Proceedings of the Jangjeon Mathematical Society
24 (2021), No. 3, pp. 309 - 321
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2021.24.3.309

# METRIC DIMENSION AND ITS VARIATIONS OF CHAIN GRAPHS

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Abstract. Chain graphs and threshold graphs are characterized as graphs with the largest spectral radius among all connected bipartite graphs (former one) and all connected graphs (latter one) with prescribed order and size. In this article, we derive results on the metric dimension of chain and threshold graphs. We present an algorithm which returns a chain graph (if one exists) having specified order and metric dimension. We define the restricted threshold dimension of chain graphs which minimizes the metric dimension of graphs obtained by adding edges while keeping the nesting property and bipartiteness. We also derive related results.

2010 Mathematics Subject Classification. 05C76, 05C40 , 05C50.

Keywords and phrases. Chain, bipartite graphs, threshold dimension

## 1. Preliminaries

Metric dimension is one of the concepts of primary significance in navigation studies in graphs. The study of navigation provides better visualization and conclusions in a graph based framework, where we assume every vertex to be a navigating agent or a robot which moves from one vertex to the other. Any of the robots can locate themselves by distinctively labeled "landmarks". For a navigating robot, the distances to the set of landmarks allow the robot to sense its position by triangulation. Accordingly, the problem of finding the fewest number and the location of landmarks, so that the distances to the landmarks uniquely determine the position of the robots in the graph is posed. A minimum set of landmarks that uniquely determine the positions of the robots is the metric basis and the minimum number is the metric dimension [15].

The problem of metric dimension was introduced in 1975 by Slater, and later in the year 1976 independently by Harary and Melter. The distance $d(u, v)$ between a pair of vertices $u$ and $v$ in $G$ is the length of the shortest path between $u$ and $v$, if one exists, else $d(u, v) = \infty$. A vertex $x \in V(G)$ resolves a pair of vertices $v, w \in V(G)$ if $d(v, x) \neq d(w, x)$. A set of vertices $S \subseteq V(G)$ resolves $G$, and $S$ is a resolving set of $G$, if every pair of distinct vertices of $G$ is resolved by some vertex in $S$. A resolving set $S$ of $G$ with the minimum cardinality is a metric basis of $G$, and $|S|$ is the metric dimension of $G$, denoted by $\beta(G)$. Determining the metric dimension of an arbitrary graph is an NP-complete problem [6]. To date, some standard classes of graphs and graphs of order $n$ with metric dimension 1, 2, $n-3$, $n-2$, and $n-1$ have been characterized in [7, 8, 11, 12, 13, 17, 19, 20].

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A graph is called a chain graph if it is bipartite and the neighborhoods of the vertices in each partite set form a chain with respect to set inclusion. The color classes $U, V$ of a chain graph $G$ can be partitioned into $h$ non-empty cells $U_1, \ldots, U_h$ and $V_1, \ldots, V_h$ such that $N_G(u) = V_1 \cup \cdots \cup V_{h-i+1}$, for any $u \in U_i$, $1 \leq i \leq h$. Due to this nesting property of edges, chain graphs are also called Double Nested Graphs (DNG for short). If $m_i = |U_i|$ and $n_i = |V_i|$, then we write $G = DNG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$. If $m_i = n_i = 1$ for all $1 \leq i \leq h$, then the graph is called half graph [9]. For other interesting properties and characterizations, the readers are referred to [2, 3, 4, 5, 9, 14].

A split graph is one whose vertex set can be divided into two subsets, out of which one forms a co-clique, the other forms a clique, and all other remaining edges (cross edges) join two vertices which belong to different subsets. A threshold graph is a special type of split graph in which there is a nesting property imposed on the cross edges. Threshold graphs are also called Nested Split Graphs (NSG for short). For a threshold graph $G$ having bipartition $U \cup V$ such that the vertices of $U$ induce a co-clique, while the vertices of $V$ induce a clique, both $U$ and $V$ are partitioned into $h$ non-empty cells such that $U = U_1 \cup U_2 \cup \cdots \cup U_h$ and $V = V_1 \cup V_2 \cup \ldots \cup V_h$ such that $N_G(u) = V_1 \cup \cdots \cup V_{h-i+1}$, for any $u \in U_i$, $1 \leq i \leq h$. If $|U_i| = m_i$ and $|V_i| = n_i$, then we write $G = NSG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$. The schematic representation of a DNG as well as an NSG are given in Figure 1.

schematic1-eps-converted-to.pdf

Figure 1. Schematic diagram

For further results on threshold graphs, the readers are referred to [1], [4], [10], [16].

