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SOME IDENTITIES OF THE DEGENERATE BERNOULLI POLYNOMIALS OF THE SECOND KIND ARISING FROM λ-SHEFFER SEQUENCES

JIN-WOO PARK[1], BYUNG MOON KIM[2], AND JONGKYUM KWON[3]

Abstract. Korobov introduced the first degenerate version of the Bernoulli polynomials of the second kind called Bernoulli polynomials of the second kind. Recently, degenerate versions of such polynomials as Bernoulli polynomials, Euler polynomials and Genocchi polynomials and so on were introduced by the many researchers. The aim of this paper is to represent the degenerate Bernoulli polynomials of the second kind by other polynomials using the λ-umbral calculus.

1. Introduction

The ordinary Bernoulli polynomials are defined by the generating function to be

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt}, \text{ (see [6, 19]).} \tag{1.1}$$

In the special case, $x=0$, $b_n(0)=b_n$ are called the *Bernoulli numbers*.

For any nonzero real number $\lambda \in \mathbb{R}$, the *degenerate exponential function* is defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \ e_\lambda(t) = (1+\lambda t)^{\frac{1}{\lambda}}, \text{ (see [4, 7, 12, 10]).} \tag{1.2}$$

Let $\log_\lambda(t)$ be the compositional inverse function of $e_\lambda(t)$ satisfying $\log_\lambda(e_\lambda(t)) = t$. Then we have

$$\log_\lambda(1+t) = \sum_{n=1}^{\infty} \lambda^{n-1}(1)_{n,\frac{1}{\lambda}}\frac{t^n}{n!}, \text{ (see [4, 7, 12]),} \tag{1.3}$$

where $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$.

By using (1.2), the higher-order degenerate Bernoulli polynomials are defined as follows:

$$\sum_{n=0}^{\infty} B_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} = \left(\frac{t}{e_\lambda(t)-1}\right)^r e_\lambda^x(t), \text{ (see [18]).} \tag{1.4}$$

When $x=0$, $B_{n,\lambda}^{(r)}(0) = B_{n,\lambda}^{(r)}$ are called the *higher-order degenerate Bernoulli numbers*. In addition, when $r=1$, we denote $B_{n,\lambda}^{(1)}(x) = B_{n,\lambda}(x)$.

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The *higher order Bernoulli polynomials of the second kind* are defined by the generating function to be

$$\sum_{n=0}^{\infty} b_n^{(r)}(x)\frac{t^n}{n!} = \left(\frac{t}{\log(1+t)}\right)^r (1+t)^x, \text{ (see [3, 4])}. \tag{1.5}$$

When $x = 0$, $b_n^{(r)}(0) = b_n^{(r)}$ are called the *higher order Bernoulli numbers of the second kind*. In addition, when $r = 1$, $b_n^{(1)}(x) = b_n(x)$ are called the *Bernoulli polynomials of the second kind*.

For $n \geq 0$, the *Stirling numbers of the first kind* $S_1(n,k)$ and *Stirling numbers of the second kind* $S_2(n,k)$, respectively, are given by

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \text{ (see [1-19])}, \tag{1.6}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$ is the falling factorial sequence.

For each positive integer k, it is well known (see [8]) that

$$\frac{1}{k!}\left(\log(1+t)\right)^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \text{ and } \frac{1}{k!}\left(e^t-1\right)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}. \tag{1.7}$$

As degenerate version of the Stirling numbers of the first and second kind in (1.6), the *degenerate Stirling numbers of the first kind* $S_{1,\lambda}(n,k)$ and the *degenerate Stirling numbers of the second kind* $S_{2,\lambda}(n,k)$ are respectively introduced by Kim-Kim (see [6, 8]) as follows:

$$\frac{1}{k!}\left(\log_\lambda(1+t)\right)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} \text{ and } \frac{1}{k!}\left(e_\lambda(t)-1\right)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}. \tag{1.8}$$

It was Gian-Carlo Rota who started to make a completely rigorous foundation for umbral calculus in the 1970s (see[18]). The umbral calculus is based on linear functionals, linear operators, and differential operators. Recently, Kim-Kim introduced degenerate Sheffer sequences and λ-Sheffer sequences. They defined the λ-linear functionals, λ-linear operators and λ-differential operators instead of the linear functionals, linear operator and differential operators used by Rota.

Carlitz introduced degenerate Stirling, Bernoulli and Eulerian numbers in 1979 (see [1]). Korobov introduced the first degenerate version of the Bernoulli polynomials of the second kind called Korobov polynomials of the first kind (see [15]). Recently, degenerate versions of such polynomials as Bernoulli polynomials, Euler polynomials and Genocchi polynomials and so on were introduced by the many researchers (see [2-21]). The aim of this paper is to represent the degenerate Bernoulli polynomials of the second kind by other polynomials using the λ-umbral calculus.

Let $\mathbb{C}$ be the field of complex numbers,

$$\mathcal{F} = \left\{ f(t) = \sum_{n=0}^{\infty} a_k \frac{t^n}{n!} \,\middle|\, a_k \in \mathbb{C} \right\},$$

and let

$$\mathbb{P} = \mathbb{C}[x] = \left\{ \sum_{k=0}^{\infty} a_l x^k \,\middle|\, a_k \in \mathbb{C} \text{ with } a_k = 0 \text{ for all but finite number of } k \right\}.$$

Let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$.

Then each $\lambda \in \mathbb{R}$ gives rise to the linear functional $\langle f(t)| \cdot \rangle_\lambda$ on $\mathbb{P}$, called λ-linear functional given by $f(t)$, which is defined by

$$\left\langle f(t) \middle| (x)_{n,\lambda} \right\rangle_\lambda = a_n, (n \geq 0), \tag{1.9}$$

and by linear extension (see [7]). From (1.9), we have

$$\left\langle t^k \middle| (x)_{n,\lambda} \right\rangle_\lambda = n!\delta_{n,k}, (n, k \geq 0), \tag{1.10}$$

where $\delta_{n,k}$ is Kronecker's symbol.

For each $\lambda \in \mathbb{R}$ and each $k \in \mathbb{N}$, Kim-Kim defined the differential operator on $\mathbb{P}$ in [7] by

$$\left(t^k\right)_\lambda (x)_{n,\lambda} = \begin{cases} (n)_k (x)_{n-k,\lambda}, & if \ k \leq n, \\ 0, & if \ k \geq n, \end{cases}$$

and for any $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$,

$$\left(f(t)\right)_\lambda (x)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k} a_k (x)_{n-k,\lambda}. \tag{1.11}$$

In addition, they showed that for $f(t), g(t) \in \mathcal{F}$, and $p(x) \in \mathbb{P}$,

$$\left\langle f(t)g(t) \middle| p(x) \right\rangle_\lambda = \left\langle g(t) \middle| (f(t))_\lambda p(x) \right\rangle_\lambda = \left\langle f(t) \middle| (g(t))_\lambda p(x) \right\rangle_\lambda. \tag{1.12}$$

The *order* $o(f(t))$ of $f(t) \in \mathcal{F} - \{0\}$ is the smallest integer k for which the coefficient of t^k does not vanish. If $o(f(t)) = 0$, then $f(t)$ is called *invertible* and such series has a multiplicative inverse $\frac{1}{f(t)}$ of $f(t)$. If $o(f(t)) = 1$, then $f(t)$ is called *delta series* and it has a compositional inverse $\bar{f}(t)$ of $f(t)$ with $\bar{f}(f(t)) = f(\bar{f}(t)) = t$.

