problem, we propose the quadratic cost functional $J(\cdot)$, which is to be minimized within $\mathcal{U}_{ad}$, $\mathcal{U}_{ad}$ is an admissible set of control variables in Eq.(2). We show the existence of $u \in \mathcal{U}_{ad}$ which minimizes the quadratic cost functional $J(\cdot)$. Then, we establish the necessary condition of optimality of the optimal control u for a distributive observation case by employing an associate adjoint system. To this end, we use the strong Gâteaux differentiability of the map $v \to J(v)$, which is ensured by the Fréchet differentiability of the solution map $v \to y(v)$. We can also use this to define the adjoint system.

We also discuss the time local uniqueness of optimal control for the cost $J(\cdot)$. It is unclear and difficult to verify that the uniqueness of optimal control in nonlinear control problems. For this purpose, we employ the idea in [4] which showed the strict convexity of the cost $J(v)$ in local time interval by making use of the second order Gâteaux differentiability of the nonlinear solution mapping $v \to y(v)$.

To discuss the uniqueness of our case, we have proved the second order Fréchet differentiability of the solution map with spatial dimensions of $n \leq 3$. However, to prove the strict convexity of the cost $J(\cdot)$, we limited the study to a one-dimensional case. As a result, we proved the local uniqueness of the optimal control in part. This is another novelty of the paper.

2. Preliminaries

Throughout this paper, we use C as a generic constant. Let X be a Banach space. We denote its topological dual by X', and the duality pairing between X' and X by $\langle \cdot, \cdot \rangle_{X',X}$. $H_0^k(\Omega)$ is the completions of $C_0^\infty(\Omega)$ in $H^k(\Omega)$ for $k \geq 1$. Let the scalar product and norm on $L^2(\Omega)$ by $(\cdot, \cdot)_2$ and $\| \cdot \|_2$, respectively. Since we assume that $\Omega \subset \mathbf{R}^n (n \leq 3)$ has a sufficiently smooth boundary Γ, we can use the regularity theory for elliptic boundary value problems (cf. Temam [19 p.150]) and the well-known Poincare's inequality to endow the norms and the scalar products on $V := H^2(\Omega) \cap H_0^1(\Omega)$ and $H_0^1(\Omega)$ as follows:

$$\|\psi\|_V = \|\Delta\psi\|_2, \quad ((\psi, \phi))_V = (\Delta\psi, \Delta\phi)_2, \quad \forall \psi, \phi \in V; \tag{3}$$

$$\|\psi\|_{H_0^1(\Omega)} = \|\nabla\psi\|_2, \quad ((\psi, \phi))_{H_0^1(\Omega)} = (\nabla\psi, \nabla\phi)_2, \quad \forall \psi, \phi \in H_0^1(\Omega). \tag{4}$$

Considering the boundary conditions of Eq.(2), we know its domain $\mathcal{A}$ ($= \Delta^2$) is given by

$$D(\mathcal{A}) = \{u \in V \mid \mathcal{A}u \in L^2(\Omega), \ \Delta u = 0 \ \text{ on } \Gamma\}.$$

By using the well known notation of $H^{-1}(\Omega) = (H_0^1(\Omega))'$, it is clear that

$$D(\mathcal{A}) \hookrightarrow V \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \hookrightarrow V'. \tag{5}$$

Each space is dense in the following one, and the injections are continuous and compact. Since $L^2(\Omega)$ is the pivot Hilbert space between V and V', in view of (3), we can endow the norm on V' as in [19 pp. 54-55]:

$$\|\phi\|_{V'} = \|\Delta^{-1}\phi\|_2, \quad \forall \phi \in V'. \tag{6}$$

According to Adams [1], the following embeddings are continuous when $n \leq 3$:

$$H_0^1(\Omega) \hookrightarrow L^p(\Omega), \quad (i.e., \ \|\psi\|_{L^p(\Omega)} \ \leq \ C\|\nabla\psi\|_2, \ \forall \psi \in H_0^1(\Omega)), \quad (1 \leq p \leq 6); \tag{7}$$

$$H_0^1(\Omega) \hookrightarrow C^0(\bar{\Omega}), \quad \text{when } n = 1, \quad (i.e., \ \|\psi\|_{C^0(\bar{\Omega})} \ \leq \ C\|\nabla\psi\|_2, \forall \psi \in H_0^1(\Omega)); \tag{8}$$

$$V \hookrightarrow C^0(\bar{\Omega}), \quad (i.e., \ \|\phi\|_{C^0(\bar{\Omega})} \ \leq \ C\|\Delta\phi\|_2, \quad \forall \phi \in V). \tag{9}$$

Let

$$W(0,T) := \{g \mid g \in L^2(0,T;V), \ g' \in L^2(Q)\} \tag{10}$$

be a Hilbert space with the norm

$$\|g\|_{W(0,T)} = \left(\|g\|^2_{L^2(0,T;V)} + \|g'\|^2_{L^2(Q)}\right)^{\frac{1}{2}},$$

where g' denotes the first order distributional derivative of g with respect to t. Since $[V,\ L^2(\Omega)]_{\frac{1}{2}} = H_0^1(\Omega)$, we can obtain the following [7, p.555]:

$$W(0,T) \hookrightarrow C([0,T]; H_0^1(\Omega)). \tag{11}$$

The following variational formulation is used to define the *weak* solution of Eq.(2).

Definition. *A function y is said to be a weak solution of Eq.(2) if $y \in W(0,T)$ and y satisfies*

$$\begin{cases} (1-\nu)\big(y'(\cdot),\phi\big)_2 + \big(\Delta y(\cdot) - y^3(\cdot) + y(\cdot) - \nu y'(\cdot), \Delta\phi\big)_2 \\ \qquad\qquad = (v(\cdot)y(\cdot),\phi)_2 + \big\langle f(\cdot),\phi\big\rangle_{V',V} \\ \qquad\qquad \textit{for all } \ \phi \in V \ \textit{ in the sense of } \mathcal{D}'(0,T), \\ y(0) = y_0. \end{cases} \tag{12}$$

There are several methods to establish the well-posedness of Eq.(2) in the sense of Hadamard; for example, the semigroup approach, as described in [6], or the Faedo-Galerkin method, as described in [7]. Since these approaches are quite standard, we can omit the details here. In the sequel, we rely on the following theorem for the well-posedness of Eq.(2).

Theorem 2.1. *Assume that $\nu \in (0,1)$, $v \in L^\infty(Q)$, $f \in L^2(0,T;V')$ and $y_0 \in H_0^1(\Omega)$. Then the Eq.(2) has a unique weak solution y in $W(0,T)$. And the solution mapping $p = (y_0, v, f) \to y(p)$ of $P \equiv H_0^1(\Omega) \times L^\infty(Q) \times L^2(0,T;V')$ into $W(0,T)$ is locally Lipschitz continuous. That is, for each $p_1 = (y_0^1, v_1, f_1) \in P$ and $p_2 = (y_0^2, v_2, f_2) \in P$ we have the inequality*

$$\begin{aligned} & \|y(p_1) - y(p_2)\|_{W(0,T)} \\ \leq\ & C(\|\nabla(y_0^1 - y_0^2)\|_2 + \|v_1 - v_2\|_{L^\infty(Q)} + \|f_1 - f_2\|_{L^2(0,T;V')}) \\ \equiv\ & C\|p_1 - p_2\|_{\mathcal{P}}, \end{aligned} \tag{13}$$

where $C > 0$ is a constant depending on the data.

We prove Theorem 2.1 by showing the following inequality

$$\begin{aligned} & \|\nabla(y(p_1;t) - y(p_2;t))\|_2^2 + \int_0^t \|\Delta(y(p_1;s) - y(p_2;s))\|_2^2 ds \\ \leq\ & C(\|\nabla(y_0^1 - y_0^2)\|_2^2 + \|v_1 - v_2\|_{L^\infty(Q)}^2 + \|f_1 - f_2\|_{L^2(0,T;V')}^2). \end{aligned} \tag{14}$$

For simplicity, we will omit writing the integral variables in the definite integral throughout the paper without any confusion.