## 2. Bounds for metric dimension of chain graphs

For a given chain graph $G$, computation of metric dimension is tedious even though the distance $d(u, v) \in \{1, 2, 3\}$ for any pair of vertices $(u, v)$ in $G$ [18]. But it is bounded by certain factors. We give bounds for the metric dimension of a chain graph. Before moving directly to the bounds, the nesting in the structure of chain graphs enables us to state some results concerned with their resolving set $S$.

**Lemma 2.1.** *Let $G = DNG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ be a chain graph, where $m_i = |U_i|$ and $n_i = |V_i|$. Let $S$ be the resolving set of the graph $G$. Suppose $m_i = k \geq 2$ (or $n_j = k \geq 2$), then $S$ contains at least $k - 1$ vertices of $U_i$ (or $V_j$) for $1 \leq i, j \leq h$.*

*Proof.* Let $m_i = |U_i| = k \geq 2$. Suppose $S$ does not contain any vertex from the cell $U_i$. Then all the vertices of $U_i$ have the same label with respect to $S$, a contradiction to the fact that $S$ is a resolving set. Suppose $S$ has $r < k - 1$ vertices from $U_i$, then the remaining $k - r > 1$ vertices have the same label with respect to $S$, a contradiction. Thus the resolving set contains at least $k - 1$ vertices of $U_i$ (or $V_j$). □

**Lemma 2.2.** *Let $G = DNG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ be a chain graph, where $m_i = |U_i|$ and $n_i = |V_i|$ for $i = 1, 2, \ldots, h$. Then any two vertices, one each from the cells $U_i$ and $U_{i+1}$ are resolved only by a vertex from the cell $V_{h-i+1}$ for $i = 1, 2, \ldots, h-1$. Equivalently, any two vertices, one each from the cells $V_j$ and $V_{j+1}$ are resolved only by a vertex from the cell $U_{h-j+1}$ for $j = 1, 2, \ldots, h-1$.*

*Proof.* The proof follows from the fact that only those vertices from the cell $V_{h-i+1}$ are adjacent to all the vertices of $U_i$, but not to any of the vertices of $U_{i+1}$ (thus having distance three between them), which results in the distinct labels. That is, for all the vertices of $U_i$, the label corresponding to a vertex from $V_{h-i+1}$ is 1 and for all the vertices of $U_{i+1}$, it is 3. □

For a chain graph $G = DNG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$, suppose $m_1 = |U_1| > 1$ and $n_1 = |V_1| > 1$. Then by Lemma 2.1, the resolving set contains at least $m_1 + n_1 - 2$ vertices. Further, the remaining vertices of $U_1$ and of $V_1$ do not resolve any of the vertices in the graph. In other words, with respect to these vertices, all the vertices of the same partite set have the label 2 and all the vertices of the other partite set have the label 1. With all the properties of the resolving set given in the above lemmas, we derive the lower and the upper bounds for the metric dimension of chain graphs.

**Theorem 2.3.** *Let $G = DNG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ be a chain graph on $n$ vertices, where $m_i = |U_i|$ and $n_i = |V_i|$ for $i = 1, 2, \ldots, h$. Let $\beta(G)$ be the metric dimension of $G$. Then*

$$n - 2h \leq \beta(G) \leq n - h - 1.$$

*Proof.* Let $S$ be the metric basis. Since $m_i, n_i \geq 1$ for $1 \leq i \leq h$, each of the $2h$ cells have at least one vertex. Then by Lemma 2.1, the metric basis $S$ contains the remaining $n - 2h$ vertices, irrespective of the cells to which they belong. Thus $\beta(G) \geq n - 2h$.

Since $|S| \geq n - 2h$, each of the cells $U_i$ and $V_j$ for $1 \leq i, j \leq h$ have exactly one vertex which is not contained in $S$. In addition to the $n - 2h$ vertices, by Lemma 2.2, the set $S$ contains a vertex from $V_{h-i+1}$ for each $1 \leq i \leq h-1$ if $|V_{h-i+1}| = 1$. If $|V_{h-i+1}| \geq 2$, then $S$ already contains at least one vertex of $V_{h-i+1}$, which resolves any pair of vertices, each from one of $U_i$ and $U_{i+1}$ for $i = 1, 2, \ldots, h-1$. The set $S$ has maximum cardinality when each $|V_{h-i+1}| = 1$. In other words, none of the $n - 2h$ vertices are included in the cells $V_{h-i+1}$ for each $i = 1, 2, \ldots, h-1$. Thus, the $n - 2h$ vertices are present in $V_1$. Hence, along with $n - 2h$ vertices, $S$ has vertices of the cells $V_2, V_3, \ldots, V_h$, thus resolving the entire graph. Thus $|S| \leq n - 2h + h - 1 = n - h - 1$. □

Consider a chain graph of order $n$. From the above theorem, we note that the lower bound $n - 2h$ is minimum when $h$ is maximum, i.e., when $h = \lceil \frac{n}{2} \rceil$. Similarly, the upper bound $n - h - 1$ is maximum when $h = 1$. In other words, the lower bound is attained by the half graph when $n$ is even and by the graph having all $m_i = n_i = 1$, except for exactly one of $m_i$ and $n_i$, which is equal to 2, when $n$ is odd. The upper bound is attained by all the complete bipartite graphs on $n$ vertices. Thus, we have the following result giving bounds for $\beta(G)$ in terms of the order $n$ irrespective of the size of the cells.