Let $f(t)$ be a delta series and let $g(t)$ be an invertible series. Then there exists a unique sequence $S_{n,\lambda}(x)$ ($\deg S_{n,\lambda}(x) = n$) of polynomials satisfying the orthogonality conditions

$$\left\langle g(t)\big(f(t)\big)^k \ \middle| \ S_{n,\lambda}(x) \right\rangle_\lambda = n!\delta_{n,k}, \quad (n, k \geq 0), \quad \text{(see [7])}. \tag{1.13}$$

Here $S_{n,\lambda}(x)$ is called the λ-Sheffer sequence for $\big(g(t), f(t)\big)$, which is denoted by $S_{n,\lambda}(x) \sim \big(g(t), f(t)\big)_\lambda$. The sequence $S_{n,\lambda}(x)$ is the λ-Sheffer sequence for $(g(t), f(t))$ if and only if

$$\frac{1}{g(\overline{f}(t))} e_\lambda^y\big(\overline{f}(t)\big) = \sum_{n=0}^{\infty} S_{n,\lambda}(y) \frac{t^n}{n!}, \quad \text{(see [7])}, \tag{1.14}$$

for all $y \in \mathbb{C}$, where $\overline{f}(t)$ is the compositional inverse of $f(t)$ such that $f(\overline{f}(t)) = \overline{f}(f(t)) = t$.

For $S_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$, $r_{n,\lambda}(x) \sim (h(t), l(t))_\lambda$, we have

$$S_{n,\lambda}(x) = \sum_{k=0}^{n} C_{n,k} r_{k,\lambda}(x), \quad (n \geq 0),$$

where

$$C_{n,k} = \frac{1}{k!} \left\langle \frac{h(\overline{f}(t))}{g(\overline{f}(t))} \big(l(\overline{f}(t))\big)^k \ \middle| \ (x)_{n,\lambda} \right\rangle_\lambda, \quad \text{(see [7])}. \tag{1.15}$$

Let $S_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ and let $h(x) = \sum_{l=0}^{n} a_l S_{l,\lambda}(x) \in \mathbb{P}$. Then by (1.13), we have

$$\begin{aligned}\left\langle g(t)\left(f(t)\right)^k \middle| h(x)\right\rangle_\lambda =& \sum_{l=0}^{n} a_l \left\langle g(t)\left(f(t)\right)^k \middle| S_{l,\lambda}(x)\right\rangle_\lambda \\ =& k! a_k,\end{aligned}$$

and thus we know that

$$a_k = \frac{1}{k!}\left\langle g(t)\left(f(t)\right)^k \middle| h(x)\right\rangle_\lambda. \tag{1.16}$$

Let $(x)_n = \sum_{k=0}^{n} c_{n,k}(x)_{k,\lambda}$. Since

$$(x)_n \sim (1, e_\lambda(t) - 1)_\lambda \text{ and } (x)_{n,\lambda} \sim (1, t)_\lambda,$$

by (1.15), we get

$$\begin{aligned}c_{n,k} =& \frac{1}{k!}\left\langle (\log_\lambda(1+t))^k \middle| (x)_{n,\lambda}\right\rangle_\lambda = \sum_{l=k}^{\infty} S_{1,\lambda}(l,k)\frac{1}{l!}\left\langle t^l \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ =& S_{1,\lambda}(n,k),\end{aligned}$$

and thus, we know that

$$(x)_n = \sum_{k=0}^{n} S_{1,\lambda}(n,k)(x)_{k,\lambda}. \tag{1.17}$$

In the similar way, we also know that

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k)(x)_k.$$

2. The degenerate Bernoulli polynomials of the second kind arising from λ-Sheffer sequences

In this section, we find some relationships between Bernoulli polynomials of the second kind and some special polynomials arising from λ-Sheffer sequences.

The *higher-order degenerate Bernoulli polynomials of the second kind* are defined as follows:

$$\left(\frac{t}{\log_\lambda(1+t)}\right)^r e_\lambda^x\left(\log_\lambda(1+t)\right) = \sum_{n=0}^{\infty} b_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}, \text{ (see [3, 4, 13]).} \tag{2.1}$$

When $x = 0$, $b_{n,\lambda}^{(r)}(0) = b_{n,\lambda}^{(r)}$ are called *higher-order degenerate Bernoulli numbers of the second kind.* In the special case $r = 1$, $b_{n,\lambda}^{(1)}(x) = b_{n,\lambda}(x)$ are called the *Bernoulli polynomials of the second kind.* Note tat if $\lambda \to 0$, then $\lim_{\lambda\to 0} b_{n,\lambda}^{(r)}(x) = b_n^{(r)}(x)$.

Theorem 2.1. *For each nonnegative integer n, we have*

$$\begin{aligned}b_{n,\lambda}(x) =& \sum_{k=0}^{n}\left(\sum_{m=0}^{n}\binom{n}{m}S_{1,\lambda}(n-m,k)b_{m,\lambda}\right)(x)_{k,\lambda} \\ =& b_{n,\lambda} + \sum_{k=1}^{n}\frac{nS_{1,\lambda}(n-1,k-1)}{k}(x)_{k,\lambda}.\end{aligned} \tag{2.2}$$

As the inversion formula of (2.2), *we have*

$$(x)_{n,\lambda} = \sum_{k=0}^{n}\left(\sum_{l=k}^{n}\binom{n}{l}B_{n-l,\lambda}S_{2,\lambda}(l,k)\right)b_{k,\lambda}(x).$$

Proof. Let $b_{n,\lambda}(x) = \sum_{k=0}^{n} c_{n,k}(x)_{k,\lambda}$. Since

$$b_{n,\lambda}(x) \sim \left(\frac{t}{e_\lambda(t)-1}, e_\lambda(t)-1\right)_\lambda \text{ and } (x)_{n,\lambda} \sim (1,t)_\lambda,$$

by (1.15), we get

$$c_{n,0} = \left\langle \frac{t}{\log_\lambda(1+t)} \middle| (x)_{n,\lambda}\right\rangle = b_{n,\lambda},$$

and each $k \geq 1$,

$$\begin{aligned}
c_{n,k} =& \frac{1}{k!}\left\langle \left(\frac{t}{\log_\lambda(1+t)}\right)(\log_\lambda(1+t))^k \middle| (x)_{n,k}\right\rangle_\lambda \\
=& \frac{1}{k}\left\langle t\frac{1}{(k-1)!}(\log_\lambda(1+t))^{k-1} \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
=& \frac{1}{k}\sum_{l=k-1}^{\infty} S_{1,\lambda}(l,k-1)\frac{1}{l!}\left\langle t^{l+1} \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
=& \frac{n}{k}S_{1,\lambda}(n-1,k-1).
\end{aligned}$$

In the other way,

$$\begin{aligned}
c_{n,k} =& \frac{1}{k!}\left\langle \left(\frac{t}{\log_\lambda(1+t)}\right)(\log_\lambda(1+t))^k \middle| (x)_{n,k}\right\rangle_\lambda \\
=& \left\langle \frac{1}{k!}(\log_\lambda(1+t))^k \middle| \left(\frac{t}{\log_\lambda(1+t)}\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
=& \sum_{m=0}^{n} b_{m,\lambda}\binom{n}{m}\left\langle \sum_{l=k}^{\infty} S_{1,\lambda}(l,k)\frac{t^l}{l!} \middle| (x)_{n-m,\lambda}\right\rangle_\lambda \\
=& \sum_{m=0}^{n}\binom{n}{m}S_{1,\lambda}(n-m,k)b_{m,\lambda}.
\end{aligned}$$

Therefore, we proved the equation (2.2).

