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VL Status Index and Co-index of Connected Graphs

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Abstract

The status $\sigma_G(u)$ of a vertex u in a connected graph G is defined as the sum of the distances between u and all other vertices of G. In this paper some relations over VL status index and VL status co-index of connected graphs are established. Furthermore distinguished examples for k-transmission regular graphs and nanostructures of VL status indices are computed.

Keywords: Status index; Degree; Connected graph; Topological indices.
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1 Introduction and Preliminaries

A topological (graph) index is a graph invariant which is mathematically derived from the graph structure. The graph theoretic models can be used to study the properties of molecules in theoretical chemistry. The oldest well known index is the Wiener index which was used to study the chemical properties of paraffin (cf. [20]). Also one of the most popular index is the Zagreb index that were used, for instance, to study the structural property models (see, for instance, [1,8,17–19]).

Let G be a connected graph with n vertices and m edges. Suppose that $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G)$ are the vertex and edge sets of G. In general the notation uv denotes the edge joining of the vertices u and v, and also $d_G(u)$ denotes the degree of a vertex u in a graph G which is the number of edges joining to u. Moreover the distance between the vertices u and v is the length of the shortest path joining u and v and is denoted by $d_G(u,v)$. For graph theoretical terminology, we may refer the books [4,5].

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In the following we will recall some topological indices that will be needed in this study.

The *status* ([9]) of a vertex u is defined as the sum of its distances from every other vertex of G and is denoted by

$$\sigma(u) = \sum_{uv \in E(G)} d(u,v).$$

In fact by considering the status of u, one may define the *Wiener index* ([20]) of a connected graph G as

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v) = \frac{1}{2} \sum_{u \in V(G)} \sigma(u).$$

On the other hand, the first and second *Zagreb indices* ([8]) of a graph G are defined by

$$M_1(G) = \sum_{uv \in E(G)} [d(u) + d(v)] \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} [d(u) \cdot d(v)].$$

while the first and second *Zagreb co-indices* (see [1,19]) are defined by

$$\overline{\mathrm{M_1}}(G) = \sum_{uv \notin E(G)} [d(u) + d(v)] \quad \text{and} \quad \overline{\mathrm{M_2}}(G) = \sum_{uv \notin E(G)} [d(u) \cdot d(v)].$$

Recently, the *first* and *second status connectivity index* ([16]) of a graph G have been introduced to study the property of benzenoid hydrocarbons which are defined by

$$S_1(G) = \sum_{uv \in E(G)} [\sigma(u) + \sigma(v)] \quad \text{and} \quad S_2(G) = \sum_{uv \in E(G)} [\sigma(u) \cdot \sigma(v)].$$

With a similar idea as in above equalities, the *first* and *second status connectivity co-index* are defined by

$$\overline{\mathrm{S_1}}(G) = \sum_{uv \notin E(G)} [\sigma(u) + \sigma(v)] \quad \text{and} \quad \overline{\mathrm{S_2}}(G) = \sum_{uv \notin E(G)} [\sigma(u) \cdot \sigma(v)].$$

In [7], it has been also recently introduced the *VL index*

$$VL(G) = \sum_{uv \in E(G)} [d_e + d_f + 4],$$

where $d_e = d(u) + d(v) - 2$ and $d_f = (d(u).d(v)) - 2$ of a graph G. As it mentioned in [7], the VL index figures out a good correlation with the physical properties of octane isomers and polychlorinated biphenyl. Presently, in [13], two more indices introduced and studied their graph theoretical properties under the name of the *VL status index $VLS(G)$* and *VL status co-index* $\overline{\mathrm{VLS}}(G)$ of a graph G that are defined by

$$VLS(G) = \frac{1}{2} \sum_{uv \in E(G)} [\sigma(u) + \sigma(v) + \sigma(u) \cdot \sigma(v)] \tag{1}$$

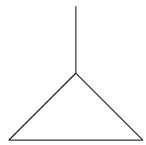

Figure 1: $VLS(G) = 42.5$ and $\overline{\text{VLS}}(G) = 29$.

and

$$\overline{\text{VLS}}(G) = \frac{1}{2} \sum_{uv \notin E(G)} [\sigma(u) + \sigma(v) + \sigma(u) \cdot \sigma(v)], \quad (2)$$

respectively. We may refer Figure 1 for a simple example of $VLS(G)$ and $\overline{\text{VLS}}(G)$.