**Corollary 2.4.** *Let $G$ be a chain graph on $n$ vertices. Let $\beta(G)$ be the metric dimension of $G$. Then*

$$\lceil \frac{n}{2} \rceil - 1 \leq \beta(G) \leq n - 2.$$

From Lemmas 2.1, 2.2 and Theorem 2.3, the remark below easily follows:

**Remark 2.5.** *Suppose $m_i, n_i > 1$ for $2 \leq i \leq h$. Then $\beta(DNG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)) = n - 2h$, where $n$ is the order of the graph $G$.*

With the above remark, we note that the metric dimension of a chain graph depends on the position of the cells in each partite set having cardinality 1. We note that the upper bound in Theorem 2.3 is attained by all the graphs where exactly one of the cells has cardinality $n - 2h + 1$ and the remaining have cardinality 1, which are given by $DNG(n - 2h + 1, 1, \ldots, 1; 1, 1, \ldots, 1)$, $DNG(1, n - 2h + 1, \ldots, 1; 1, 1, \ldots, 1)$, $\ldots$, and $DNG(1, 1, \ldots, n - 2h + 1; 1, 1, \ldots, 1)$. The lower bound is attained only by the graphs satisfying the relation $n \geq 4h - 2$. When $n \geq 4h - 2$, the lower bound is attained by all the graphs in which every cell is of cardinality greater than 1. But, when $n < 4h - 2$, the lower bound given in the above theorem is too small. If we consider an instance where $n = 11$ and $h = 5$, Theorem 2.3 gives $\beta(G) \geq 1$, which is trivial. Thus we improve the lower bound for $\beta(G)$ when $n < 4h - 2$.

**Theorem 2.6.** *Let $G = DNG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ be a chain graph on $n$ vertices, where $m_i = |U_i|$ and $n_i = |V_i|$ for $i = 1, 2, \ldots, h$. Let $\beta(G)$ be the metric dimension of $G$. Suppose $n < 4h - 2$, then*

$$\beta(G) \geq \begin{cases} \dfrac{n-2}{2}, & \text{if } n \text{ is even} \\ \dfrac{n-1}{2}, & \text{otherwise.} \end{cases}$$

*Proof.* Let $n = 2h + k$, where $1 \leq k \leq 2h - 3$. From Theorem 2.3, $\beta(G) \geq k$. We know that each of the $2h$ cells contains at least one vertex. The remaining $k$ vertices are distributed to $2h$ cells such that $\beta(G)$ is minimum. For the resolving set to have minimum cardinality, we distribute the $k$ vertices to those cells, the vertices of which are required to resolve some pair of vertices. Since the vertices of $U_1, V_1$ do not resolve any pair of vertices, we distribute the $k$ vertices to the remaining $2h - 2$ cells. We follow the given procedure to distribute the $k$ vertices to $2h - 2$ cells such that it resolves the maximum number of pairs of vertices of $G$. Suppose that the first vertex is included in the cell $U_2$, then in order to resolve a pair of vertices, each from one of $U_1$ and $U_2$, we need a vertex from $V_h$ and thus the second vertex is included in $V_h$. Now, if the third vertex is included in $U_3$, then the next one is added to $V_{h-1}$, vertex of which is needed to resolve the vertices of $U_2$ and $U_3$. Continuing like this, we end up with the following cases.
Case i: When $k$ is even: The $\frac{k}{2}$ vertices are added to the cells $U_2, U_3, \ldots, U_{\frac{k+2}{2}}$ and $V_h, V_{h-1}, \ldots, V_{h-\frac{k-2}{2}}$. We note that each of the partite sets of $G$ has

$h-\frac{k}{2}$ cells with cardinality one, out of which vertices of $U_1, V_1$ do not resolve any vertices of $G$. Thus

$$\begin{aligned}\beta(G) &\geq k+h-1-\frac{k}{2}\\ &= \frac{n-2h}{2}+h-1\\ &= \frac{n-2}{2}.\end{aligned}$$