Proof of Theorem 2.1. With the standard Faedo-Galerkin method as in [19, p.151, Theorem 4.2] which is applied to a general pattern formation equation, we can see that the weak solution y of Eq.(2) exists in $L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;V)$ under the data condition $p = (y_0, v, f) \in H_0^1(\Omega) \times L^\infty(Q) \times L^2(0,T;V')$. As is well known, especially as described in [6], the operator $-((1-\nu)I - \nu\Delta)^{-1}\Delta$ is bounded, self-adjoint and positive linear operator on $L^2(\Omega)$ and moreover on the Hilbert spaces $D(A^\theta)$ ($\theta \in \mathbf{R}$), ($A = -\Delta$ with Dirichlet boundary condition). Therefore, the operator $((1-\nu)I - \nu\Delta)^{-1}$ is a bounded linear operator from V' to $L^2(\Omega)$ and from $L^2(\Omega)$ to V (cf. [2]). Thus, as we shall see later, we can infer that y' of Eq.(2) exists in $L^2(Q)$. As a result, we can know that the weak solution y of Eq.(2) exists in $W(0,T)$.

Based on the above, we prove the inequality (14). For the purpose, we denote $y_1 - y_2 \equiv y(p_1) - y(p_2)$ by ψ. Then, we can know from Eq.(2) that ψ satisfies the following equation in weak sense:

$$\text{(15)} \quad \begin{cases} (1-\nu)\psi' + \Delta\Big(\Delta\psi - \psi(y_1^2 + y_1y_2 + y_2^2) + \psi - \nu\psi'\Big) = v_1\psi \\ \qquad\qquad\qquad\qquad +(v_1 - v_2)y_2 + f_1 - f_2 \quad \text{in} \ \ Q, \\ \psi = \Delta\psi = 0 \quad \text{on} \ \ \Sigma, \\ \psi(0) = y_0^1 - y_0^2 \quad \text{in} \ \ \Omega. \end{cases}$$

By multiplying both sides of (15) by 2ψ, we have

$$\begin{aligned} & \frac{d}{dt}\Big((1-\nu)\|\psi\|_2^2 + \nu\|\nabla\psi\|_2^2\Big) + 2\|\Delta\psi\|_2^2 \\ = \ & 2\|\nabla\psi\|_2^2 + 2(\psi(y_1^2 + y_1y_2 + y_2^2), \Delta\psi)_2 + 2(v_1\psi, \psi)_2 \\ & +2((v_1 - v_2)y_2, \psi)_2 + 2\langle f_1 - f_2, \psi\rangle_{V',V}. \end{aligned} \tag{16}$$

To obtain the estimate of the second term of the right hand side of (16), we consider

$$\begin{aligned} & \|\psi(y_1^2 + y_1y_2 + y_2^2)\|_2 \\ \leq \ & \|\psi y_1^2\|_2 + \|\psi y_1 y_2\|_2 + \|\psi y_2^2\|_2 \\ \leq \ & (\|y_1\|_{L^6(\Omega)}^2 + \|y_1\|_{L^6(\Omega)}\|y_2\|_{L^6(\Omega)} + \|y_2\|_{L^6(\Omega)}^2)\|\psi\|_{L^6(\Omega)} \\ \leq \ & (\text{with } (7)) \\ \leq \ & C(\|\nabla y_1\|_2^2 + \|\nabla y_1\|_2\|\nabla y_2\|_2 + \|\nabla y_2\|_2^2)\|\nabla\psi\|_2 \\ \leq \ & C(\|y_1\|_{C([0,T];H_0^1(\Omega))} + \|y_2\|_{C([0,T];H_0^1(\Omega))})^2\|\nabla\psi\|_2 \\ \leq \ & (\text{with } (11)) \\ \leq \ & C(\|y_1\|_{W(0,T)} + \|y_2\|_{W(0,T)})^2\|\nabla\psi\|_2 \end{aligned} \tag{17}$$

by which we can obtain

$$\begin{aligned} 2|(\psi(y_1^2 + y_1y_2 + y_2^2), \Delta\psi)_2| \ & \leq \ 2\|\psi(y_1^2 + y_1y_2 + y_2^2)\|_2\|\Delta\psi\|_2 \\ & \leq \ (\text{with } (17)) \\ & \leq \ C\|\nabla\psi\|_2\|\Delta\psi\|_2 \\ & \leq \ (\text{with the Young inequality}) \\ & \leq \ \frac{1}{2}\|\Delta\psi\|_2^2 + C\|\nabla\psi\|_2^2. \end{aligned} \tag{18}$$

The other terms of the right hand side of (16) can be estimated as follows:

$$\begin{aligned} 2|(v_1\psi,\psi)_2| &\le 2\|v_1\|_{L^\infty(Q)}\|\psi\|_2^2 \\ &\le \text{(with (5))} \\ &\le C\|\nabla\psi\|_2^2; \quad (19) \\ 2|((v_1-v_2)y_2,\psi)_2| &\le 2\|v_1-v_2\|_{L^\infty(Q)}\|y_2\|_2\|\psi\|_2 \\ &\le \|v_1-v_2\|_{L^\infty(Q)}^2\|y_2\|_2^2+\|\psi\|_2^2 \\ &\le \text{(with (5) and } W(0,T)\hookrightarrow C([0,T];L^2(\Omega))) \\ &\le C(\|v_1-v_2\|_{L^\infty(Q)}^2+\|\nabla\psi\|_2^2); \quad (20) \\ 2|\langle f_1-f_2,\psi\rangle_{V',V}| &\le 2\|f_1-f_2\|_{V'}\|\Delta\psi\|_2 \\ &\le 8\|f_1-f_2\|_{V'}^2+\frac{1}{2}\|\Delta\psi\|_2^2. \quad (21) \end{aligned}$$

By combining (16) and (18) - (21), we can get

$$\begin{aligned} &\frac{d}{dt}\Big((1-\nu)\|\psi\|_2^2+\nu\|\nabla\psi\|_2^2\Big)+\|\Delta\psi\|_2^2 \\ \le\ & C(\|\nabla\psi\|_2^2+\|v_1-v_2\|_{L^\infty(Q)}^2+\|f_1-f_2\|_{V'}^2). \quad (22) \end{aligned}$$

By integrating (22) on $[0,t]$ and applying the Gronwall's lemma to it, we can obtain

$$\begin{aligned} &\|\nabla\psi(t)\|_2^2+\int_0^t\|\Delta\psi\|_2^2ds \\ \le\ & C(\|\nabla\psi(0)\|_2^2+\|v_1-v_2\|_{L^\infty(Q)}^2+\|f_1-f_2\|_{L^2(0,T;V')}^2). \quad (23) \end{aligned}$$

This proves (14). From (15) we have

$$\psi'=I_1+I_2+I_3, \quad (24)$$

where

$$\begin{aligned} I_1 &= -((1-\nu)I-\nu\Delta)^{-1}\Delta(\Delta\psi-\psi(y_1^2+y_1y_2+y_2^2)+\psi), \\ I_2 &= ((1-\nu)I-\nu\Delta)^{-1}(v_1\psi+(v_1-v_2)y_2), \\ I_3 &= ((1-\nu)I-\nu\Delta)^{-1}(f_1-f_2). \end{aligned}$$