This paper is organized as follows: Section 1 covers some reminders about the indices that will be needed at the remaining part. In Section 2, there will be given some results related to VL status (co-)index in terms of different parameters and indices. The final section will be constructed on the results about VL status index and co-index of some transmission regular graphs and nanostructures.

2 VL status index and coindex results

In this section, by considering the indices given in Eqs. (1) and (2), we state and prove some results on connected graphs.

Theorem 2.1. *Let G be a connected graph on n vertices. Then*

$$\overline{\text{VLS}}(G) = (n-1)W(G) + (W(G))^2 - \frac{1}{4} \sum_{u \in V(G)} (\sigma(u))^2 - VLS(G).$$

Proof. By Eq. (2), we have

$$\begin{aligned}
\overline{\text{VLS}}(G) &= \frac{1}{2} \sum_{uv \notin E(G)} [\sigma(u) + \sigma(v) + \sigma(u) \cdot \sigma(v)] \\
&= \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} [\sigma(u) + \sigma(v) + \sigma(u) \cdot \sigma(v)] \\
&\quad - \frac{1}{2} \sum_{uv \in E(G)} [\sigma(u) + \sigma(v) + \sigma(u) \cdot \sigma(v)] \\
&= \frac{1}{2}(n-1) \sum_{u \in V(G)} \sigma(u) + \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} \sigma(u) \cdot \sigma(v) - VLS(G) \\
&= (n-1)W(G) + \frac{1}{2}\left[2(W(G))^2 - \frac{1}{2} \sum_{u \in V(G)} (\sigma(u))^2\right] - VLS(G) \\
&= (n-1)W(G) + (W(G))^2 - \frac{1}{4} \sum_{u \in V(G)} (\sigma(u))^2 - VLS(G).
\end{aligned}$$

Hence we obtain the required equality in the statement of theorem. □

Theorem 2.2. *Let G be a connected graph with n vertices and m edges. Then*

$$VLS(G) = mn(2n-2) + \frac{(1-2n)}{2} M_1(G) + \frac{M_2(G)}{2}.$$

Proof. For any vertex u of G, there exist $d(u)$ vertices which are at distance 1 from u and the remaining $(n-1-d(u))$ vertices are at distance at least 2. Therefore $\sigma(u) = 2n-2-d(u)$. After that by Eq. (1), we get

$$\begin{aligned} VLS(G) &= \frac{1}{2} \sum_{uv \in E(G)} [\sigma(u) + \sigma(v) + \sigma(u) \cdot \sigma(v)] \\ &= \frac{1}{2} \sum_{uv \in E(G)} [2n-2-d(u) + 2n-2-d(v) + (2n-2-d(u)) \\ &\quad \cdot (2n-2-d(v))] \end{aligned}$$

which actually gives

$$\begin{aligned} VLS(G) &\leq \frac{1}{2} \sum_{uv \in E(G)} [4n^2 - 4n + (1-2n)(d(u)+d(v)) + d(u)d(v)] \\ &= mn(2n-2) + \frac{(1-2n)}{2} M_1(G) + \frac{M_2(G)}{2}, \end{aligned}$$

as required. □

Corollary 2.3. *Let G be a connected graph with n vertices, m edges and $diam(G) \leq 2$. Then*

$$\overline{\text{VLS}}(G) = n^2(n^2-2n+1) + m(m-1) + 5mn - 4mn^2 - \left(\frac{3}{4} - n\right) M_1(G) - \frac{M_2(G)}{2}.$$

Proof. For any graph G with $diam(G) \leq 2$, the status $\sigma(u) = 2n-2-d(u)$ and so the Wiener index is

$$W(G) = m + 2\left[\frac{n(n-1)}{2} - m\right] = n(n-1) - m.$$

Also the first and second status connectivity indices are $S_1(G) = 4m(n-1) - M_1(G)$ and $S_2(G) = 4m(n-1)^2 - 2(n-1)M_1(G) + M_2(G)$. Therefore, by Theorem 2.1, we get