Case ii: When $k$ is odd: The $\frac{k-1}{2}$ vertices are added to the cells $U_2, U_3, \dots, U_{\frac{k+3}{2}}$ and $V_h, V_{h-1}, \dots, V_{h-\frac{k-3}{2}}$. Further, by Lemma 2.2, we need a vertex from the cell $V_{h-\frac{k-1}{2}}$ to resolve a pair of vertices, each from one of the cells $U_{\frac{k+1}{2}}$ and $U_{\frac{k+3}{2}}$. The partite set $U$ has $h-\frac{k+1}{2}$ cells with cardinality 1, out of which $U_1$ does not resolve any vertices of the graph. Thus

$$\begin{aligned}\beta(G) &\geq k+\left(h-1-\frac{k+1}{2}\right)+1\\ &= \frac{k+2h-1}{2}\\ &= \frac{n-1}{2}.\end{aligned}$$

□

For the instance taken above, when $n = 11$ and $h = 5$, the Theorem 2.6 gives better bound $\beta(G) \geq 5$ than that of Theorem 2.3. We also note that the lower bound in the above theorem depends only on $n$. The following remark is a consequence of the above theorem.

**Remark 2.7.** *For a given chain graph of fixed order $n$, whenever the number of cells $h > \frac{n+2}{4}$, the lower bound for $\beta(G)$ is achieved at a constant value, which is independent of $h$.*

## 3. Further results

In the above section, we have noted the graphs on $n$ vertices attaining the least metric dimension. That is, if $G = DNG(1, 1, \dots, 1; 1, 1, \dots, 1)$, a half graph on $n$ vertices, then $\beta(G) = \frac{n-2}{2}$. Similarly, for all the chain graphs $DNG(m_1, m_2, \dots, m_h; n_1, n_2, \dots, n_h)$ on $n$ vertices where exactly one of $m_i$ and $n_i$ is 2 and others are 1, the metric dimension is $\beta(G) = \frac{n-1}{2}$. Further, if each of $m_i$ and $n_i$ is greater than 1, then $\beta(G) = n-2h$. Apart from these graphs, we look into another case.

**Theorem 3.1.** *Let $G = DNG(m_1, m_2, \dots, m_h; 1, 1, \dots, 1)$ be a chain graph on $n$ vertices, where $m_i = |U_i|$ and $|V_i| = 1$ for $1 \leq i \leq h$. Let $\beta(G)$ be the metric dimension of $G$. Then $\beta(G) = n-h-1$.*

*Proof.* Let $S$ be the metric basis of the graph $G$. Then by Theorem 2.3, $|S| \geq \sum_{i=1}^{h}(m_i - 1)$. We note that if $|U_i| = k$, then $S$ has at least $k-1$ vertices from $U_i$ for each $1 \leq i \leq h$. In order to resolve the remaining $h$ vertices of the partite set which are in distinct cells $U_i$, from Lemma 2.2, $S$ contains $h-1$ vertices of the cells $V_j, 2 \leq j \leq h$. Thus

$$\begin{aligned}\beta(G) &= \sum_{i=1}^{h}(m_i - 1) + (h-1)\\ &= n - h - 1. \end{aligned}$$ □

We exhibit an alternative metric basis for the graph mentioned in the above theorem, except when $h = 1$. When $G = DNG(m_1, m_2, \ldots, m_h; 1, 1, \ldots, 1)$, since $\beta(G) = n - h - 1$, one can easily note that even $|U| - 1$ vertices of the partite set $U$ are sufficient to resolve the entire graph $G$, where $U = \sum_{i=1}^{h} U_i$.

**Lemma 3.2.** *Let $G = DNG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ be a chain graph, where $m_i = |U_i|$ and $n_i = |V_i|$ for $1 \leq i \leq h$. Let $S$ be the metric basis of $G$. Then $S$ contains vertices of exactly one partite set, say $U$ if and only if $n_i = 1$ for all $1 \leq i \leq h$.*

*Proof.* Let $S$ contain the vertices of the partite set $U$. That is, let $S = \{u_1, u_2, \ldots, u_k\}$, where $u_i \in U$ for all $1 \leq i \leq k$. We note that $|U \setminus S| = 1$. Suppose $|U \setminus S| = k > 1$. Then all the $k$ vertices in $U \setminus S$ have label $(2, 2, \ldots, 2)$, which is a contradiction to the fact that $S$ is a metric basis. By Theorem 2.3, it follows that $|V_i| = 1$ for all $1 \leq i \leq h$. Thus $G = DNG(m_1, m_2, \ldots, m_h; 1, 1, \ldots, 1)$. The converse follows from Theorem 3.1. □

We know that every threshold graph $H = NSG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ can be obtained from the chain graph $G = DNG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ by making any one of the partite sets complete. In the next theorem, we give the relation between the metric dimension of a threshold graph and the corresponding chain graph from which it is obtained.