Since $((1-\nu)I-\nu\Delta)^{-1}\Delta$ is bounded, self-adjoint linear operator on $L^2(\Omega)$ and $((1-\nu)I-\nu\Delta)^{-1}$ is bounded linear operator from $L^2(\Omega)$ into V and from V' into $L^2(\Omega)$, we can infer the following:

$$
\begin{aligned}
\|I_1\|_2 &\le (\text{by } ((1-\nu)I-\nu\Delta)^{-1}\Delta \in \mathcal{L}(L^2(\Omega))) \\
&\le C\|\Delta\psi - \psi(y_1^2+y_1y_2+y_2^2)+\psi\|_2 \\
&\le C(\|\Delta\psi\|_2 + \|\psi(y_1^2+y_1y_2+y_2^2)\|_2 + \|\psi\|_2) \\
&\le (\text{with (5) and (17)}) \\
&\le C(\|\Delta\psi\|_2 + \|\nabla\psi\|_2); \qquad (25)\\
\|I_2\|_2 &\le C\|I_2\|_V \\
&\le (\text{by } ((1-\nu)I-\nu\Delta)^{-1} \in \mathcal{L}(L^2(\Omega),V)) \\
&\le C\|v_1\psi + (v_1-v_2)y_2\|_2 \\
&\le C(\|v_1\|_{L^\infty(Q)}\|\psi\|_2 + \|v_1-v_2\|_{L^\infty(Q)}\|y_2\|_2) \\
&\le (\text{with (5) and } W(0,T)\hookrightarrow C([0,T];L^2(\Omega))) \\
&\le C(\|\nabla\psi\|_2 + \|v_1-v_2\|_{L^\infty(Q)}); \qquad (26)\\
\|I_3\|_2 &\le (\text{by } ((1-\nu)I-\nu\Delta)^{-1} \in \mathcal{L}(V',L^2(\Omega))) \\
&\le C\|f_1-f_2\|_{V'}. \qquad (27)
\end{aligned}
$$

From (23) and (25)-(27) we can obtain

$$
\begin{aligned}
\|\psi'\|^2_{L^2(Q)} &\le \|I_1+I_2+I_3\|^2_{L^2(Q)} \\
&\le 3(\|I_1\|^2_{L^2(Q)} + \|I_2\|^2_{L^2(Q)} + \|I_3\|^2_{L^2(Q)}) \\
&\le (\text{with } (25)-(27)) \\
&\le C(\|\Delta\psi\|^2_{L^2(Q)} + \|\nabla\psi\|^2_{L^2(Q)} \\
&\qquad + \|v_1-v_2\|^2_{L^\infty(Q)} + \|f_1-f_2\|^2_{L^2(0,T;V')}) \\
&\le (\text{with (23)}) \\
&\le C(\|\nabla\psi(0)\|_2^2 + \|v_1-v_2\|^2_{L^\infty(Q)} + \|f_1-f_2\|^2_{L^2(0,T;V')}). \qquad (28)
\end{aligned}
$$

Finally, by (23) and (28) we can conclude the following

$$
\begin{aligned}
&\|\psi\|^2_{W(0,T)} \\
&\le C(\|\nabla\psi(0)\|_2^2 + \|v_1-v_2\|^2_{L^\infty(Q)} + \|f_1-f_2\|^2_{L^2(0,T;V')}). \qquad (29)
\end{aligned}
$$

This completes the proof.

3. Fréchet differentiability of the nonlinear solution map

In this section, we study the Fréchet differentiability of the nonlinear solution map. The Fréchet differentiability of the solution map plays an important role in many applications. Let $\mathcal{F} = L^\infty(Q)$. We consider the nonlinear solution map from $u \in \mathcal{F}$ to $y(u) \in W(0,T)$, where $y(u)$ is the solution of

$$
(30)\quad \begin{cases}
(1-\nu)y'(u) + \Delta\Big(\Delta y(u) - y^3(u) + y(u) - \nu y'(u)\Big) \\
\qquad\qquad\qquad\qquad = uy(u) + f \quad \text{in } Q, \\
y(u) = \Delta y(u) = 0 \quad \text{on } \Sigma, \\
y(u;0,x) = y_0(x) \quad \text{in } \Omega.
\end{cases}
$$

From Theorem 2.1, for fixed $(y_0,f) \in H_0^1(\Omega)\times L^2(0,T;V')$, we know that the solution map $\mathcal{F} \to W(0,T)$, which maps from the term $u \in \mathcal{F}$ of Eq.(30)

to $y(u) \in W(0,T)$, is well defined and continuous. We define the Fréchet differentiability of the nonlinear solution map as follows.

Definition. *The solution map $u \to y(u)$ of $\mathcal{F}$ into $W(0,T)$ is said to be Fréchet differentiable on $\mathcal{F}$ if for any $u \in \mathcal{F}$, there exists a $T(u) \in \mathcal{L}(\mathcal{F}, W(0,T))$ such that, for any $w \in \mathcal{F}$,*

$$\frac{\|y(u+w) - y(u) - T(u)w\|_{W(0,T)}}{\|w\|_{\mathcal{F}}} \to 0 \quad as \quad \|w\|_{\mathcal{F}} \to 0. \tag{31}$$

The operator $T(u)$ is called the Fréchet derivative of y at u, which we denote by $Dy(u)$, and $T(u)w = Dy(u)w \in W(0,T)$ is called the Fréchet derivative of y at u in the direction of $w \in \mathcal{F}$.

Theorem 3.1. *The solution map $u \to y(u)$ of $\mathcal{F}$ into $W(0,T)$ is Fréchet differentiable on $\mathcal{F}$ and the Fréchet derivative of $y(u)$ at u in the direction $w \in \mathcal{F}$ (that is $z = Dy(u)w$) is the solution of*

$$\begin{cases} (1-\nu)z' + \Delta\Big(\Delta z - 3y^2(u)z + z - \nu z'\Big) = uz + wy(u) & in \;\; Q, \\ z = \Delta z = 0 \qquad on \;\; \Sigma, \\ z(0,x) = 0, \qquad in \;\; \Omega. \end{cases} \tag{32}$$

We prove this theorem by two steps:

(i) For any $w \in \mathcal{F}$, Eq.(32) has a unique solution $z \in W(0,T)$. That is, there exists an operator $T \in \mathcal{L}(\mathcal{F}, W(0,T))$ satisfying $Tw = z (= z(w))$.

(ii) We show that $\|y(u+w) - y(u) - z\|_{W(0,T)} = o(\|w\|_{\mathcal{F}})$ as $\|w\|_{\mathcal{F}} \to 0$.

Proof. (i) To estimate the solution z of Eq.(32), we take the scalar product of Eq.(32) with z in $L^2(\Omega)$:

$$\begin{aligned} & \frac{(1-\nu)}{2}\frac{d}{dt}\|z\|_2^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla z\|_2^2 + \|\Delta z\|_2^2 \\ = \;& (3y^2(u)z, \Delta z)_2 + \|\nabla z\|_2^2 + (uz + wy(u), z)_2. \end{aligned} \tag{33}$$

The right hand side of (33) can be estimated as follows:

$$\begin{aligned} |(3y^2(u)z, \Delta z)_2| &\le 3\|y^2(u)z\|_2\|\Delta z\|_2 \\ &\le \text{(with the Young inequality)} \\ &\le \frac{1}{2}\|\Delta z\|_2^2 + 18\|y^2(u)z\|_2^2 \\ &\le \frac{1}{2}\|\Delta z\|_2^2 + 18\|y(u)\|_{L^6(\Omega)}^4\|z\|_{L^6(\Omega)}^2 \\ &\le \text{(with (7))} \\ &\le \frac{1}{2}\|\Delta z\|_2^2 + C\|\nabla y(u)\|_2^4\|\nabla z\|_2^2 \\ &\le \frac{1}{2}\|\Delta z\|_2^2 + C\|y(u)\|_{C([0,T];H_0^1(\Omega))}^4\|\nabla z\|_2^2 \\ &\le \text{(with (11))} \\ &\le \frac{1}{2}\|\Delta z\|_2^2 + C\|y(u)\|_{W(0,T)}^4\|\nabla z\|_2^2 \\ &\le \frac{1}{2}\|\Delta z\|_2^2 + C\|\nabla z\|_2^2; \end{aligned} \tag{34}$$

$$
\begin{aligned}
|(uz + wy(u), z)_2| &\leq |(uz, z)_2| + |(wy(u), z)_2| \\
&\leq \|u\|_{\mathcal{F}}\|z\|_2^2 + \|wy(u)\|_2\|z\|_2 \\
&\leq \text{(with the Young inequality)} \\
&\leq (1 + \|u\|_{\mathcal{F}})\|z\|_2^2 + \|wy(u)\|_2^2 \\
&\leq C\|z\|_2^2 + \|wy(u)\|_2^2. \qquad (35)
\end{aligned}
$$