$$\begin{aligned} \overline{\text{VLS}}(G) &= (n-1)[n(n-1)-m] + [n(n-1)-m]^2 \\ &\quad - \frac{1}{4} \sum_{u \in V(G)} (2n-2-d(u))^2 \frac{1}{2} [S_1(G) + S_2(G)] \\ &= (n-1)[n(n-1)-m] + [n(n-1)-m]^2 \\ &\quad - \frac{1}{4} [n(2n-2)^2 - 4m(2n-2) + M_1(G)] \\ &\quad - \frac{1}{2} [4m(n-1) - M_1(G) + 4m(n-1)^2 - 2(n-1)M_1(G) + M_2(G)] \end{aligned}$$

which imply

$$= n^4 - 2n^3 + m^2 + n^2 - m + 5mn - 4mn^2 - \left(\frac{3}{4} - n\right)M_1(G) - \frac{M_2(G)}{2}$$
$$= n^2(n^2 - 2n + 1) + m(m-1) + 5mn - 4mn^2 - \left(\frac{3}{4} - n\right)M_1(G) - \frac{M_2(G)}{2}.$$

Hence the result. □

The above corollary can also be obtained in terms of Zagreb co-indices as in the following .

Corollary 2.4. *Let G be a connected graph with n vertices, m edges and $diam(G) \leq 2$. Then*

$$\overline{\mathrm{VLS}}(G) = n(n-1)[n(n-1) - 2m] - \left(n - \frac{1}{2}\right)\overline{\mathrm{M}_1}(G) + \frac{\overline{\mathrm{M}_2}(G)}{2}.$$

Proof. As we done in the proof of Corollary 2.3, the status is given by $\sigma(u) = 2n - 2 - d(u)$ for any graph G with $diam(G) \leq 2$. Thus, by considering Eq. (2), we get

$$\begin{aligned}
\overline{\mathrm{VLS}}(G) &= \frac{1}{2}\sum_{uv \notin E(G)} [\sigma(u) + \sigma(v) + \sigma(u)\cdot\sigma(v)] \\
&= \frac{1}{2}\sum_{uv \notin E(G)} [2n - 2 - d(u) + 2n - 2 - d(v) + (2n - 2 - d(u)) \\
&\quad \cdot (2n - 2 - d(v))] \\
&= \frac{1}{2}\Bigg[\left(\frac{n(n-1)}{2} - m\right)(4n-4) - \sum_{uv \notin E(G)} (d(u) + d(v)) + \left(\frac{n(n-1)}{2} - m\right) \\
&\quad (2n-2)^2 - (2n-2)\sum_{uv \notin E(G)} (d(u) + d(v)) + \sum_{uv \notin E(G)} (d(u)\cdot d(v))\Bigg] \\
&= \frac{1}{2}[2(n-1)(n(n-1) - 2m) - \overline{\mathrm{M}_1}(G) + 2(n-1)^2(n(n-1) - 2m) \\
&\quad - 2(n-1)\overline{\mathrm{M}_1}(G) + \overline{\mathrm{M}_2}(G)] \\
&= n(n-1)[n(n-1) - 2m] - \left(n - \frac{1}{2}\right)\overline{\mathrm{M}_1}(G) + \frac{\overline{\mathrm{M}_2}(G)}{2}
\end{aligned}$$

as required. □

Corollary 2.5. *For a graph G with n vertices and m edges, let $\overline{\mathrm{G}}$ be the connected complement of G. Then*

$$VLS(\overline{\mathrm{G}}) \geq \frac{1}{2}\left[\left(\frac{n(n-1)}{2} - m\right)(n^2 - 1) + n\overline{\mathrm{M}_1}(G) + \overline{\mathrm{M}_2}(G)\right].$$

Equality holds if and only if $diam(\overline{\mathrm{G}}) \leq 2$.