Conversely, we assume that $(x)_{n,\lambda} = \sum_{k=0}^{n} d_{n,k}b_{k,\lambda}$. Then

$$\begin{aligned}
d_{n,k} =& \frac{1}{k!}\left\langle \frac{t}{e_\lambda(t)-1}(e_\lambda(t)-1)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
=& \left\langle \frac{t}{e_\lambda(t)-1} \middle| \left(\frac{1}{k!}(e_\lambda(t)-1)^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
=& \sum_{l=k}^{n}\binom{n}{l}S_{2,\lambda}(l,k)\left\langle \sum_{m=0}^{\infty} B_{m,\lambda}\frac{t^m}{m!} \middle| (x)_{n-l,\lambda}\right\rangle_\lambda \\
=& \sum_{l=k}^{n}\binom{n}{l}S_{2,\lambda}(l,k)B_{n-l,\lambda}.
\end{aligned}$$

Thus, our proof is completed. □

Theorem 2.2. *For each $n \geq 0$, we have*

$$b_{n,\lambda}(x) = \sum_{k=0}^{n} \left(\sum_{l=k}^{n} \binom{n}{l} S_{1,\lambda}(l,k) b_{n-l,\lambda}^{(2)} \right) B_{k,\lambda}(x). \tag{2.3}$$

As the inversion formula of (2.3), *we have*

$$B_{n,\lambda}(x) = \sum_{k=0}^{n} \left(\sum_{l=k}^{n} \binom{n}{l} S_{2,\lambda}(l,k) B_{n-l,\lambda}^{(2)} \right) b_{k,\lambda}(x).$$

Proof. Let $b_{n,\lambda}(x) = \sum_{k=0}^{n} c_{n,k} B_{n,\lambda}(x)$. Note that

$$B_{n,\lambda}(x) \sim \left(\frac{e_\lambda(t)-1}{t}, t \right)_\lambda .$$

By (1.15), we get

$$\begin{aligned} c_{n,k} =& \frac{1}{k!} \left\langle \frac{\frac{t}{\log_\lambda(1+t)}}{\frac{\log_\lambda(1+t)}{t}} \left(\log_\lambda(1+t)\right)^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\ =& \left\langle \left(\frac{t}{\log_\lambda(1+t)} \right)^2 \middle| \left(\frac{1}{k!} \left(\log_\lambda(1+t)\right)^k \right)_\lambda (x)_{n,\lambda} \right\rangle_\lambda \\ =& \sum_{l=k}^{n} \binom{n}{l} S_{1,\lambda}(l,k) \left\langle \sum_{m=0}^{\infty} b_{m,\lambda}^{(2)} \frac{t^m}{m!} \middle| (x)_{n-l,\lambda} \right\rangle_\lambda \\ =& \sum_{l=k}^{n} \binom{n}{l} S_{1,\lambda}(l,k) b_{n-l,\lambda}^{(2)}. \end{aligned}$$

Conversely, let $B_{n,\lambda}(x) = \sum_{k=0}^{n} d_{n,k} b_{k,\lambda}(x)$. Then

$$\begin{aligned} d_{n,k} =& \frac{1}{k!} \left\langle \frac{\frac{t}{e_\lambda(t)-1}}{\frac{e_\lambda(t)-1}{t}} \left(e_\lambda(t)-1\right)^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\ =& \left\langle \left(\frac{t}{e_\lambda(t)-1} \right)^2 \middle| \left(\frac{1}{k!} \left(e_\lambda(t)-1\right)^k \right)_\lambda (x)_{n,\lambda} \right\rangle_\lambda \\ =& \sum_{l=k}^{n} \binom{n}{l} S_{2,\lambda}(l,k) \left\langle \sum_{m=0}^{\infty} B_{m,\lambda}^{(2)} \frac{t^m}{m!} \middle| (x)_{n-l,\lambda} \right\rangle_\lambda \\ =& \sum_{l=k}^{n} \binom{n}{l} S_{2,\lambda}(l,k) B_{n-l,\lambda}^{(2)}. \end{aligned}$$

Hence our proofs are completed. □

Note that by (2.1), we get

$$\begin{aligned}\sum_{n=0}^{\infty} b_{n,\lambda}(x)\frac{t^n}{n!} =&\frac{t}{\log_\lambda(1+t)}e_\lambda^x\left(\log_\lambda(1+t)\right)\\ =&\left(\sum_{n=0}^{\infty} b_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n} S_{1,\lambda}(n,k)(x)_{k,\lambda}\frac{t^n}{n!}\right)\\ =&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}S_{1,\lambda}(m,k)b_{n-m,\lambda}(x)_{k,\lambda}\right)\frac{t^n}{n!},\end{aligned}$$

and so we know that

$$b_{n,\lambda}(x) = \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}S_{1,\lambda}(m,k)b_{n-m,\lambda}(x)_{k,\lambda}. \tag{2.4}$$

The *higher order degenerate Daehee polynomials* are defined by the generating function to be

$$\left(\frac{\log_\lambda(1+t)}{t}\right)^r(1+t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}, \text{ (see [16]).}$$

In the special case $x = 0$, $D_{n,\lambda}^{(r)} = D_{n,\lambda}^{(r)}(0)$ are called the *higher order degenerate Daehee numbers.* When $r = 1$, $D_{n,\lambda}^{(1)}(x) = D_{n,\lambda}(x)$ are called *degenerate Daehee polynomials.*

Note that

$$\frac{\log_\lambda(1+t)}{t} = \frac{1}{t}\sum_{n=1}^{\infty}\frac{(1)_{n,1/\lambda}\lambda^{n-1}}{n!}t^n = \sum_{n=0}^{\infty}\frac{(1)_{n+1,1/\lambda}\lambda^n}{(n+1)!}t^n. \tag{2.5}$$

Theorem 2.3. *For each nonnegative integer n, we have*

$$\begin{aligned}b_{n,\lambda}(x) =&\sum_{k=0}^{n}\binom{n}{k}b_{n-k,\lambda}^{(2)}D_{k,\lambda}(x)\\ =&\sum_{k=0}^{n}\left((k+1)\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\frac{S_{1,\lambda}(m,l)S_{2,\lambda}(l+1,k+1)b_{n-m,\lambda}}{l+1}\right)D_{k,\lambda}(x).\end{aligned} \tag{2.6}$$

As the inversion formula of (2.6)*, we have*

$$D_{n,\lambda}(x) = \sum_{k=0}^{n}\binom{n}{k}D_{n-k,\lambda}^{(2)}b_{k,\lambda}(x).$$

Proof. Let $b_{n,\lambda}(x) = \sum_{k=0}^{n} c_{n,k}D_{k,\lambda}(x)$. Since

$$D_{n,\lambda}(x) \sim \left(\frac{e_\lambda(t)-1}{t}, e_\lambda(t)-1\right)_\lambda,$$

we have

$$\begin{aligned}
c_{n,k} =&\frac{1}{k!}\left\langle \frac{\frac{t}{\log_\lambda(1+t)}}{\frac{\log_\lambda(1+t)}{t}} t^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
=&\frac{1}{k!}\left\langle t^k \middle| \left(\left(\frac{t}{\log_\lambda(1+t)}\right)^2\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
=&\frac{1}{k!}\sum_{m=0}^{n} b^{(2)}_{m,\lambda}\frac{1}{m!}\left\langle t^k \middle| (t^m)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
=&\frac{1}{k!}\sum_{m=0}^{n} b^{(2)}_{m,\lambda}\frac{1}{m!}\left\langle t^{k+m} \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
=&\binom{n}{k} b^{(2)}_{n-k,\lambda}.
\end{aligned}$$

On the other hand, by (1.13) and (2.4), we get

$$\begin{aligned}
c_{n,k} =&\frac{1}{k!}\left\langle \frac{e_\lambda(t)-1}{t}(e_\lambda(t)-1)^k \middle| b_{n,,\lambda}(x)\right\rangle_\lambda \\
=&\frac{1}{k!}\left\langle \frac{1}{t}(e_\lambda(t)-1)^{k+1} \middle| b_{n,\lambda}(x)\right\rangle_\lambda \\
=&\frac{1}{k!}\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}S_{1,\lambda}(m,l)b_{n-m,\lambda}\left\langle \frac{1}{t}(e_\lambda(t)-1)^{k+1} \middle| (t)_\lambda \frac{1}{l+1}(x)_{l+1,\lambda}\right\rangle_\lambda \\
=&(k+1)\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}S_{1,\lambda}(m,l)b_{n-m,\lambda}\frac{1}{l+1}\left\langle \frac{1}{(k+1)!}(e_\lambda(t)-1)^{k+1} \middle| (x)_{l+1,\lambda}\right\rangle_\lambda \\
=&(k+1)\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\frac{S_{1,\lambda}(m,l)S_{2,\lambda}(l+1,k+1)b_{n-m,\lambda}}{l+1}.
\end{aligned}$$