Considering from (33) to (35), we can obtain the following:

$$
\begin{aligned}
&\frac{d}{dt}((1-\nu)\|z\|_2^2 + \nu\|\nabla z\|_2^2) + \|\Delta z\|_2^2 \\
&\leq C(\|z\|_2^2 + \|\nabla z\|_2^2) + C\|wy(u)\|_2^2. \qquad (36)
\end{aligned}
$$

By integrating (36) over $[0, t]$, we obtain

$$
\begin{aligned}
&\|z(t)\|_2^2 + \|\nabla z(t)\|_2^2 + \int_0^t \|\Delta z\|_2^2 ds \\
&\leq C\int_0^t (\|z\|_2^2 + \|\nabla z\|_2^2)ds + C\|wy(u)\|_{L^2(Q)}^2. \qquad (37)
\end{aligned}
$$

Then, by applying Gronwall's lemma to (37), we can have

$$
\|z(t)\|_2^2 + \|\nabla z(t)\|_2^2 + \int_0^t \|\Delta z\|_2^2 ds \leq C\|wy(u)\|_{L^2(Q)}^2. \qquad (38)
$$

Since

$$
\begin{aligned}
z' &= -((1-\nu)I - \nu\Delta)^{-1}\Delta(\Delta z - 3y^2(u)z + z) \\
&\quad +((1-\nu)I - \nu\Delta)^{-1}(uz + wy(u)), \qquad (39)
\end{aligned}
$$

we can use similar calculations in (25) and (26) to obtain the following from (39):

$$
\begin{aligned}
\|z'\|_2 &\leq C(\|\Delta z\|_2 + \|\nabla z\|_2 + \|z\|_2 + \|wy(u)\|_2) \\
&\leq \text{(with (5))} \\
&\leq C(\|\Delta z\|_2 + \|wy(u)\|_2). \qquad (40)
\end{aligned}
$$

Hence, we can have the following from (40):

$$
\begin{aligned}
\|z'\|_{L^2(Q)} &\leq C(\|\Delta z\|_{L^2(Q)} + \|wy(u)\|_{L^2(Q)}) \\
&\leq \text{(with (38))} \\
&\leq C\|wy(u)\|_{L^2(Q)}. \qquad (41)
\end{aligned}
$$

Therefore, from (38) and (41), we know that $z \in W(0,T)$, and the solution $z(= z(w))$ of Eq.(32) satisfies

$$
\begin{aligned}
\|z(w)\|_{W(0,T)} &\leq C\|wy(u)\|_{L^2(Q)} \\
&\leq C\|w\|_{\mathcal{F}}\|y(u)\|_{L^2(Q)} \\
&\leq C\|y(u)\|_{W(0,T)}\|w\|_{\mathcal{F}} \\
&\leq C\|(y_0, f, u)\|_{\mathcal{P}}\|w\|_{\mathcal{F}}. \qquad (42)
\end{aligned}
$$

Hence, from Eq.(32) and (42), the mapping $w \in \mathcal{F} \mapsto z(w) \in W(0,T)$ is linear and bounded. From this, we can infer that there exists $T \in \mathcal{L}(\mathcal{F}, W(0,T))$ such that $Tw = z(w)$ for each $w \in \mathcal{F}$.

(ii) We set the difference $\delta = y(u+w) - y(u) - z$. Based on the following

$$\begin{aligned} & y^3(u+w) - y^3(u) - 3y^2(u)z \\ = \;& \delta(y^2(u+w) + y(u+w)y(u) + y^2(u)) \\ & + z(y^2(u+w) + y(u+w)y(u) - 2y^2(u)) \\ = \;& \delta(y^2(u+w) + y(u+w)y(u) + y^2(u)) \\ & + z(y(u+w) - y(u))(y(u+w) + 2y(u)), \end{aligned} \tag{43}$$

we know from (43) that δ satisfies

$$\begin{cases} (1-\nu)\delta' + \Delta\Big(\Delta\delta - \delta\mathcal{G}(y(u+w), y(u)) + \delta - \nu\delta'\Big) \\ \qquad\qquad = (u+w)\delta + wz + \Delta\Big(z\mathcal{H}(y(u+w), y(u))\Big) \quad \text{in } \; Q, \\ \delta = \Delta\delta = 0 \quad \text{on } \; \Sigma, \\ \delta(0,x) = 0 \quad \text{in } \; \Omega \end{cases} \tag{44}$$

in the weak sense, where

$$\begin{aligned} \mathcal{G}(y(u+w), y(u)) &= y^2(u+w) + y(u+w)y(u) + y^2(u), \\ \mathcal{H}(y(u+w), y(u)) &= (y(u+w) - y(u))(y(u+w) + 2y(u)). \end{aligned}$$

If we follow similar arguments as in the proof of Theorem 2.1 or (i), then we can arrive at

$$\|\delta\|_{W(0,T)} \leq C\|wz + \Delta(z\mathcal{H}(y(u+w), y(u)))\|_{L^2(0,T;V')}. \tag{45}$$

From Theorem 2.1 and (42), we can deduce the following:

$$\begin{aligned} \|wz\|_{L^2(Q)} &\leq \|w\|_{\mathcal{F}}\|z\|_{L^2(Q)} \\ &\leq C\|w\|_{\mathcal{F}}\|z\|_{W(0,T)} \\ &\leq (\text{with } (42)) \\ &\leq C\|w\|^2_{\mathcal{F}}; \end{aligned} \tag{46}$$

$$\begin{aligned} & \|z\mathcal{H}(y(u+w), y(u))\|^2_{L^2(Q)} \\ \leq \;& \int_0^T \|z\|^2_{L^6(\Omega)}\|y(u+w) - y(u)\|^2_{L^6(\Omega)}\|y(u+w) + 2y(u)\|^2_{L^6(\Omega)}dt \\ \leq \;& \|z\|^2_{C([0,T];L^6(\Omega))}\|y(u+w) - y(u)\|^2_{C([0,T];L^6(\Omega))} \\ & \times\|y(u+w) + 2y(u)\|^2_{L^2(0,T;L^6(\Omega))} \\ \leq \;& (\text{with } (7)) \\ \leq \;& C\|z\|^2_{C([0,T];H^1_0(\Omega))}\|y(u+w) - y(u)\|^2_{C([0,T];H^1_0(\Omega))} \\ & \times\|y(u+w) + 2y(u)\|^2_{L^2(0,T;H^1_0(\Omega))} \\ \leq \;& (\text{with } (11) \text{ and Theorem 2.1}) \\ \leq \;& C\|z\|^2_{W(0,T)}\|u + w - u\|^2_{\mathcal{F}} \\ & \times(\|(y_0, u+w, f)\|^2_{\mathcal{P}} + \|(y_0, u, f)\|^2_{\mathcal{P}}) \\ \leq \;& (\text{with } (42)) \\ \leq \;& C\|w\|^4_{\mathcal{F}}. \end{aligned} \tag{47}$$

Hence, from (45) to (47), we can obtain

$$\begin{aligned}
\|\delta\|_{W(0,T)} &\le C\|wz+\Delta(z\mathcal{H}(y(u+w),y(u)))\|_{L^2(0,T;V')}\\
&\le C(\|wz\|_{L^2(0,T;V')}+\|\Delta(z\mathcal{H}(y(u+w),y(u)))\|_{L^2(0,T;V')})\\
&\le \text{(with (6))}\\
&\le C(\|wz\|_{L^2(Q)}+\|z\mathcal{H}(y(u+w),y(u))\|_{L^2(Q)})\\
&\le \text{(with (46) and (47))}\\
&\le C\|w\|^2_{\mathcal{F}}, \qquad (48)
\end{aligned}$$

which implies that $\|\delta\|_{W(0,T)}=o(\|w\|_{\mathcal{F}})$ as $\|w\|_{\mathcal{F}}\to 0$.

This completes the proof.