Proof. For any vertex u in $\overline{\mathrm{G}}$ there are $n-1-d_G(u)$ vertices which are at distance 1 and the remaining $d_G(u)$ vertices are at least 2. Therefore

$$\sigma_{\overline{\mathrm{G}}}(u) \geq [n-1-d_G(u)] + 2d_G(u) = n-1+d_G(u).$$

Therefore

$$\begin{aligned}
VLS(\overline{\mathrm{G}}) &= \frac{1}{2}\sum_{uv\in E(\overline{\mathrm{G}})} [\sigma_{\overline{\mathrm{G}}}(u) + \sigma_{\overline{\mathrm{G}}}(v) + \sigma_{\overline{\mathrm{G}}}(u)\cdot\sigma_{\overline{\mathrm{G}}}(v)] \\
&\geq \frac{1}{2}\sum_{uv\in E(\overline{\mathrm{G}})} [n-1+d_G(u)+n-1+d_G(v)+(n-1+d_G(u)) \\
&\quad \cdot (n-1+d_G(v))] \\
&= \frac{1}{2}\sum_{uv\notin E(G)} n^2-1+n(d_G(u)+d_G(v))+d_G(u)\cdot d_G(v) \\
&= \frac{1}{2}\Bigg[\left(\frac{n(n-1)}{2}-m\right)(n^2-1)+n\sum_{uv\notin E(G)}(d_G(u)+d_G(v)) \\
&\quad + \sum_{uv\notin E(G)}(d_G(u)\cdot d_G(v))\Bigg] \\
&= \frac{1}{2}\left[\left(\frac{n(n-1)}{2}-m\right)(n^2-1)+n\overline{\mathrm{M}_1}(G)+\overline{\mathrm{M}_2}(G)\right].
\end{aligned}$$

We note that if the diameter of $\overline{\mathrm{G}}$ is 1 or 2, then the equality holds.

Conversely, let $VLS(\overline{\mathrm{G}}) = \frac{1}{2}\left[\left(\frac{n(n-1)}{2}-m\right)(n^2-1)+n\overline{\mathrm{M}_1}(G)+\overline{\mathrm{M}_2}(G)\right]$ and suppose that $diam(\overline{\mathrm{G}}) \geq 3$. Then there exists at least one pair of vertices, say u_1 and u_2 such that $d_{\overline{\mathrm{G}}}(u_1,u_2) \geq 3$. Therefore $\sigma_{\overline{\mathrm{G}}}(u_1) \geq d_{\overline{\mathrm{G}}}(u_1)+3+2(n-2-d_{\overline{\mathrm{G}}}(u_1)) = n+d_G(u_1)$. Similarly $\sigma_{\overline{\mathrm{G}}}(u_2) \geq n+d_G(u_2)$ and for all other vertices u of $\overline{\mathrm{G}}$, $\sigma_{\overline{\mathrm{G}}}(u) \geq n-1+d_G(u)$. Partition the edge set of $\overline{\mathrm{G}}$ into three sets E_1, E_2 and E_3, where

$$\begin{aligned}
E_1 &= \{u_1v \mid \sigma_{\overline{\mathrm{G}}}(u_1) \geq n+d_G(u_1) \quad and \quad \sigma_{\overline{\mathrm{G}}}(v) \geq n-1+d_G(v)\}, \\
E_2 &= \{u_2v \mid \sigma_{\overline{\mathrm{G}}}(u_2) \geq n+d_G(u_2) \quad and \quad \sigma_{\overline{\mathrm{G}}}(v) \geq n-1+d_G(v)\} \quad \text{and} \\
E_3 &= \{uv \mid \sigma_{\overline{\mathrm{G}}}(u) \geq n-1+d_G(u) \quad and \quad \sigma_{\overline{\mathrm{G}}}(v) \geq n-1+d_G(v)\}.
\end{aligned}$$

It is easy to check that $|E_1| = d_{\overline{\mathrm{G}}}(u_1)$, $|E_2| = d_{\overline{\mathrm{G}}}(u_2)$ and $|E_3| = \binom{n}{2}-m-d_{\overline{\mathrm{G}}}(u_1)-d_{\overline{\mathrm{G}}}(u_2)$. Thus