Conversely, we assume that $D_{n,\lambda}(x) = \sum_{k=0}^{n} d_{n,k}b_{k,\lambda}(x)$. Then, by (2.5), we get

$$\begin{aligned}
d_{n,k} =&\frac{1}{k!}\left\langle \left(\frac{\log_\lambda(1+t)}{t}\right)^2 t^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
=&\binom{n}{k}\left\langle \sum_{m=0}^{\infty} D^{(2)}_{m,\lambda}\frac{t^m}{m!} \middle| (x)_{n-k,\lambda}\right\rangle_\lambda \\
=&\binom{n}{k} D^{(2)}_{n-k,\lambda},
\end{aligned}$$

and thus, our proofs are completed. □

The *unsigned Lah number* $L(n,k)$ counts the number of ways a set of n elements can be partitioned into k nonempty linearly ordered subsets and has the explicit formula

$$L(n,k) = \binom{n-1}{k-1}\frac{n!}{k!}, \text{ see ([9]).} \tag{2.7}$$

By (2.7), we can derive the generating function of $L(n,k)$ to be

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k=\sum_{n=k}^{\infty}L(n,k)\frac{t^n}{n!},\ (k\geq 0),\ \text{(see [9])}.$$

Recently, Kim-Kim introduced the *degenerate Lah-Bell polynomials* as follows:

$$e_\lambda^x\left(\frac{t}{1-t}\right)=\sum_{n=0}^{\infty}B_{n,\lambda}^L(x)\frac{t^n}{n!},\ \text{(see [9])}. \tag{2.8}$$

In the special case, $x=1$, $B_{n,\lambda}^L=B_{n,\lambda}^L(1)$ are called the *degenerate Lah-Bell numbers*. From (2.8), we get

$$B_{n,\lambda}^L(x)=\sum_{m=0}^{n}L(n,m)(x)_{m,\lambda}. \tag{2.9}$$

For each nonnegative integer k,

$$\left(\frac{t}{1+t}\right)^k=\sum_{l=0}^{\infty}\frac{(-1)^l<k>_l}{l!}t^{l+k}, \tag{2.10}$$

where $<x>_0=1$, $<x>_l=x(x+1)(x+2)\cdots(x-(l-1))$, $(l\geq 1)$. By (2.11), we note that

$$\begin{aligned}
&\left(\frac{\log_\lambda(1+t)}{1+\log_\lambda(1+t)}\right)^k\\
=&\sum_{l=0}^{\infty}\frac{(-1)^l<k>_l}{l!}\left(\log_\lambda(1+t)\right)^{l+k}\\
=&\sum_{l=0}^{\infty}\frac{(-1)^l<k>_l}{l!}(l+k)!\frac{1}{(l+k)!}\left(\log_\lambda(1+t)\right)^{l+k}\\
=&\sum_{l=0}^{\infty}\frac{(-1)^l<k>_l}{l!}(l+k)!\sum_{m=l+k}^{\infty}S_{1,\lambda}(m,l+k)\frac{t^m}{m!}\\
=&\sum_{r=0}^{\infty}\sum_{l=0}^{r}\frac{(-1)^l<k>_l}{l!}(l+k)!S_{1,\lambda}(r+k,l+k)\frac{t^{r+k}}{(r+k)!}.
\end{aligned} \tag{2.11}$$

Theorem 2.4. *For each nonnegative integer n, we have*

$$b_{n,\lambda}(x)=\sum_{k=0}^{n}\left(\sum_{l=0}^{n-k}\sum_{m=l+k}^{n}\binom{n}{m}\binom{l+k}{l}(-1)^l<k>_l S_{1,\lambda}(m,l+k)b_{n-m,\lambda}\right)B_{k,\lambda}^L(x). \tag{2.12}$$

As the inversion formula of (2.12), *we have*

$$B_{n,\lambda}^L(x)=\sum_{k=0}^{n}\left(\sum_{m=0}^{n}\sum_{l=k}^{m}\binom{m}{l}L(n,m)S_{2,\lambda}(l,k)B_{m-l,\lambda}\right)b_{k,\lambda}(x).$$

Proof. Let $b_{n,\lambda}(x)=\sum_{k=0}^{n}c_{n,k}B_{k,\lambda}^L(x)$. Since

$$B_{n,\lambda}^L(x)\sim\left(1,\frac{t}{1+t}\right)_\lambda,$$

by (1.15) and (2.11), we get

$$\begin{aligned}
c_{n,k} =& \frac{1}{k!}\left\langle \frac{1}{\frac{\log_\lambda(1+t)}{t}}\left(\frac{\log_\lambda(1+t)}{1+\log_\lambda(1+t)}\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
=& \frac{1}{k!}\left\langle \frac{t}{\log_\lambda(1+t)} \middle| \left(\left(\frac{\log_\lambda(1+t)}{1+\log_\lambda(1+t)}\right)^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
=& \frac{1}{k!}\left\langle \frac{t}{\log_\lambda(1+t)} \middle| \left(\sum_{l=0}^{\infty}\frac{(-1)^l <k>_l}{l!}\left(\log_\lambda(1+t)\right)^{l+k}\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
=& \sum_{l=0}^{n-k}\frac{(-1)^l <k>_l (l+k)!}{l!k!}\left\langle \sum_{p=0}^{\infty} b_{p,\lambda}\frac{t^p}{p!} \middle| \left(\frac{1}{(l+k)!}\left(\log_\lambda(1+t)\right)^{l+k}\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
=& \sum_{l=0}^{n-k}\sum_{m=l+k}^{n}\binom{n}{m}\binom{l+k}{l}(-1)^l <k>_l S_{1,\lambda}(m,l+k)\left\langle \sum_{p=0}^{\infty} b_{p,\lambda}\frac{t^p}{p!} \middle| (x)_{n-m,\lambda}\right\rangle_\lambda \\
=& \sum_{l=0}^{n-k}\sum_{m=l+k}^{n}\binom{n}{m}\binom{l+k}{l}(-1)^l <k>_l S_{1,\lambda}(m,l+k) b_{n-m,\lambda}.
\end{aligned}$$

Conversely, we assume that $B^L_{n,\lambda}(x) = \sum_{k=0}^{n} d_{n,k} b_{k,\lambda}(x)$. Then, by (1.16), we get

$$\begin{aligned}
d_{n,k} =& \frac{1}{k!}\left\langle \frac{t}{e_\lambda(t)-1}\left(e_\lambda(t)-1\right)^k \middle| B^L_{n,\lambda}(x)\right\rangle_\lambda \\
=& \sum_{m=0}^{n} L(n,m)\left\langle \frac{t}{e_\lambda(t)-1} \middle| \left(\frac{1}{k!}\left(e_\lambda(t)-1\right)^k\right)_\lambda (x)_{m,\lambda}\right\rangle_\lambda \\
=& \sum_{m=0}^{n}\sum_{l=k}^{m}\binom{m}{l} L(n,m) S_{2,\lambda}(l,k)\left\langle \sum_{a=0}^{\infty} B_{a,\lambda}\frac{t^a}{a!} \middle| (x)_{m-l,\lambda}\right\rangle_\lambda \\
=& \sum_{m=0}^{n}\sum_{l=k}^{m}\binom{m}{l} S_{2,\lambda}(l,k) L(n,m) B_{m-l,\lambda},
\end{aligned}$$

and so our proofs are completed. □

The *degenerate Euler polynomials* are defined by the generating function to be

$$\frac{2}{e_\lambda(t)+1}e^x_\lambda(t) = \sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}, \text{ (see [14]).}$$

When $x = 0$, $\mathcal{E}_{n,\lambda} = \mathcal{E}_{n,\lambda}(0)$ are called the *degenerate Euler numbers.*

Theorem 2.5. *For each $n \geq 0$, we have*

$$b_{n,\lambda}(x) = \sum_{k=0}^{n}\left(\frac{1}{2}\sum_{l=k}^{n}\binom{n}{l}\left(2b_{n-l,\lambda} + (n-l)b_{n-l-1,\lambda}\right)S_{1,\lambda}(l,k)\right)\mathcal{E}_{k,\lambda}(x). \quad (2.13)$$