**Theorem 3.3.** *Let $G = DNG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ be a chain graph, where $|U| = \sum_{i=1}^{h} m_i$ and $|V| = \sum_{i=1}^{h} n_i$. Let $H = NSG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ be the threshold graph obtained from $G$ by making $U$ complete. Suppose $\beta(G), \beta(H)$ are the metric dimensions of the graphs $G$ and $H$, respectively. Then $\beta(G) = \beta(H)$ except when $|U| = 1$.*

*Proof.* Let $S$ be the metric basis of $G = DNG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$. Suppose $S = \{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_l\}$, where $l + k = \beta(G)$, $u_i \in U$, and $v_j \in V$ for $1 \leq i \leq k, 1 \leq j \leq l$. In the graph $H$, since the distance between every pair of vertices in $U$ is 1, for every vertex $u_i \in U$, the label 2 corresponding to the vertex $u_j \in S$ $(j \neq i)$ is replaced by the label 1. All other labels remain unchanged. Thus $S$ forms the metric basis for the graph $H$ unless $S$ contains the vertices of $U$ alone. If $S$ contains the vertices of $U$ alone, then by Lemma 3.2, it is $DNG(m_1, m_2, \ldots, m_h; 1, 1, \ldots, 1)$, which has an alternative metric basis containing the vertices of both the partite sets for all $h \geq 2$. When $h = 1$, $G = DNG(n-1; 1)$ and the corresponding threshold graph is $H = NSG(n-1; 1)$ where $\beta(G) = n-2$ and $\beta(H) = n-1$.

Thus, $\beta(NSG(n-1;1)) = \beta(DNG(n-1;1)) + 1$ and in rest of the cases the metric dimension remains unchanged. □

## 4. Inverse metric dimension problem

Computation of metric dimension of a given graph gets computationally difficult with an increasing number of cells with cardinality 1. By contrast, the term inverse metric dimension problem refers to the problem of constructing a graph $G$ of order $n$, given the metric dimension $\beta(G) = c$. The inverse metric dimension problem was posed in the literature a couple of years ago. But in this article, we restrict our attention to one particular class. For all the integers $c \in (\lceil \frac{n}{2} \rceil, n-2)$, there exists at least one chain graph of order $n$ with metric dimension $c$.

We present an algorithm concerned with the classical inverse metric dimension problem relating to the construction of a chain graph of order $n$ having given metric dimension $c$. Suppose $c = \lceil \frac{n}{2} \rceil$, then the graph is $G = DNG(\underbrace{1,\ldots,1}_{\frac{n}{2}\text{ times}};\underbrace{1,\ldots,1}_{\frac{n}{2}\text{ times}})$ when $n$ is even and is $DNG(\underbrace{1,1,\ldots,1}_{\frac{n-1}{2}\text{ times}};\underbrace{1,\ldots,1}_{\frac{n-3}{2}\text{ times}},2)$ when $n$ is odd. Similarly, if $c = n-2$, then the output is the star graph $DNG(1;n-1)$. We use the following facts and proceed to obtain the realizing chain graph for all the intermediate values in the interval $(\lceil \frac{n}{2} \rceil, n-2)$.

**Remark 4.1.** *For a given $n$, even though $h$ takes the values from 1 to $\lfloor \frac{n}{2} \rfloor$, all the intermediate values for the metric dimension $c$ from $\lceil \frac{n}{2} \rceil + 1$ to $n-3$ are covered when $h$ reaches $\lfloor \frac{n}{4} \rfloor$. Thus, we reduce the length of the search interval by half in the algorithm by varying $h$ from 2 to $\lfloor \frac{n}{4} \rfloor$ instead of 2 to $\lfloor \frac{n}{2} \rfloor$.*

**Remark 4.2.** *For any two successive values $h-1$ and $h$, the bounds for $c$ are $(n-2h+2, n-h)$ and $(n-2h, n-h-1)$, respectively. Thus, for each $h$, all the values except $n-2h$ and $n-2h+1$ are already covered in the interval corresponding to $h-1$. Hence in the algorithm, for every $h$, only the first two values are taken into consideration as the next values have been covered previously. Thus the iteration is continued till the first time we encounter the value $c$. Once we encounter the first occurrence of the integer $c$ for some $h$, we give the corresponding graph.*

**Algorithm:**

| | |
|---|---|
| Input: | The integer $c$ and the number of vertices $n$ |
| Output | Returns a chain graph $G$ on $n$ vertices with $\beta(G) = c$ |

## 5. Restricted threshold dimension

For a graph with a given number of vertices, the metric basis determines the location of landmarks. But naturally the question arises if the number of landmarks can be reduced by the effect of addition/deletion of edges. The variation of metric dimension of a graph with that of its subgraphs have been studied in the literature. In the same context, the notion of threshold