$$\begin{aligned}
VLS(\overline{\mathrm{G}}) &= \frac{1}{2}\sum_{uv\in E(\overline{\mathrm{G}})} [\sigma_{\overline{\mathrm{G}}}(u) + \sigma_{\overline{\mathrm{G}}}(v) + \sigma_{\overline{\mathrm{G}}}(u)\cdot\sigma_{\overline{\mathrm{G}}}(v)] \\
&= \frac{1}{2}\sum_{uv\in E_1} [\sigma_{\overline{\mathrm{G}}}(u) + \sigma_{\overline{\mathrm{G}}}(v) + \sigma_{\overline{\mathrm{G}}}(u)\cdot\sigma_{\overline{\mathrm{G}}}(v)] \\
&\quad + \frac{1}{2}\sum_{uv\in E_2} [\sigma_{\overline{\mathrm{G}}}(u) + \sigma_{\overline{\mathrm{G}}}(v) + \sigma_{\overline{\mathrm{G}}}(u)\cdot\sigma_{\overline{\mathrm{G}}}(v)] \\
&\quad + \frac{1}{2}\sum_{uv\in E_3} [\sigma_{\overline{\mathrm{G}}}(u) + \sigma_{\overline{\mathrm{G}}}(v) + \sigma_{\overline{\mathrm{G}}}(u)\cdot\sigma_{\overline{\mathrm{G}}}(v)]
\end{aligned}$$

which gives

$$= \frac{1}{2}\sum_{uv\in E_1}[n+d_G(u)+n-1+d_G(v)+(n+d_G(u))\cdot(n-1+d_G(v))]$$
$$+\frac{1}{2}\sum_{uv\in E_2}[n+d_G(u)+n-1+d_G(v)+(n+d_G(u))\cdot(n-1+d_G(v))]$$
$$+\frac{1}{2}\sum_{uv\in E_3}[n-1+d_G(u)+n-1+d_G(v)+(n-1+d_G(u))$$
$$\cdot(n-1+d_G(v))]$$
$$= \frac{1}{2}\Big[\frac{n}{2}(n^3-n^2-n+1)+m(1-n^2)+n(d_{\overline{G}}(u_1)+d_{\overline{G}}(u_2))$$
$$+d_G(v)(d_{\overline{G}}(u_1)+d_{\overline{G}}(u_2))+n\overline{M_1}(G)+\overline{M_2}(G)\Big]\ .$$

The above process gives a contradiction. Hence $diam(\overline{G})\leq 2$. □

3 *VL* status index and co-index of some transmission regular graphs

A bijection α on $V(G)$ is called an *automorphism* of G if it preserves $E(G)$. In other words α defines an automorphism if for each $u,v\in V(G)$, we have $e=uv\in E(G)$ if and only if $\alpha(e)=\alpha(u).\alpha(v)\in E(G)$. Let $Aut(G)=\{\alpha\,|\,\alpha:V(G)\to V(G)$ is a bijection which preserves the adjacency$\}$. It is known $Aut(G)$ forms an algebraic group under the composition of mappings. On the other hand a graph G is called *vertex-transitive* if for every two vertices u and v of G, there exists an automorphism α of G such that $\alpha(u)=\alpha(v)$. It is known that any vertex-transitive graph is vertex degree regular, transmission regular and self-centered. Indeed the graph depicted in Figure 2 is 14-transmission regular graph but not degree regular and therefore not vertex-transitive (see [2]).

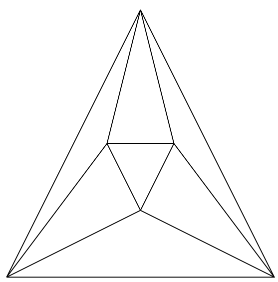

Figure 2: The transmission regular but not degree regular graph with the smallest order

Lemma 3.1 ([15, 16]). *Let G be a connected k-transmission regular graph with m edges. Then*

$$S_1(G)=2mk \quad and \quad S_2(G)=mk^2.$$

Lemma 3.2. *For a connected k-transmission regular graph G with m edges,* $VLS(G) = \frac{m}{2}(k^2 + 2k)$.

Proposition 3.3 ([3]). *Let G be a connected graph on n vertices with the automorphism group* $Aut(G)$ *and the vertex set* $V(G)$*. Let* $V_1, V_2, \ldots, V_t$ *be all orbits of the action* $Aut(G)$ *on* $V(G)$*. Suppose also that for each* $1 \leq i \leq t$*, each* k_i *is the transmission of vertex in the orbit* V_i*. Then*

$$W(G) = \frac{1}{2}\sum_{i=1}^{t}|V_i|k_i.$$

Specially if G is vertex-transitive (i.e. $t = 1$*), then* $W(G) = \frac{1}{2}nk$*, where k denotes the transmission of each vertex of G.*