As the inversion formula of (2.13), *we have*

$$\begin{aligned}\mathcal{E}_{n,\lambda}(x) =&\sum_{k=0}^{n}\left(\sum_{l=k}^{n}\sum_{a=0}^{n-l}\binom{n}{l}\binom{n-l}{a}S_{2,\lambda}(l,k)\mathcal{E}_{a,\lambda}B_{n-l-a,\lambda}\right)b_{k,\lambda}(x)\\ =&\sum_{k=0}^{n}\left(\sum_{m=0}^{n}\sum_{l=k}^{m}\binom{n}{m}\binom{m}{l}S_{2,\lambda}(l,k)\mathcal{E}_{n-m,\lambda}B_{m-l,\lambda}\right)b_{k,\lambda}(x).\end{aligned}$$

Proof. Let $b_{n,\lambda}(x) = \sum_{k=0}^{n} c_{n,k}\mathcal{E}_{k,\lambda}(x)$. Since

$$\mathcal{E}_{n,\lambda} \sim \left(\frac{e_\lambda(t)+1}{2}, t\right)_\lambda,$$

by (1.11) and (1.15), we get

$$\begin{aligned}c_{n,k} =&\frac{1}{k!}\left\langle \frac{\frac{t+2}{2}}{\frac{\log_\lambda(1+t)}{t}}\left(\log_\lambda(1+t)\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda\\ =&\frac{1}{2}\left\langle \frac{t}{\log_\lambda(1+t)}(t+2)\middle| \left(\frac{1}{k!}\left(\log_\lambda(1+t)\right)^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda\\ =&\frac{1}{2}\sum_{l=k}^{n}\binom{n}{l}S_{1,\lambda}(l,k)\left\langle \frac{t}{\log_\lambda(1+t)}\middle| (t+2)_\lambda(x)_{n-l,\lambda}\right\rangle_\lambda\\ =&\frac{1}{2}\sum_{l=k}^{n}\binom{n}{l}S_{1,\lambda}(l,k)\left\langle \frac{t}{\log_\lambda(1+t)}\middle| 2(x)_{n-l,\lambda}+(n-l)(x)_{n-l-1,\lambda}\right\rangle_\lambda\\ =&\frac{1}{2}\sum_{l=k}^{n}\left(2b_{n-l,\lambda}+(n-l)b_{n-l-1,\lambda}\right)\binom{n}{l}S_{1,\lambda}(l,k).\end{aligned}$$

Conversely, we assume that $\mathcal{E}_{n,\lambda}(x) = \sum_{k=0}^{n} d_{n,k}b_{k,\lambda}(x)$. Then

$$\begin{aligned}d_{n,k} =&\frac{1}{k!}\left\langle \frac{\frac{t}{e_\lambda(t)-1}}{\frac{e_\lambda(t)+1}{2}}(e_\lambda-1)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda\\ =&\left\langle \frac{2t}{(e_\lambda(t)-1)(e_\lambda(t)+1)}\middle| \left(\frac{1}{k!}(e_\lambda(t)-1)^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda\\ =&\sum_{l=k}^{n}\binom{n}{l}S_{2,\lambda}(l,k)\left\langle \frac{t}{e_\lambda(t)-1}\middle| \left(\frac{2}{e_\lambda(t)+1}\right)_\lambda (x)_{n-l,\lambda}\right\rangle_\lambda\\ =&\sum_{l=k}^{n}\sum_{a=0}^{n-l}\binom{n}{l}\binom{n-l}{a}S_{2,\lambda}(l,k)\mathcal{E}_{a,\lambda}\left\langle \sum_{b=0}^{\infty}B_{b,\lambda}\frac{t^b}{b!}\middle| (x)_{n-l-a,\lambda}\right\rangle_\lambda\\ =&\sum_{l=k}^{n}\sum_{a=0}^{n-l}\binom{n}{l}\binom{n-l}{a}S_{2,\lambda}(l,k)\mathcal{E}_{a,\lambda}B_{n-l-a,\lambda}.\end{aligned}$$

On the other hand,

$$\begin{aligned} d_{n,k} =&\frac{1}{k!}\left\langle \frac{t}{e_\lambda(t)-1}(e_\lambda(t)-1)^k \middle| \mathcal{E}_{n,\lambda}(x)\right\rangle_\lambda \\ =&\frac{1}{k!}\left\langle \frac{t}{e_\lambda(t)-1}(e_\lambda(t)-1)^k \middle| \sum_{m=0}^{n}\binom{n}{m}\mathcal{E}_{n-m,\lambda}(x)_{m,\lambda}\right\rangle_\lambda \\ =&\sum_{m=0}^{n}\binom{n}{m}\mathcal{E}_{n-m,\lambda}\left\langle \frac{t}{e_\lambda(t)-1}\middle| \left(\frac{1}{k!}(e_\lambda(t)-1)^k\right)_\lambda (x)_{m,\lambda}\right\rangle_\lambda \\ =&\sum_{m=0}^{n}\sum_{l=k}^{m}\binom{n}{m}\binom{m}{l}\mathcal{E}_{n-m,\lambda}S_{2,\lambda}(l,k)\left\langle \sum_{a=0}^{\infty}B_{a,\lambda}\frac{t^a}{a!}\middle| (x)_{m-l,\lambda}\right\rangle_\lambda \\ =&\sum_{m=0}^{n}\sum_{l=k}^{m}\binom{n}{m}\binom{m}{l}\mathcal{E}_{n-m,\lambda}S_{2,\lambda}(l,k)B_{m-l,\lambda}, \end{aligned}$$

and so our proofs are completed. □

The *Changhee polynomials* are defined by the generating function to be

$$\frac{2}{(1+t)+1}(1+t)^x = \frac{2}{e_\lambda(\log_\lambda(1+t))+1}e_\lambda^x(\log_\lambda(1+t)) = \sum_{n=0}^{\infty}Ch_n(x)\frac{t^n}{n!}, \text{ (see [5]).}$$

When $x=0$, $Ch_n = Ch_n(0)$ are called the *Changhee numbers.*

Theorem 2.6. *For each $n \geq 0$, we have*

$$b_{n,\lambda}(x) = \sum_{k=0}^{n}\left(\frac{1}{2}\binom{n}{k+1}(k+1)b_{n-k-1,\lambda} + \binom{n}{k}b_{n-k,\lambda}\right)Ch_k(x). \tag{2.14}$$

As the inversion formula of (2.14), we have

$$\begin{aligned} Ch_n(x) =&\sum_{k=0}^{n}\left(\sum_{l=0}^{n-k}\binom{n}{k}\binom{n-k}{l}Ch_l D_{n-k-l,\lambda}\right)b_{k,\lambda}(x) \\ =&\sum_{k=0}^{n}\left(\sum_{l=0}^{n-k}\frac{(-1)^l l!}{2^l}\binom{n}{l+k}\binom{l+k}{l}D_{n-l-k,\lambda}\right)b_{k,\lambda}(x) \\ =&\sum_{k=0}^{n}\left(\sum_{l=0}^{n}\sum_{a=0}^{l}\sum_{b=k}^{a}\binom{n}{l}\binom{a}{b}S_{1,\lambda}(l,a)S_{2,\lambda}(b,k)Ch_{n-l}B_{a-b,\lambda}\right)b_{k,\lambda}(x). \end{aligned}$$

Proof. Let $b_{n,\lambda}(x) = \sum_{k=0}^{n}c_{n,k}Ch_k(x)$. Note that

$$Ch_n(x) \sim \left(\frac{e_\lambda(t)+1}{2}, e_\lambda(t)-1\right)_\lambda.$$

By (1.11) and (1.15), we get

$$\begin{aligned}
c_{n,k} =&\frac{1}{k!}\left\langle \frac{\frac{t+2}{2}}{\frac{\log_\lambda(1+t)}{t}}t^k \middle| (x)_{n,k}\right\rangle_\lambda \\
=&\frac{1}{2k!}\left\langle \frac{t}{\log_\lambda(1+t)}\middle| \left(t^{k+1}+2t^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
=&\frac{1}{2}\left\langle \frac{t}{\log_\lambda(1+t)}\middle| \binom{n}{k+1}(k+1)(x)_{n-k-1,\lambda}\right\rangle_\lambda + \left\langle \frac{t}{\log_\lambda(1+t)}\middle| \binom{n}{k}(x)_{n-k,\lambda}\right\rangle_\lambda \\
=&\frac{1}{2}\binom{n}{k+1}(k+1)b_{n-k-1,\lambda}+\binom{n}{k}b_{n-k,\lambda}.
\end{aligned}$$