**Algorithm 1** function Metric$(c, n)$

**Input:** $c, n$
**Output:** A chain graph $G$ if exists

**if** $c \notin (\lceil \frac{n}{2} \rceil, n-2)$ **then**
  **print** 'There is no chain graph $G$ on $n$ vertices with $\beta(G) = c$.'
**else if** $n \cong 0 \pmod 2$ and $c = \frac{n}{2}$ **then**
  $G = DNG(\underbrace{1, 1, \ldots, 1}_{\frac{n}{2} \text{ times}}; \underbrace{1, 1, \ldots, 1}_{\frac{n}{2} \text{ times}})$
  **return** $G$
**else if** $n \cong 1 \pmod 2$ and $c = \frac{n-1}{2}$ **then**
  $G = DNG(\underbrace{1, 1, \ldots, 1}_{\frac{n-1}{2} \text{ times}}; \underbrace{1, 1, \ldots, 1}_{\frac{n-3}{2} \text{ times}}, 2)$
  **return** $G$
**else if** $c == n-2$ **then**
  $G = DNG(1; n-1)$
**else**
  **for** $h = 2 : \lceil \frac{n}{4} \rceil$ **do**
    **if** $c \notin (n-2h, n-h-1)$ **then**
      **continue**
    **else**
      $k = c - n + 2h$
    **end if**
    **if** $k == 0$ **then**
      $G = DNG(1, \underbrace{2, 2, \ldots, 2}_{h-1 \text{ times}}; 1, \underbrace{2, 2, \ldots, 2}_{h-2 \text{ times}}, n-4h+4)$
      **return** $G$
    **else**
      $G = DNG(1, 1, \underbrace{2, \ldots, 2}_{h-2 \text{ times}}; 1, \underbrace{2, 2, \ldots, 2}_{h-2 \text{ times}}, n-4h+5)$
      **return** $G$
    **end if**
  **end for**
**end if**

dimension of a graph $G$, denoted by $\tau(G)$ is introduced. The threshold dimension of a graph $G$ is defined as

$$\tau(G) = \min\{\, \beta(H) : \text{H contains } G \text{ as a spanning subgraph} \,\}$$

A graph $G$ is called irreducible if $\beta(G) = \tau(G)$, otherwise it is reducible. Since we are restricting our attention to a particular class of graphs, we would like to note the variation of the metric dimension of a chain graph on the addition of edges without compromising the bipartiteness and nesting property of the neighborhoods. In other words, we define the threshold dimension of a graph with the restriction of maintaining the bipartiteness and nesting property and call it restricted threshold dimension. Formally, we define restricted threshold dimension of a chain graph $G$, denoted by

$\tau_r(G)$, as follows:

$\tau_r(G) = \min\{\, \beta(H) : H$ is a chain graph having $G$ as a spanning subgraph$\,\}$.

The restricted threshold dimension of a chain graph enables us to identify if the number of landmarks can be minimized on interconnecting the vertices by edges without compromising the nesting property. In this context, we call a chain graph $G$ irreducible if $\tau_r(G) = \beta(G)$.

Since half graphs on $n$ vertices attain the least value of metric dimension when $n$ is even, half graphs are irreducible. Similarly, all the graphs $DNG(1, 1, \ldots, 1; n_1, n_2, \ldots, n_h)$ where exactly one of $n_i$ $(1 \le i \le h)$ is 2 and the rest are 1 are also irreducible. In the next series of theorems, we characterize graphs having restricted threshold dimension $\lceil \frac{n}{2} \rceil - 1$. Before deriving the above-mentioned results, we prove the following theorem. For a given chain graph of order $n$, the possible edges which can be added to $G$ in order that the resultant graph is also a chain graph.

**Theorem 5.1.** *Let $G = DNG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ be a chain graph where $m_i = |U_i|$ and $n_i = |V_i|$ for $1 \le i \le h$. The graph $H$ obtained by adding an edge $e = (u, v)$ to $G$ is a chain graph if and only if $u \in U_i$ and $v \in V_{h-i+2}$ for some $2 \le i \le h$.*

*Proof.* Let $H$ be the chain graph obtained by adding an edge $e$ to the graph $G$. Without loss of generality, let $u \in U_i$ for some $1 \le i \le h$. Since $N_G(u) = V_1 \cup V_2 \cup \cdots \cup V_{h-i+1}$, it is clear that $v \in V_{h-i+k}$ for some $k \ge 2$. Suppose $k \ge 3$, then $N_G(u) = V_1 \cup V_2 \cup \cdots \cup V_{h-i+1} \cup \{v\}$ where $v \in V_{h-i+k}$ for some $k \ge 3$. Consider a vertex $x \in U_{i-1}$, then $N_H(x) = V_1 \cup V_2 \cup \cdots \cup V_{h-i+1} \cup V_{h-i+2}$. Then neither $N_H(x) \subseteq N_H(u)$ nor $N_H(u) \subseteq N_H(x)$, which is a contradiction. Thus $k = 2$.
Conversely, let $H$ be a chain graph obtained by adding an edge $e = (u, v)$ where $u \in U_i$ and $v \in V_{h-i+2}$ for some $2 \le i \le h$. Then $N_H(u) = V_1 \cup V_2 \cup \cdots \cup V_{h-i+1} \cup \{v\}$ where $v \in V_{h-i+2}$. Clearly, $N_H(U_h) \subseteq N_H(U_{h-1}) \subseteq \cdots \subseteq N_H(U_{i+1}) \subseteq N_H(u) \subseteq N_H(U_i) \subseteq \cdots \subseteq N_H(U_2) \subseteq N_H(U_1)$. □