4. Quadratic cost bilinear optimal control problems

In this section, we study the quadratic cost bilinear optimal control problems for a viscous Cahn-Hilliard equation. Let the following be the set of the admissible controls:

$$\mathcal{U}_{ad}=\{u\in\mathcal{F}|\ a\ \le\ u\ \le b \text{ a.e. in } Q\}. \qquad (49)$$

To perform our variational analysis, $L^2(Q)$ norms of $\mathcal{U}_{ad}$ is preferable, even though $\mathcal{U}_{ad}$ is subset of $\mathcal{F}$. Using Theorem 2.1, we can uniquely define the solution mapping $\mathcal{U}_{ad}\ \to\ W(0,T)$, which maps the term $v\in\mathcal{U}_{ad}$ to the solution $y(v)\in W(0,T)$, which satisfies the following equation:

$$\begin{cases}
(1-\nu)y'(v)+\Delta\Big(\Delta y(v)-y^3(v)+y(v)-\nu y'(v)\Big)\\
\qquad\qquad\qquad\qquad\qquad = vy(v)+f \quad \text{in} \quad Q,\\
y(v)=\Delta y(v)=0 \qquad \text{on } \Sigma,\\
y(v;0,x)=y_0(x) \quad \text{in } \Omega,
\end{cases} \qquad (50)$$

where $y_0\in H^1_0(\Omega),\ f\in L^2(0,T;V')$ and v is a control variable.

The quadratic cost function associated with the control system (50) is:

$$J(v)\ =\ \frac{1}{2}\|y(v)-Y_d\|^2_{L^2(Q)}+\frac{\alpha}{2}\|v\|^2_{L^2(Q)}, \qquad (51)$$

where $Y_d\in L^2(Q)$ is a desired value, and the positive constant α is the weight of the second term on the right hand side of (51). As mentioned, the bilinear optimal control problem can be summarized as follows:

- Verify the existence of an admissible control $u\in\mathcal{U}_{ad}$ such that

$$\inf\{J(v)|v\in\mathcal{U}_{ad}\}=J(u). \qquad (52)$$

- Give a characterization of such a u (optimality condition).

Such a u in (52) is called an optimal control for the problem with the cost function (51).

4.1. **Existence of optimal controls.**

To prove the existence of optimal controls, we need the following compactness lemma.

Lemma 4.1. *(Aubin-Lions) Let $X\hookrightarrow Y\hookrightarrow Z$ be Banach spaces with X and Z being reflexive. If the imbedding $X\hookrightarrow Y$ is compact, then for any $1<p,q<\infty$, every bounded set of $L^p(0,T;X)\cap W^{1,q}(0,T;Z)$ is relatively compact in $L^p(0,T;Y)$.*

Proof. See Teman [20; p. 271].

The existence of optimal controls for the cost (51) can be stated by the following theorem.

Theorem 4.2. *Assume that the hypotheses of Theorem 2.1 are satisfied. Then there exists at least one optimal control u for the control problem (50) with the cost (51).*

Proof. Set $J = \inf_{v\in\mathcal{U}_{ad}} J(v)$. Since $\mathcal{U}_{ad}$ is non-empty, there is a sequence $\{u_n\}$ in $\mathcal{U}_{ad}$ such that

$$\inf_{v\in\mathcal{U}_{ad}} J(v) = \lim_{n\to\infty} J(u_n) = J.$$

Since $\{u_n\}$ is bounded in $\mathcal{F}$ and $\mathcal{U}_{ad}$ is bounded, closed, and convex, we can choose a subsequence (denote again by $\{u_n\}$) of $\{u_n\}$ and find a $u \in \mathcal{U}_{ad}$ such that

$$u_n \rightharpoonup u \quad \text{weakly - star in} \quad \mathcal{U}_{ad} \tag{53}$$

as $n \to \infty$. From now on, each state $y_n = y(u_n) \in W(0,T)$ corresponding to v_n is the solution of

$$\begin{cases} (1-\nu)y_n' + \Delta\Big(\Delta y_n - y_n^3 + y_n - \nu y_n'\Big) = u_n y_n + f \quad \text{in} \quad Q, \\ y_n = \Delta y_n = 0 \quad \text{on} \ \Sigma, \\ y_n(0) = y_0 \quad \text{in} \ \Omega. \end{cases} \tag{54}$$

It follows from Theorem 2.1 that

$$\begin{aligned} \|y_n\|_{W(0,T)} &\leq C(\|\nabla y_0\|_2 + \|f\|_{L^2(0,T;V')} + \|u_n\|_{\mathcal{F}}) \\ &\leq C(\|\nabla y_0\|_2 + \|f\|_{L^2(0,T;V')} + |a| \vee |b|). \end{aligned} \tag{55}$$

Therefore, by the extraction theorem of Rellich's, we can find a subsequence of $\{y_n\}$ denoted again by $\{y_n\}$ and find a $y \in W(0,T)$ such that

$$y_n \rightharpoonup y \ \text{weakly in} \ W(0,T) \quad \text{as } n \to \infty. \tag{56}$$

Since $V \hookrightarrow H_0^1(\Omega)$ is compact, we can use Lemma 4.1, in which $p = q = 2$, $X = V$, $Y = H_0^1(\Omega)$ and $Z = L^2(\Omega)$ to see from (55) that $\{y_n\}$ is precompact in $L^2(0,T;H_0^1(\Omega))$. Hence, there exist a subsequence $\{y_{n_k}\} \subset \{y_n\}$ such that

$$y_{n_k} \to y \quad \text{strongly in} \ L^2(0,T;H_0^1(\Omega)) \ \text{as } k \to \infty. \tag{57}$$

From (53) and (57), we can also extract a subsequence, if necessary, denoted again by v_n such that

$$u_n y_n \rightharpoonup uy \quad \text{weakly in} \quad L^2(Q) \quad \text{as } n \to \infty. \tag{58}$$

Let y be the weak limit in (56). We shall show that there exist a subsequence $\{y_{n_k}\} \subset \{y_n\}$ such that

$$\Delta y_{n_k}^3 \to \Delta y^3 \quad \text{strongly in} \ L^2(0,T;V') \ \text{as } k \to \infty : \tag{59}$$

For any given $\phi \in L^2(0,T;V)$, we can deduce

$$
\begin{aligned}
& \left|\int_0^T \langle \Delta(y_n^3 - y^3), \phi\rangle_{V',V} dt\right| \\
= & \left|\int_0^T (y_n^3 - y^3, \Delta\phi)_2 dt\right| \\
= & \left|\int_0^T \Big((y_n - y)(y_n^2 + y_n y + y^2), \Delta\phi\Big)_2 dt\right| \\
\le & \|(y_n - y)(y_n^2 + y_n y + y^2)\|_{L^2(Q)} \|\phi\|_{L^2(0,T;V)} \\
\le & \text{(with (17))} \\
\le & C(\|y_n\|_{W(0,T)} + \|y\|_{W(0,T)})^2 \|\nabla(y_n - y)\|_{L^2(Q)} \|\phi\|_{L^2(0,T;V)} \\
\le & \text{(with (55) and (56))} \\
\le & C\|y_n - y\|_{L^2(0,T;H_0^1(\Omega))} \|\phi\|_{L^2(0,T;V)} \qquad (60)
\end{aligned}
$$

by which we can get

$$\|\Delta(y_n^3 - y^3)\|_{L^2(0,T;V')} \le C\|y_n - y\|_{L^2(0,T;H_0^1(\Omega))}. \tag{61}$$

Thus, we can verify (59) from (57) and (61).