Conversely, we assume that $Ch_n(x)=\sum_{k=0}^n d_{n,k}b_{k,\lambda}(x)$. Then

$$\begin{aligned}
d_{n,k} =&\frac{1}{k!}\left\langle \frac{\frac{\log_\lambda(1+t)}{t}}{\frac{t+2}{2}}t^k \middle| (x)_{n,k}\right\rangle_\lambda \\
=&\frac{1}{k!}\left\langle \frac{\log_\lambda(1+t)}{t}\frac{2}{t+2}\middle| \left(t^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
=&\binom{n}{k}\left\langle \frac{\log_\lambda(1+t)}{t}\middle| \left(\frac{2}{t+2}\right)_\lambda (x)_{n-k,\lambda}\right\rangle_\lambda \\
=&\sum_{l=0}^{n-k}\binom{n}{k}\binom{n-k}{l}Ch_l\left\langle \frac{\log_\lambda(1+t)}{t}\middle| (x)_{n-k-l,\lambda}\right\rangle_\lambda \\
=&\sum_{l=0}^{n-k}\binom{n}{k}\binom{n-k}{l}Ch_l D_{n-k-l,\lambda}.
\end{aligned}$$

On the other hand, since

$$2t^k(t+2)^{-1}=2t^k\sum_{l=0}^\infty\binom{-1}{l}2^{-1-l}t^l=\sum_{l=0}^\infty\frac{(-1)^l}{2^l}t^{l+k}, \tag{2.15}$$

$$\begin{aligned}
d_{n,k} =&\frac{1}{k!}\left\langle \frac{\log_\lambda(1+t)}{t}\frac{2t^k}{t+2}\middle| (x)_{n,\lambda}\right\rangle_\lambda = \frac{1}{k!}\left\langle \frac{\log_\lambda(1+t)}{t}\middle| \left(\frac{2t^k}{t+2}\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
=&\frac{1}{k!}\sum_{l=0}^{n-k}\frac{(-1)^l}{2^l}\left\langle \frac{\log_\lambda(1+t)}{t}\middle| \left(t^{l+k}\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
=&\frac{1}{k!}\sum_{l=0}^{n-k}\frac{(-1)^l}{2^l}(n)_{l+k}\left\langle \frac{\log_\lambda(1+t)}{t}\middle| (x)_{n-l-k,\lambda}\right\rangle_\lambda \\
=&\sum_{l=0}^{n-k}\frac{(-1)^l}{2^l}\binom{n}{l+k}\binom{l+k}{l}l!D_{n-l-k,\lambda}.
\end{aligned}$$

In addition, since $Ch_n(x)=\sum_{l=0}^{n}\sum_{a=0}^{l}\binom{n}{l}Ch_{n-l}S_{1,\lambda}(l,a)(x)_{a,\lambda}$, by (1.16), we get

$$\begin{aligned}d_{n,k}=&\frac{1}{k!}\left\langle \frac{t}{e_\lambda(t)-1}\left(e_\lambda(t)-1\right)^k \middle| Ch_n(x)\right\rangle_\lambda\\ =&\sum_{l=0}^{n}\sum_{a=0}^{l}\binom{n}{l}Ch_{n-l}S_{1,\lambda}(l,a)\left\langle \frac{t}{e_\lambda(t)-1}\middle|\left(\frac{1}{k!}\left(e_\lambda(t)-1\right)^k\right)_\lambda (x)_{a,\lambda}\right\rangle_\lambda\\ =&\sum_{l=0}^{n}\sum_{a=0}^{l}\sum_{b=k}^{a}\binom{a}{b}\binom{n}{l}Ch_{n-l}S_{1,\lambda}(l,a)S_{2,\lambda}(b,k)\left\langle \frac{t}{e_\lambda(t)-1}\middle|(x)_{a-b,\lambda}\right\rangle_\lambda\\ =&\sum_{l=0}^{n}\sum_{a=0}^{l}\sum_{b=k}^{a}\binom{n}{l}\binom{a}{b}Ch_{n-l}S_{1,\lambda}(l,a)S_2(b,k)B_{a-b,\lambda},\end{aligned}$$

and hence our proofs are completed. □

The *Mittag-Leffler polynomials* are defined by the generating function to be

$$\left(\frac{1+t}{1-t}\right)^x=e_\lambda^x\left(\log_\lambda\left(\frac{1+t}{1-t}\right)\right)=\sum_{k=0}^{n}M_k(x)\frac{t^n}{n!},\ \text{(see [7, 19])}.$$

Theorem 2.7. *For each nonnegative integer n, we have*

$$b_{n,\lambda}(x)=\sum_{k=0}^{n}\left(\sum_{l=0}^{n-k}\frac{(-1)^l<k>_l}{2^{k+l}}\binom{n}{l+k}\binom{l+k}{l}b_{n-l-k,\lambda}\right)M_k(x). \tag{2.16}$$

As the inversion formula of (2.16), *we have*

$$M_n(x)=\sum_{k=0}^{n}\left(\sum_{m=0}^{n-k}\binom{m+k}{k}2^{m+k}D_{m,\lambda}L(n,m+k)\right)b_{k,\lambda}(x).$$

Proof. Let $b_{n,\lambda}(x)=\sum_{k=0}^{n}c_{n,k}M_k(x)$. Then, noting that

$$M_n(x)\sim\left(1,\frac{e_\lambda(t)-1}{e_\lambda(t)+1}\right)_\lambda \ \text{ and } \left(\frac{t}{t+2}\right)^k=\sum_{l=0}^{\infty}\frac{(-1)^l<k>_l}{2^{k+l}}\frac{t^{l+k}}{l!},$$

we get

$$\begin{aligned}c_{n,k}=&\frac{1}{k!}\left\langle \frac{1}{\frac{\log_\lambda(1+t)}{t}}\left(\frac{t}{t+2}\right)^k\middle|(x)_{n,\lambda}\right\rangle_\lambda\\ =&\frac{1}{k!}\left\langle \frac{t}{\log_\lambda(1+t)}\middle|\left(\left(\frac{t}{t+2}\right)^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda\\ =&\sum_{l=0}^{n-k}\frac{(-1)^l<k>_l}{2^{l+k}}\binom{n}{l+k}\binom{l+k}{l}\left\langle\sum_{m=0}^{\infty}b_{m,\lambda}\frac{t^m}{m!}\middle|(x)_{n-l-k,\lambda}\right\rangle_\lambda\\ =&\sum_{l=0}^{n-k}\frac{(-1)^l<k>_l}{2^{l+k}}\binom{n}{l+k}\binom{l+k}{l}b_{n-l-k,\lambda}.\end{aligned}$$