We use the procedure given in the above theorem and add the edges to get irreducible chain graphs. In other words, we characterize the chain graphs which can be transformed into irreducible chain graphs, which in turn characterizes the graphs with restricted threshold dimension $\lceil \frac{n}{2} \rceil - 1$.

**Theorem 5.2.** *Let $DNG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ be a chain graph of order $n$, where $n$ is even. Let $\tau_r(G)$ be the restricted threshold dimension of $G$. Then $\tau_r(G) = \frac{n}{2} - 1$ if the following conditions are satisfied*

*(i).* $\sum_{i=1}^{h} m_i = \sum_{i=1}^{h} n_i = \frac{n}{2}$

*(ii).* $m_1 = n_1 = 1$

*(iii).* $\deg(v_i) - \sum_{j=i+1}^{h} |V_j| \le 1$ *for* $1 \le i \le h$ *where* $v_i \in V_i$.

*Proof.* Let $G$ be the graph satisfying the conditions given in the above theorem. Since $(i)$ is true, it follows that $|U| = |V|$ where $|U| = \sum_{i=1}^{h} m_i$ and

$|V| = \sum_{i=1}^{h} n_i$. In particular, since $(ii)$ is true, $|U_1| = |V_1| = 1$. On adding the edges in the manner given in Theorem 5.1, we make $m_i = n_i = 1$. Since $(iii)$ is satisfied, $\deg(v) \leq m_h + 1$, for all $v \in V_{h-1}$, which enable us to add edges sequentially to the vertices of $V_h$, making the vertices to have distinct degrees given by 1,2, $\ldots$, $m_h$. This results in $m_h = 1$ and adding the other vertices of $V_h$ to either new cells containing the singletons or the existing cell. Similarly, since $\deg(v) \leq m_h + m_{h-1} + 1$ for all $v \in V_{h-2}$, adding edges to the vertices of $V_{h-1}$, resulting in distinct vertex degrees $m_h + 1, m_h + 2, \ldots, m_h + m_{h-1}$. On successfully iterating this, we get a graph where each partite set has vertices of distinct degrees given by $1, 2, \ldots, \frac{n}{2}$, which results in the graph $DNG(\underbrace{1, 1, \ldots, 1}_{h \text{ times}}; \underbrace{1, 1, \ldots, 1}_{h \text{ times}})$. Since $\beta(DNG(\underbrace{1, 1, \ldots, 1}_{h \text{ times}}; \underbrace{1, 1, \ldots, 1}_{h \text{ times}})) = \frac{n}{2} - 1$ and is the least value of the metric dimension among all chain graphs on $n$ vertices, $\tau_r(G) = \frac{n}{2} - 1$. □

In the above theorem, we have characterized one class of graphs on even number of vertices, which are spanning subgraphs of half graphs. Further, we know that $\beta(DNG(\underbrace{1, 1, \ldots, 1}_{h \text{ times}}; n_1, n_2, \ldots, n_h)) = \frac{n-1}{2}$ where exactly one of $n_i$ is two and rest are one. In the next theorem, we characterize the graphs $G$ on odd number of vertices, which are spanning subgraphs of $DNG(\underbrace{1, 1, \ldots, 1}_{h \text{ times}}; n_1, n_2, \ldots, n_h)$ where exactly one of $n_i$ is 2 and rest are one. In other words, we characterize a class of chain graphs whose restricted threshold dimension is $\frac{n-1}{2}$.

**Theorem 5.3.** *Let $DNG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ be a chain graph of order $n$, where $n$ is odd. Let $\sum_{i=1}^{h} m_i = |U|$ and $\sum_{i=1}^{h} n_i = |V|$ and $\tau_r(G)$ be the restricted threshold dimension of $G$. Then $\tau_r(G) = \frac{n-1}{2}$ if the following conditions are satisfied:*

*(i)* $\left| \sum_{i=1}^{h} m_i - \sum_{i=1}^{h} n_i \right| = 1$.
*(ii)* $m_1 = 1$ *and* $n_1 \leq 2$.
*(iii) If* $|V| < |U|$, *then* $\deg(v_i) - \sum_{j=i+1}^{h} |V_j| \leq 1$ *for* $1 \leq i \leq h$, *where* $v_i \in V_i$.