With (56), (58) and (59), we replace y_n in Eq.(54) by y_{n_k}, if necessary, and take the limit as $k \to \infty$. Then, by the standard argument in Dautray and Lions [7, pp.515-517], we can know that the limit $y \in W(0,T)$ is a weak solution of

$$
\begin{cases}
(1-\nu)y' + \Delta\Big(\Delta y - y^3 + y - \nu y'\Big) = uy + f \quad \text{in} \quad Q, \\
y = \Delta y = 0 \quad \text{on } \Sigma, \\
y(0) = y_0 \quad \text{in } \Omega.
\end{cases} \tag{62}
$$

Also since the equation (62) has a unique weak solution $y \in W(0,T)$ by Theorem 2.1, we conclude that $y = y(u)$ in $W(0,T)$ by the uniqueness of solutions, which implies

$$y(u_n) \rightharpoonup y(u) \quad \text{weakly in } W(0,T) \text{ as } n \to \infty. \tag{63}$$

By using a similar argument in (57), we can apply Lemma 4.1 to (63) to obtain the following

$$y(u_n) \to y(u) \text{ strongly in } L^2(0,T;H_0^1(\Omega)) \text{ as } n \to \infty. \tag{64}$$

Since $\|\cdot\|_{L^2(Q)}$ is weakly lower semi continuous, it follows from (53) that

$$\liminf_{n\to\infty} \|u_n\|_{L^2(Q)} \ge \|u\|_{L^2(Q)}. \tag{65}$$

Hence, from (64) and (65), we can arrive at

$$J = \liminf_{n\to\infty} J(u_n) \ge J(u).$$

But since $J(u) \ge J$ by definition, we conclude that $J(u) = \inf_{v\in\mathcal{U}_{ad}} J(v)$. This completes the proof.

4.2. **Necessary condition of bilinear optimal controls.**

We now derive necessary conditions that any optimal control must satisfy. To derive these necessary conditions, we must differentiate the cost function (51). The solution map $v \to y(v)$ of $\mathcal{U}_{ad}$ into $W(0,T)$ is said to be Gâteaux differentiable at $v = u$ if for any $w \in \mathcal{F}$ that satisfies $u + \lambda w \in \mathcal{U}_{ad}$, there exists a $Dy(u) \in \mathcal{L}(\mathcal{U}, W(0,T))$ such that

$$\left\| \frac{1}{\lambda}(y(u+\lambda w) - y(u)) - Dy(u)w \right\|_{W(0,T)} \to 0 \quad \text{as} \quad \lambda \to 0.$$

The operator $Dy(u)$ denotes the Gâteaux derivative of $y(u)$ at $v = u$ and the function $Dy(u)w \in W(0,T)$ is called the Gâteaux derivative in the direction $w \in \mathcal{F}$.

Now we formulate the following adjoint equation to describe the necessary optimality conditions for the observation given in the cost (51):

$$(66) \qquad \begin{cases} -(1-\nu)p' + \Delta\Big(\Delta p + p + \nu p'\Big) - 3y^2(u)\Delta p \\ \qquad\qquad\qquad\qquad = up + y(u) - Y_d \quad \text{in} \quad Q, \\ p = \Delta p = 0 \quad \text{on} \quad \Sigma, \\ p(T,x) = 0 \quad \text{in} \quad \Omega. \end{cases}$$

Proposition 4.3. *Eq.(66) has a unique weak solution $p \in W(0,T)$.*

Proof. The time reversed equation of Eq.(66) ($t \to T - t$ in Eq.(66)) is given by

$$(67) \qquad \begin{cases} (1-\nu)\tilde{p}' + \Delta\Big(\Delta\tilde{p} + \tilde{p} - \nu\tilde{p}'\Big) - 3\tilde{y}^2(\tilde{u})\Delta\tilde{p} \\ \qquad\qquad\qquad\qquad = \tilde{u}\tilde{p} + \tilde{y}(\tilde{u}) - \tilde{Y}_d \quad \text{in} \quad Q, \\ \tilde{p} = \Delta\tilde{p} = 0 \quad \text{on} \quad \Sigma, \\ \tilde{p}(0,x) = 0 \quad \text{in} \quad \Omega, \end{cases}$$

where $\tilde{\psi}(\cdot) = \psi(T - \cdot)$. By multiplying both sides of Eq.(67) by $2\tilde{p}$, we have

$$\begin{aligned} &\frac{d}{dt}\Big((1-\nu)\|\tilde{p}\|_2^2 + \nu\|\nabla\tilde{p}\|_2^2\Big) + 2\|\Delta\tilde{p}\|_2^2 \\ (68) \qquad &= 2\|\nabla\tilde{p}\|_2^2 + (6\tilde{y}^2(\tilde{u})\Delta\tilde{p}, \tilde{p})_2 + 2(\tilde{u}\tilde{p}, \tilde{p})_2 + 2(\tilde{y}(\tilde{u}) - \tilde{Y}_d, \tilde{p})_2. \end{aligned}$$

We can deduce:

$$\begin{aligned} |(6\tilde{y}^2(\tilde{u})\Delta\tilde{p}, \tilde{p})_2| &\le 2\|\Delta\tilde{p}\|_2\|3\tilde{y}^2(\tilde{u})\tilde{p}\|_2 \\ &\le \|\Delta\tilde{p}\|_2^2 + \|3\tilde{y}^2(\tilde{u})\tilde{p}\|_2^2 \\ &\le \|\Delta\tilde{p}\|_2^2 + 9\|\tilde{y}(\tilde{u})\|_{L^6(\Omega)}^4\|\tilde{p}\|_{L^6(\Omega)}^2 \\ &\le \text{(with (7))} \\ &\le \|\Delta\tilde{p}\|_2^2 + C\|\nabla\tilde{y}(\tilde{u})\|_2^4\|\nabla\tilde{p}\|_2^2 \\ &\le \|\Delta\tilde{p}\|_2^2 + C\|\tilde{y}(\tilde{u})\|_{C([0,T];H_0^1(\Omega))}^4\|\nabla\tilde{p}\|_2^2 \\ &\le \text{(by (11) and } \tilde{y}(\tilde{u}) \in W(0,T)) \\ (69) \qquad &\le \|\Delta\tilde{p}\|_2^2 + C\|\nabla\tilde{p}\|_2^2. \end{aligned}$$

After routine estimations of the other terms of the right hand side of (68), we can obtain the following from (69):

$$\begin{aligned} & \frac{d}{dt}\Big((1-\nu)\|\tilde{p}\|_2^2+\nu\|\nabla\tilde{p}\|_2^2\Big)+\|\Delta\tilde{p}\|_2^2 \\ \text{(70)} \qquad \leq \ & C(\|\tilde{p}\|_2^2+\|\nabla\tilde{p}\|_2^2+\|\tilde{y}(\tilde{u})-\tilde{Y}_d\|_2^2). \end{aligned}$$

By integrating (70) over $[0,t]$ and applying the Bellmann Gronwall's inequality to the integrated inequality, we have

$$\tilde{p}\in L^\infty(0,T;H_0^1(\Omega))\cap L^2(0,T;V). \tag{71}$$

Using similar arguments as in the proof of Theorem 2.1, we can obtain the following from Eq.(67) and (71):

$$\tilde{p}\in W(0,T). \tag{72}$$

This is sufficient for the existence result that is obtained by implementing a Faedo-Galerkin method and passing the limit.

This completes the proof.

We now discuss the first-order optimality conditions for the optimal control problem (50) with the cost (51).