Conversely, we assume that $M_n(x) = \sum_{k=0}^{n} d_{n,k} b_{k,\lambda}(x)$. Then

$$\begin{aligned} d_{n,k} =& \frac{1}{k!}\left\langle \frac{\log_\lambda\left(\frac{1+t}{1-t}\right)}{\frac{1+t}{1-t}-1}\left(\frac{1+t}{1-t}-1\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ =&\left\langle \frac{1}{k!}\sum_{m=0}^{\infty} D_{m,\lambda}\frac{1}{m!}\left(\frac{2t}{1-t}\right)^{m+k}\middle| (x)_{n,\lambda}\right\rangle_\lambda \\ =&\sum_{m=0}^{n-k} D_{m,\lambda}\binom{m+k}{k}2^{m+k}\left\langle \frac{1}{(m+k)!}\left(\frac{t}{1-t}\right)^{m+k}\middle| (x)_{n,\lambda}\right\rangle_\lambda \\ =&\sum_{m=0}^{n-k}\binom{m+k}{k} D_{m,\lambda}2^{m+k}L(n,m+k), \end{aligned}$$

and so our proofs are completed. □

Note that

$$\begin{aligned} \frac{1}{k!}\left(\log_\lambda\left(1+\log_\lambda(1+t)\right)\right)^k =& \sum_{l=k}^{\infty} S_{1,\lambda}(l,k)\frac{1}{l!}\left(\log_\lambda(1+t)\right)^l \\ =& \sum_{l=k}^{\infty}\sum_{m=l}^{\infty} S_{1,\lambda}(l,k)S_{1,\lambda}(m,l)\frac{t^m}{m!} \\ =& \sum_{n=k}^{\infty}\sum_{l=0}^{n-k} S_{1,\lambda}(l+k,k)S_{1,\lambda}(n,l+k)\frac{t^n}{n!}, \end{aligned} \tag{2.17}$$

and, similarly to (2.17), we have

$$\frac{1}{k!}\left(e_\lambda\left(e_\lambda(t)-1\right)-1\right)^k = \sum_{n=k}^{\infty}\sum_{l=0}^{n-k} S_{2,\lambda}(l+k,k)S_{2,\lambda}(n,l+k)\frac{t^n}{n!}. \tag{2.18}$$

The *degenerate Bell polynomials* are defined by the generating function to be

$$e_\lambda^x\left(e_\lambda(t)-1\right) = \sum_{n=0}^{\infty} Bel_{n,\lambda}(x)\frac{t^n}{n!}, \text{ (see [12, 15]).} \tag{2.19}$$

From (2.19)

$$Bel_{n,\lambda}(x) = \sum_{m=0}^{n}(x)_{m,\lambda}S_{2,\lambda}(n,m), \text{ (see [12, 15]).} \tag{2.20}$$

Theorem 2.8. *For each nonnegative integer n, we have*

$$\begin{aligned} b_{n,\lambda}(x) =& \sum_{k=0}^{n}\left(\sum_{a=k}^{n}\sum_{l=0}^{a-k}\binom{n}{a}S_{1,\lambda}(l+k,k)S_{1,\lambda}(a,l+k)b_{n-a,\lambda}\right)Bel_{k,\lambda}(x) \\ =& \sum_{k=0}^{n}\left(\sum_{m=k}^{n}\sum_{l=k}^{m}\binom{n}{m}S_{1,\lambda}(m,l)S_{1,\lambda}(l,k)b_{n-m,\lambda}\right)Bel_{k,\lambda}(x). \end{aligned} \tag{2.21}$$

As the inversion formula of (2.21), *we have*

$$Bel_{n,\lambda}(x) = \sum_{k=0}^{n}\left(\sum_{a=k}^{n}\sum_{l=0}^{a-k}\sum_{m=0}^{n-a}\binom{n}{a}S_{2,\lambda}(l+k,k)S_{2,\lambda}(a,l+k)S_{2,\lambda}(n-a,m)B_{m,\lambda}\right)b_{k,\lambda}(x)$$
$$= \sum_{k=0}^{n}\left(\sum_{m=0}^{n}\sum_{l=k}^{m}\binom{m}{l}S_{2,\lambda}(n,m)S_{2,\lambda}(l,k)B_{m-l,\lambda}\right)b_{k,\lambda}(x).$$

Proof. Let $b_{n,\lambda}(x) = \sum_{k=0}^{n} c_{n,k}Bel_{k,\lambda}(x)$. Since

$$b_{n,\lambda}(x) \sim \left(\frac{t}{e_\lambda(t)-1}, e_\lambda(t)-1\right)_\lambda \text{ and } Bel_{n,\lambda}(x) \sim (1, \log_\lambda(1+t))_\lambda,$$

by (1.15) and (2.17), we get

$$c_{n,k} = \frac{1}{k!}\left\langle \frac{1}{\frac{\log_\lambda(1+t)}{t}}\left(\log_\lambda\left(1+\log_\lambda(1+t)\right)\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda$$
$$= \left\langle \frac{t}{\log_\lambda(1+t)} \middle| \left(\frac{1}{k!}\left(\log_\lambda\left(1+\log_\lambda(1+t)\right)\right)^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda$$
$$= \sum_{a=k}^{n}\sum_{l=k}^{a-k} S_{1,\lambda}(l+k,k)S_{1,\lambda}(a,l+k)\binom{n}{a}\left\langle \frac{t}{\log_\lambda(1+t)} \middle| (x)_{n-a,\lambda}\right\rangle_\lambda$$
$$= \sum_{a=k}^{n}\sum_{l=k}^{a-k}\binom{n}{a}S_{1,\lambda}(l+k,k)S_{1,\lambda}(a,l+k)b_{n-a,\lambda}.$$

On the other hand, by (1.16) and (2.4), we have

$$c_{n,k} = \frac{1}{k!}\left\langle (\log_\lambda(1+t))^k \middle| b_{n,\lambda}(x)\right\rangle_\lambda$$
$$= \sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}S_{1,\lambda}(m,l)b_{n-m,\lambda}\left\langle \frac{1}{k!}(\log_\lambda(1+t))^k \middle| (x)_{l,\lambda}\right\rangle_\lambda$$
$$= \sum_{m=k}^{n}\sum_{l=k}^{m}\binom{n}{m}S_{1,\lambda}(m,l)S_{1,\lambda}(l,k)b_{n-m,\lambda}.$$

Conversely, we assume that $Bel_{n,\lambda}(x)=\sum_{k=0}^{n}d_{n,k}b_{k,\lambda}(x)$. Then, by (2.18), we get

$$\begin{aligned}
d_{n,k}=&\frac{1}{k!}\left\langle \frac{e_\lambda(t)-1}{e_\lambda(e_\lambda(t)-1)-1}\left(e_\lambda(e_\lambda(t)-1)-1\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda\\
=&\left\langle \frac{e_\lambda(t)-1}{e_\lambda\left(e_\lambda(t)-1\right)-1}\middle| \left(\frac{1}{k!}\left(e_\lambda\left(e_\lambda(t)-1\right)-1\right)^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda\\
=&\sum_{a=k}^{n}\sum_{l=0}^{a-k}S_{2,\lambda}(l+k,k)S_{2,\lambda}(a,l+k)\binom{n}{a}\left\langle \frac{e_\lambda(t)-1}{e_\lambda\left(e_\lambda(t)-1\right)-1}\middle| (x)_{n-a,\lambda}\right\rangle_\lambda\\
=&\sum_{a=k}^{n}\sum_{l=0}^{a-k}S_{2,\lambda}(l+k,k)S_{2,\lambda}(a,l+k)\binom{n}{a}\\
&\times\left\langle \sum_{m=0}^{\infty}B_{m,\lambda}\frac{1}{m!}\left(e_\lambda(t)-1\right)^m\middle| (x)_{n-a,\lambda}\right\rangle_\lambda\\
=&\sum_{a=k}^{n}\sum_{l=0}^{a-k}S_{2,\lambda}(l+k,k)S_{2,\lambda}(a,l+k)\binom{n}{a}\sum_{b=0}^{\infty}\sum_{m=0}^{b}B_{m,\lambda}S_{2,\lambda}(b,m)\frac{1}{b!}\left\langle t^b\middle| (x)_{n-a,\lambda}\right\rangle_\lambda\\
=&\sum_{a=k}^{n}\sum_{l=0}^{a-k}\sum_{m=0}^{n-a}\binom{n}{a}S_{2,\lambda}(l+k,k)S_{2,\lambda}(a,l+k)S_{2,\lambda}(n-a,m)B_{m,\lambda}.
\end{aligned}$$