*Proof.* Similar to the previous theorem, the graph satisfying the above conditions can be transformed into the graph $DNG(\underbrace{1, 1, \ldots, 1}_{h \text{ times}}; n_1, n_2, \ldots, n_h)$, where exactly one of $n_i (1 \leq i \leq h)$ is 2 and the rest are 1, in which exactly one pair of vertices in $V$ have identical degree. □

We note another class of irreducible chain graphs given by $DNG(1, \underbrace{2, \ldots, 2}_{\frac{n-2}{4} \text{ times}}; 1, \underbrace{2, \ldots, 2}_{\frac{n-2}{4} \text{ times}})$, which has metric dimension $\frac{n}{2} - 1$ (by Remark 2.5). The following theorem characterizes the chain graphs which are transformed into the above mentioned irreducible graph.

**Theorem 5.4.** *Let $DNG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$ be a chain graph of order $n = 2k$, where $k$ is an odd integer. Let $\sum_{i=1}^{h} m_i = |U|$ and $\sum_{i=1}^{h} n_i = |V|$ and $\tau_r(G)$ be the restricted threshold dimension of $G$. Then $\tau_r(G) = \frac{n}{2} - 1$ if the following conditions are satisfied.*

*(i)* $|U| = |V|$.
*(ii)* $m_1 = 1$ *and* $n_1 \leq 2$.
*(iii)* $m_h, n_h \geq 2$.
*(iv)* $\deg(v_i) - \sum_{j=i+1}^{h} |V_j| \leq \begin{cases} 1, & \text{if } \sum_{j=i+1}^{h} |V_j| \text{ is odd} \\ 0, & \text{if } \sum_{j=i+1}^{h} |V_j| \text{ is even.} \end{cases}$

*Proof.* Since $m_1 = n_1 = 1, \deg(v) = 1$ for all $v \in V_h, U_h$. Also as $m_h, n_h \geq 2$, there are at least four vertices of degree one. The degree of remaining $m_h - 2$ vertices of $V_h$ can be increased by addition of edges. Suppose $|V_h| = m_h$ is odd, since $\deg(v) - |V_h| \leq 1$ for all $v \in V_{h-1}$, it is possible to add edges sequentially to the vertices of $V_h$ which results in vertex degrees $1, 1, 3, 3, 5, 5, \ldots, m_h - 1, m_h - 1, m_h$. This make $|V_h| = 2$ and adding the other vertices of $V_h$ to either new cells of cardinality at most two or to the existing cell. Let the new cells created in every step be named $W_i's$ without affecting the original labels $V_{h-1}, V_{h-2}, \ldots, V_1$. Now consider the vertices of $V_{h-1}$, suppose $|V_{h-1}| = m_{h-1}$ is odd. Since $\deg(v) - |V_h| - |V_{h-1}| \leq 0$ for all $v \in V_{h-2}$ as $\sum_{j=h-1}^{h} |V_j|$ is even, it is possible to add edges sequentially to the vertices of $V_{h-1}$ making the vertex degrees $m_h, m_h + 2, m_h + 2, m_h + 4, m_h + 4, \ldots, m_h + m_{h-1} - 1, m_h + m_{h-1} - 1$. But, if $|V_{h-1}| = m_{h-1}$ is even, since $\deg(v) - |V_h| - |V_{h-1}| \leq 1$ for all $v \in V_{h-2}$, it is possible to add edges sequentially to the vertices of $V_{h-1}$ making the vertex degrees $m_h, m_h + 2, m_h + 2, m_h + 4, m_h + 4, \ldots, m_h + m_{h-1} - 1, m_h + m_{h-1} - 1, m_h + m_{h-1} + 1$. Similarly, suppose $|V_h| = m_h$ is even, depending upon the parity of $m_h - 1$ if it is even or odd, we add the edges in the similar procedure. Continuing this, as mentioned above, it is possible to reduce the given graph into $DNG(1, \underbrace{2, \ldots, 2}_{\frac{n-2}{4} \text{ times}}; 1, \underbrace{2, \ldots, 2}_{\frac{n-2}{4} \text{ times}})$, which has the least metric dimension $(\frac{n}{2} - 1)$. Thus $\tau_r(G) = \frac{n}{2} - 1$. □

**Conclusion.** Metric dimension, in a way, induces some more structure on graphs representing the network. In this article, we have dealt with the metric dimension of structured networks, represented by chain graphs. This contains the construction of structured networks with the given metric dimension, whenever possible and also decides when it is possible. Thus, it opens a possible new field of constructing networks with a given structure. We conclude this article with the scope of future work of characterizing all chain graphs of order $n$ with restricted threshold dimension $k$, for any integer $k > \lceil \frac{n}{2} \rceil - 1$.

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