Theorem 4.4. *Let $u\in\mathcal{U}_{ad}$ be an optimal control for the cost (51) and let $y(u)\in W(0,T)$ be the corresponding state solution of Eq.(50). There exists a weak solution $p\in W(0,T)$ of Eq.(66) such that u satisfies:*

$$u=\max\Big\{a,\min\Big\{-\frac{y(u)p}{\alpha},b\Big\}\Big\}.$$

Proof. Let $u\in\mathcal{U}_{ad}$ be an optimal control with the cost (51) and let $y(u)$ be the corresponding weak solution of Eq.(50). From Theorem 3.1, the map $v\to y(v)$ is Fréchet differentiable, so the map $v\to y(v)$ is also Gâteaux differentiable at $v=u$ in the (possible) direction $w\in\mathcal{F}$. Let $u+\epsilon w$ be another control in $\mathcal{U}_{ad}$ and let $y(u+\epsilon w)$ be the corresponding solution to Eq.(50). Then since the cost (51) achieves its minimum at u, we have:

$$\begin{aligned} 0 \ &\leq \ DJ(u)w \\ &= \ \lim_{\epsilon\to0^+}\frac{J(u+\epsilon w)-J(u)}{\epsilon} \\ &= \ \lim_{\epsilon\to0^+}\frac{1}{2}\int_0^T\Big(y(u+\epsilon w)+y(u)-2Y_d,\ \frac{y(u+\epsilon w)-y(u)}{\epsilon}\Big)_2dt \\ &\quad +\lim_{\epsilon\to0^+}\frac{\alpha}{2}\int_0^T\Big(2(u,w)_2+\epsilon\|w\|_2^2\Big)dt \\ \text{(73)} \qquad &= \ \int_0^T(y(u)-Y_d,z)_2dt+\ \alpha\int_0^T(u,w)_2dt, \end{aligned}$$

where z is the weak solution of Eq.(32). We multiply both sides of the weak form of Eq.(66) by z, which is a solution of Eq.(32), and integrate it over

$[0,T]$. Then, we have

$$
\begin{aligned}
&-\int_0^T ((1-\nu)p', z)_2 dt + \int_0^T \Big\langle \Delta\Big(\Delta p + p + \nu p'\Big), z\Big\rangle_{V',V} dt \\
(74)\qquad &-\nu \int_0^T (3y^2(u)\Delta p, z)_2 dt = \int_0^T (up + y(u) - Y_d, z)_2 dt.
\end{aligned}
$$

By integration by parts and the terminal value of the weak solution p of Eq.(66), (74) can be rewritten as

$$
\begin{aligned}
&\int_0^T \Big\langle p,\ (1-\nu)z' + \Delta\Big(\Delta z - 3y^2(u)z + z - \nu z'\Big) - uz\Big\rangle_{V,V'} dt \\
(75)\qquad = &\int_0^T (y(u) - Y_d, z)_2 dt.
\end{aligned}
$$

Since z is the solution of Eq.(32), we can obtain the following from (75):

$$
(76)\qquad \int_0^T (y(u) - Y_d, z)_2 dt = \int_0^T (p, wy(u))_2 dt.
$$

Therefore, we can deduce that (73) and (76) imply

$$
(77)\qquad DJ(u)w = \int_0^T (p, wy(u))_2 dt + \alpha \int_0^T (u, w)_2 dt.
$$

Hence, we can obtain the following from (73) and (77):

$$
(78)\qquad \int_0^T (\alpha u + y(u)p, w)_2 dt \geq 0, \quad w \in \mathcal{F}.
$$

By considering the sign of the variations w in (78), which depend on u, we can deduce the following from (78) (possibly not unique):

$$
(79)\qquad u = \max\Big\{a, \min\Big\{ -\frac{y(u)p}{\alpha}, b\Big\}\Big\}.
$$

This completes the proof.

4.3. **Local uniqueness of an optimal control.**

We note that the uniqueness of an optimal control in nonlinear equation is not assured. However, it is worthwhile to note the partial results. For instance, we can refer to the results in [4, 18] to obtain the local uniqueness of an optimal control. To verify the local uniqueness of an optimal control, we can refer to the strict convexity arguments of a quadratic cost function (cf. [22]). To this end, we consider the following result.

Proposition 4.5. *The map $v \to y(v)$ of $\mathcal{F}$ into $W(0,T)$ is second-order Fréchet differentiable at $v = u$, and such the second order Fréchet derivative of $y(v)$ at $v = u$ in the direction $w \in \mathcal{F}$, say $\phi = D^2y(u)(w,w)$, is a unique* *solution of the following problem*

$$(80)\quad \begin{cases} (1-\nu)\phi' + \Delta\Big(\Delta\phi - 3y^2(u)\phi - 6y(u)z^2 + \phi - \nu\phi'\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad = u\phi + 2wz \quad \text{in} \quad Q, \\ \phi = \Delta\phi = 0 \quad \text{on} \quad \Sigma, \\ \phi(0,x) = 0 \quad \text{in} \quad \Omega, \end{cases}$$

where z is the solution of Eq.(32).

To prove Proposition 4.5, it is sufficient to show the following

$$(81)\qquad \|Dy(u+w)w - Dy(u)w - \phi\|_{W(0,T)} = o(\|w\|_{\mathcal{F}}) \;\; \text{as} \;\; \|w\|_{\mathcal{F}} \to 0.$$

First, we prove the following lemma.

Lemma 4.6. *For the weak solution ϕ of Eq.(80), the following holds*

$$(82)\qquad \|\phi\|_{W(0,T)} \leq C\|w\|_{\mathcal{F}}^2.$$

Proof. By similar arguments in the proof of (i) of Theorem 3.1, we can show that

$$\begin{aligned} \|\phi\|_{W(0,T)} &\leq C\|\Delta(6y(u)z^2) + 2wz\|_{L^2(0,T;V')} \\ &\leq C(\|\Delta(6y(u)z^2)\|_{L^2(0,T;V')} + \|2wz\|_{L^2(0,T;V')}) \\ &\leq (\text{with } (6)) \\ (83)\qquad &\leq C(\|y(u)z^2\|_{L^2(Q)} + \|wz\|_{L^2(Q)}). \end{aligned}$$

We note the following:

$$\begin{aligned} \|y(u)z^2\|_{L^2(Q)}^2 &\leq \int_0^T \|y(u)\|_{L^6(\Omega)}^2 \|z\|_{L^6(\Omega)}^4 dt \\ &\leq (\text{with } (7)) \\ &\leq C\int_0^T \|y(u)\|_{H_0^1(\Omega)}^2 \|z\|_{H_0^1(\Omega)}^4 dt \\ &\leq CT\|y(u)\|_{C([0,T];H_0^1(\Omega))}^2 \|z\|_{C([0,T];H_0^1(\Omega))}^4 \\ &\leq (\text{by } (11) \text{ and } y(u) \in W(0,T)) \\ &\leq C\|z\|_{W(0,T)}^4 \\ &\leq (\text{with } (42)) \\ (84)\qquad &\leq C\|w\|_{\mathcal{F}}^4. \end{aligned}$$

Hence, from (46), (83) and (84), we can have (82).
This completes the proof.

Proof of Proposition 4.5. From Eq.(32), we can deduce that $z_{u+w} = Dy(u+w)w$ is the weak solution of the following equation:

$$(85)\quad \begin{cases} (1-\nu)z'_{u+w} + \Delta\Big(\Delta z_{u+w} - 3y^2(u+w)z_{u+w} + z_{u+w} - \nu z'_{u+w}\Big) \\ \qquad\qquad\qquad\qquad = (u+w)z_{u+w} + wy(u+w) \quad \text{in} \;\; Q, \\ z_{u+w} = \Delta z_{u+w} = 0 \quad \text{on} \;\; \Sigma, \\ z_{u+w}(0,x) = 0, \quad \text{in} \;\; \Omega. \end{cases}$$

From (42), we can verify the following

$$\|z_{u+w}\|_{W(0,T)} \le C\|w\|_{\mathcal{F}}. \tag{86}$$

From Eq.(32), Eq.(80) and Eq.(85), $\delta = z_{u+w} - z - \phi$ satisfies the following equation

$$\begin{cases} (1-\nu)\delta' + \Delta\Big(\Delta\delta - 3y^2(u)\delta + \delta - \nu\delta'\Big) \\ \qquad\qquad\qquad\qquad = (u+w)\delta + I_1 + I_2 \quad \text{in } \; Q, \\ \delta = \Delta\delta = 0 \qquad \text{on } \; \Sigma, \\ \delta(0,x) = 0, \qquad \text{in } \; \Omega \end{cases} \tag{87}$$

in the weak sense, where

$$\begin{aligned} I_1 &= w\phi + w(y(u+w) - y(u) - z), \\ I_2 &= \Delta\Big((3y^2(u+w) - 3y^2(u))z_{u+w} - 6y(u)z^2\Big). \end{aligned}$$