On the other hand, by (1.16), (2.8) and (2.20), we have

$$\begin{aligned}
d_{n,k}=&\frac{1}{k!}\left\langle \frac{t}{e_\lambda(t)-1}\left(e_\lambda(t)-1\right)^k\middle| Bel_{n,\lambda}(x)\right\rangle_\lambda\\
=&\sum_{m=0}^{n}S_{2,\lambda}(n,m)\left\langle \frac{t}{e_\lambda(t)-1}\middle| \left(\frac{1}{k!}\left(e_\lambda(t)-1\right)^k\right)_\lambda (x)_{m,\lambda}\right\rangle_\lambda\\
=&\sum_{m=0}^{n}\sum_{l=k}^{m}S_{2,\lambda}(n,m)S_{2,\lambda}(l,k)\binom{m}{l}\left\langle \frac{t}{e_\lambda(t)-1}\middle| (x)_{m-l,\lambda}\right\rangle_\lambda\\
=&\sum_{m=0}^{n}\sum_{l=k}^{m}S_{2,\lambda}(n,m)S_{2,\lambda}(l,k)\binom{m}{l}\left\langle \sum_{a=0}^{\infty}B_{a,\lambda}\frac{t^a}{a!}\middle| (x)_{m-l,\lambda}\right\rangle_\lambda\\
=&\sum_{m=0}^{n}\sum_{l=k}^{m}\binom{m}{l}S_{2,\lambda}(n,m)S_{2,\lambda}(l,k)B_{m-l,\lambda},
\end{aligned}$$

and thus our proofs are completed. □

In [10], Kim-Kim defined the *degenerate Frobenius-Euler polynomils of order* r by the generating function to be

$$\left(\frac{1-u}{e_\lambda(t)-u}\right)^r e_\lambda^x(t)=\sum_{n=0}^{\infty}h_{n,\lambda}^{(r)}(x|u)\frac{t^n}{n!},\ (u\neq 1,\ u\in\mathbb{C}),\ (n\geq 0).$$

In the special case $x=0$, $h_{n,\lambda}^{(r)}(u)=h_{n,\lambda}^{(r)}(0|u)$ are called the *degenerate Frobenius-Euler numbers of order* r.

Theorem 2.9. *For each nonnegative integer* n, *we have*

$$b_{n,\lambda}(x)=\sum_{k=0}^{n}\left(\sum_{l=k}^{n}\sum_{m=0}^{r}\binom{n}{l}\binom{r}{m}\frac{(n-l)_m}{(1-u)^m}S_{1,\lambda}(l,k)b_{n-l-m,\lambda}\right)h_{k,\lambda}^{(r)}(x). \tag{2.22}$$

As the inversion formula of (2.22), *we have*

$$h_{n,\lambda}^{(r)}(x)=\sum_{k=0}^{n}\left(\sum_{m=k}^{n}\sum_{l=0}^{m}\binom{n}{m}\binom{m}{l}S_{2,\lambda}(m-l,k)h_{n-m,\lambda}^{(r)}(u)B_{l,\lambda}\right)b_{k,\lambda}(x).$$

Proof. Let $b_{n,\lambda}(x)=\sum_{k=0}^{n}c_{n,k}h_{k,\lambda}^{(r)}(x|u)$. Since

$$b_{n,\lambda}(x)\sim\left(\frac{t}{e_\lambda(t)-1},e_\lambda(t)-1\right)_\lambda \text{ and } h_{n,\lambda}^{(r)}(x|u)\sim\left(\left(\frac{e_\lambda(t)-u}{1-u}\right)^r,t\right)_\lambda,$$

by (1.12) and (1.15), we get

$$\begin{aligned}
c_{n,k}=&\frac{1}{k!}\left\langle\frac{\left(\frac{(1+t)-u}{1-u}\right)^r}{\frac{\log_\lambda(1+t)}{t}}\left(\log_\lambda(1+t)\right)^k\middle|(x)_{n,\lambda}\right\rangle_\lambda\\
=&\frac{1}{(1-u)^r}\left\langle\frac{t}{\log_\lambda(1+t)}((1+t)-u)^r\middle|\left(\frac{1}{k!}\left(\log_\lambda(1+t)\right)^k\right)_\lambda(x)_{n,\lambda}\right\rangle_\lambda\\
=&\frac{1}{(1-u)^r}\sum_{l=k}^{n}S_{1,\lambda}(l,k)\binom{n}{l}\left\langle\frac{t}{\log_\lambda(1+t)}\middle|\left((t+(1-u))^r\right)_\lambda(x)_{n-l,\lambda}\right\rangle_\lambda\\
=&\frac{1}{(1-u)^r}\sum_{l=k}^{n}S_{1,\lambda}(l,k)\binom{n}{l}\left\langle\frac{t}{\log_\lambda(1+t)}\middle|\left(\sum_{m=0}^{r}\binom{r}{m}(1-u)^{r-m}t^m\right)_\lambda(x)_{n-l,\lambda}\right\rangle_\lambda\\
=&\frac{1}{(1-u)^r}\sum_{l=k}^{n}\sum_{m=0}^{r}\binom{n}{l}\binom{r}{m}S_{1,\lambda}(l,k)(1-u)^{r-m}(n-l)_m\left\langle\frac{t}{\log_\lambda(1+t)}\middle|(x)_{n-l-m,\lambda}\right\rangle_\lambda\\
=&\sum_{l=k}^{n}\sum_{m=0}^{r}\binom{n}{l}\binom{r}{m}\frac{(n-l)_m}{(1-u)^m}S_{1,\lambda}(l,k)b_{n-l-m,\lambda}.
\end{aligned}$$

Conversely, we assume that $h_{n,\lambda}^{(r)}(x|u)=\sum_{k=0}^{n}d_{n,k}b_{k,\lambda}(x)$. Then, by (1.15) and (1.16), we have

$$\begin{aligned}
d_{n,k}=&\frac{1}{k!}\left\langle\frac{t}{e_\lambda(t)-1}\left(e_\lambda(t)-1\right)^k\middle|h_{n,\lambda}^{(r)}(x|u)\right\rangle_\lambda\\
=&\frac{1}{k!}\left\langle\frac{t}{e_\lambda(t)-1}\left(e_\lambda(t)-1\right)^k\middle|\sum_{m=0}^{n}\binom{n}{m}h_{n-m,\lambda}^{(r)}(u)(x)_{m,\lambda}\right\rangle_\lambda\\
=&\sum_{m=k}^{n}\binom{n}{m}h_{n-m,\lambda}^{(r)}(u)\left\langle\frac{1}{k!}\left(e_\lambda(t)-1\right)^k\middle|\left(\frac{t}{e_\lambda(t)-1}\right)_\lambda(x)_{m,\lambda}\right\rangle_\lambda\\
=&\sum_{m=k}^{n}\sum_{l=0}^{m}\binom{n}{m}\binom{m}{l}S_{2,\lambda}(m-l,k)h_{n-m,\lambda}^{(r)}(u)B_{l,\lambda}.
\end{aligned}$$

□

3. Conclusion

In this paper, we represented the degenerate Bernoulli polynomials of the second kind in terms of various special polynomials and derived the inversion formulas of those identities by using the λ- Sheffer sequences. We addressed the well-known special polynomials and numbers: the degenerate falling factorial, the degenerate Bernoulli polynomials, degenerate Daehee polynomials, the degenerate Lah-Bell polynomials, degenerate Euler polynomials, the Changhee polynomials, the Mittag-Leffer polynomials, the degenerate Bell polynomials, the degenerate Frobenius-Euler polynomials of order r. It is one of our future projects to continue to investigate the degenerate special numbers and polynomials by using the λ-umbral calculus.

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6. Competing interests

The authors declare that they have no competing interests.

7. Authors' contributions

JWP and JK conceived of the framework and structured the whole manuscript; JWP wrote the paper; JK and BMK checked the results of the manuscript. All authors read and approved the final paper.

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[1] Department of Mathematics Education, Daegu University, Gyeongsan-si, 38453, Republic of Korea.
Email address: a0417001@knu.ac.kr

[2] Department of Mechanical System Engineering, Dongguk University, 123 Dongdae-ro, Gyungju-si, Gyeongsangbuk-do, 38066, Republic of Korea.
Email address: kbm713@dongguk.ac.kr

[3] Department of Mathematics Educations, Gyeongsang National University ,JinJu, 52828, Republic of Korea.
Email address: mathkjk26@gnu.ac.kr