By similar arguments to those in the proof of Theorem 2.1, we can deduce

$$\begin{aligned} \|\delta\|_{W(0,T)} &\le C\|I_1 + I_2\|_{L^2(0,T;V')} \\ &\le C(\|I_1\|_{L^2(0,T;V')} + \|I_2\|_{L^2(0,T;V')}) \\ &\le (\text{with } (6)) \\ &\le C\Big(\|w\phi + w(y(u+w) - y(u) - z)\|_{L^2(Q)} \\ &\qquad + \|(3y^2(u+w) - 3y^2(u))z_{u+w} - 6y(u)z^2\|_{L^2(Q)}\Big). \end{aligned} \tag{88}$$

We can then verify the following:

$$\begin{aligned} \|w\phi\|_{L^2(Q)} &\le \|w\|_{\mathcal{F}}\|\phi\|_{L^2(Q)} \\ &\le \|w\|_{\mathcal{F}}\|\phi\|_{W(0,T)} \\ &\le (\text{with } (82)) \\ &\le C\|w\|^3_{\mathcal{F}}; \end{aligned} \tag{89}$$

$$\begin{aligned} &\|w(y(u+w) - y(u) - z)\|_{L^2(Q)} \\ &\le \|w\|_{\mathcal{F}}\|y(u+w) - y(u) - z\|_{L^2(Q)} \\ &\le \|w\|_{\mathcal{F}}\|y(u+w) - y(u) - z\|_{W(0,T)} \\ &\le (\text{with } (48)) \\ &\le C\|w\|^3_{\mathcal{F}}; \end{aligned} \tag{90}$$

$$\begin{aligned} &\|(3y^2(u+w) - 3y^2(u))z_{u+w}\|_{L^2(Q)} \\ &\le (\text{by similar arguments in } (47)) \\ &\le C\|w\|_{\mathcal{F}}\|z_{u+w}\|_{W(0,T)} \\ &\le (\text{with } (86)) \\ &\le C\|w\|^2_{\mathcal{F}}; \end{aligned} \tag{91}$$

$$\begin{aligned} \|6y(u)z^2\|_{L^2(Q)} &\le (\text{with } (84)) \\ &\le C\|w\|^2_{\mathcal{F}}. \end{aligned} \tag{92}$$

From (88) to (92), we can deduce

$$\|\delta\|_{W(0,T)} \le C(\|w\|^3_{\mathcal{F}} + \|w\|^2_{\mathcal{F}}) \tag{93}$$

which implies

$$\|\delta\|_{W(0,T)} = o(\|w\|_{\mathcal{F}}) \quad \text{as} \quad \|w\|_{\mathcal{F}} \to 0. \tag{94}$$

This completes the proof.

Lemma 4.7. *When $n = 1$, for the weak solution ϕ of Eq.(80), the following holds*

$$\|\phi\|_{W(0,T)} \le C\|w\|^2_{L^2(Q)}. \tag{95}$$

Proof. Considering $n = 1$, let $z(w)$ be the weak solution of Eq.(32). Then, we can obtain the following from (42):

$$\begin{aligned} \|z(w)\|_{W(0,T)} &\le C\|wy(u)\|_{L^2(Q)} \\ &\le C\|y(u)\|_{C^0(\bar{Q})}\|w\|_{L^2(Q)} \\ &\le (\text{since } n = 1, \text{ by (8) and (11)}) \\ &\le C\|y(u)\|_{W(0,T)}\|w\|_{L^2(Q)} \\ &\le C\|w\|_{L^2(Q)}. \end{aligned} \tag{96}$$

Let ϕ be the weak solution of Eq.(80). From (83), we can deduce with (96) the following:

$$\begin{aligned} \|\phi\|_{W(0,T)} &\le C(\|y(u)z^2\|_{L^2(Q)} + \|wz\|_{L^2(Q)}) \\ &\le C(\|y(u)\|_{L^2(Q)}\|z\|^2_{C^0(\bar{Q})} + \|w\|_{L^2(Q)}\|z\|_{C^0(\bar{Q})}) \\ &\le (\text{by (8), (11) and } W(0,T) \hookrightarrow L^2(Q)) \\ &\le C(\|y(u)\|_{W(0,T)}\|z\|^2_{W(0,T)} + \|w\|_{L^2(Q)}\|z\|_{W(0,T)}) \\ &\le (\text{with (96)}) \\ &\le C\|w\|^2_{L^2(Q)}. \end{aligned} \tag{97}$$

This completes the proof.

We prove the following partial results on the local uniqueness of the optimal control.

Theorem 4.8. *When $n = 1$ and T is small enough, then there is a unique optimal control for the cost (51).*

Proof. We show the local uniqueness by proving the strict convexity of the map $v \in \mathcal{U}_{ad} \to J(v)$. Therefore, as in [22], we need to show for all $u, v \in \mathcal{U}_{ad}$, $(u \ne v)$,

$$D^2J(u + \xi(v - u))(v - u, v - u) > 0, \qquad (0 < \xi < 1). \tag{98}$$

For simplicity, we denote $y(u+\xi(v-u))$, $z(u+\xi(v-u))$ and $\phi(u+\xi(v-u))$ by $y(\xi)$, $z(\xi)$ and $\phi(\xi)$, respectively. We calculate

$$\begin{aligned} & DJ(u + \xi(v - u))(v - u) \\ = {} & \lim_{l\to 0} \frac{J(u + (\xi + l)(v - u)) - J(u + \xi(v - u))}{l} \\ = {} & \int_0^T \big(y(\xi) - Y_d,\ z(\xi)\big)_2 ds + \alpha \int_0^T (u + \xi(v - u)), v - u)_2 dt. \end{aligned} \tag{99}$$

Since, from Proposition 4.5, the map $v \to y(v)$ is second-order Fréchet differentiable. Thus, the map $v \to y(v)$ is also the second order Gâteaux differentiable at $v = u+\xi(v-u)$ in the direction $v-u$. Therefore, from (99), we can obtain the second Gâteaux derivative of J as follows:

$$\begin{aligned} & D^2J(u+\xi(v-u))(v-u, v-u) \\ = & \lim_{\mu\to 0} \frac{DJ(u+(\xi+\mu)(v-u))(v-u) - DJ(u+\xi(v-u))(v-u)}{\mu} \\ = & \int_0^T \big(y(\xi) - Y_d,\ \phi(\xi)\big)_2 ds + \|z(\xi)\|^2_{L^2(Q)} + \alpha\|v-u\|^2_{L^2(Q)}, \end{aligned} \tag{100}$$

where $\phi(\xi)$ is the weak solution of Eq.(80), in which u and w are replaced by $u+\xi(v-u)$ and $v-u$, respectively. Then, by Lemma 4.7 and (100), we can deduce that

$$\begin{aligned} & D^2J(u+\xi(v-u))(v-u, v-u) \\ \geq & -\|\phi(\xi)\|_{C([0,T];L^2(\Omega))} \int_0^T \|y(\xi) - Y_d\|_2 ds \\ & + \|z(\xi)\|^2_{L^2(Q)} + \alpha\|v-u\|^2_{L^2(Q)} \\ \geq & -C\sqrt{T}\|\phi(\xi)\|_{W(0,T)}\|y(\xi) - Y_d\|_{L^2(Q)} \\ & + \|z(\xi)\|^2_{L^2(Q)} + \alpha\|v-u\|^2_{L^2(Q)} \\ \geq & \text{ (with (95))} \\ \geq & \Big(\alpha - C\sqrt{T}\|y(\xi) - Y_d\|_{L^2(Q)}\Big)\|v-u\|^2_{L^2(Q)} + \|z(\xi)\|^2_{L^2(Q)}, \end{aligned} \tag{101}$$

where we can take $T > 0$ to be small enough so that the right hand side of (101) is strictly greater than 0. Therefore we obtain the strict convexity of the quadratic cost $J(v),\ v \in \mathcal{U}_{ad}$, which prove this theorem.

Remark 4.9. As noted in [4], if one assumes that $\alpha > 0$ is sufficiently large instead of assuming T is sufficiently small, then one can also obtain the strict convexity of the cost function $J(\cdot)$ and the resulting uniqueness of the optimal control.

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