Proposition 3.4 ([15, 16]). *Let G be a connected graph on n vertices with the automorphism group* $Aut(G)$ *and the vertex set* $V(G)$*. Let* $V_1, V_2, \ldots, V_t$ *be all orbits of the action* $Aut(G)$ *on* $V(G)$*. Suppose also that for each* $1 \leq i \leq t$*,* d_i *and* k_i *are the vertex degree and the transmission of vertices in the orbit* V_i*, respectively. Then*

$$S_1(G) = \sum_{i=1}^{t}|V_i|d_ik_i, \quad and \quad \overline{S_1}(G) = \sum_{i=1}^{t}\left(|V_i|k_i\left(1 - \frac{d_i}{n-1}\right)\right).$$

Specially if G is vertex-transitive (i.e. $t = 1$*), then*

$$S_1(G) = ndk, \qquad S_2(G) = \tfrac{1}{2}ndk^2,$$
$$\overline{S_1}(G) = 2\tbinom{n}{2}k - ndk, \qquad \overline{S_2}(G) = \left(\tbinom{n}{2} - \tfrac{nd}{2}\right)k^2,$$

where d and k are the degree and the transmission of each vertex of G, respectively.

The following result follows from Proposition 3.4.

Theorem 3.5. *Let G be a connected graph on n vertices with the automorphism group* $Aut(G)$ *and the vertex set* $V(G)$*. Let d and k are the degree and the transmission of each vertex of G, respectively. Then*

$$VLS(G) = \frac{ndk}{4}(k+2) \quad and \quad \overline{\text{VLS}}(G) = \frac{ndk}{4}(k+2).$$

The following is a direct consequence of Theorem 2.1, Lemma 3.2 and Theorem 3.5.

Corollary 3.6. *Let G be a connected k-transmission regular graph with m edges. Then*

$$\overline{\text{VLS}}(G) = \frac{1}{4}[nk(n-1)(2+k) - 2mk(2+k)].$$

The vertex set of the *hypercube* H_n consists of all n-tuples $(b_1, b_2, \ldots, b_n)$ with $b_i \in \{0, 1\}$, and any two vertices are adjacent if the corresponding tuples differ in precisely one place. Moreover H_n has exactly 2^n vertices and $n2^{n-1}$ edges. In [6] it has been proved that H_n is vertex-transitive and $\sigma_{H_n}(u) = n2^{n-1}$ for every vertex u.

Theorem 3.7. *For a hypercube H_n, we obtain*

$$VLS(H_n) = \frac{n^2}{2}(n2^{3n-3} + 2^{2n-1}) \quad and$$
$$\overline{\text{VLS}}(H_n) = \frac{n^2}{2}[2^n(2n-5) + 2^{2n-2}(2n^2 - n - 1)].$$

The *Kneser graph $KG_{p,k}$* is the graph whose vertices correspond to the k-element subsets of a set of p elements, and any two vertices are adjacent if and only if the two corresponding sets are disjoint. Clearly we must impose the restriction $p \geq 2k$. The Kneser graph $KG_{p,k}$ has $\binom{p}{k}$ vertices and it is actually regular of degree $\binom{p-k}{k}$. Therefore, by [14], the number of edges of $KG_{p,k}$ is $\frac{1}{2}\binom{p}{k}\binom{p-k}{k}$. Moreover the Kneser graph $KG_{n,1}$ is complete on n vertices, and $KG_{8,1}$ is known as 8-complete graph (see Figure 3).

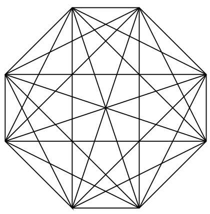

Figure 3: 8-complete graph

Lemma 3.8 ([14]). *The Kneser graph $KG_{p,k}$ is vertex-transitive and for each k-subset A, there exists*

$$\sigma_{KG_{p,k}}(A) = \frac{2W(KG_{p,k})}{\binom{p}{k}}.$$

The next result follows from Lemma 3.8.

Theorem 3.9. *For a Kneser graph $KG_{p,k}$, we have*

$$VLS(KG_{p,k}) = \frac{1}{2}\left[2W(KG_{p,k})\binom{p-k}{k} + \binom{p-k}{k}\left(\frac{2(W(KG_{p,k}))^2}{\binom{p}{k}}\right)\right]$$

and

$$\overline{\text{VLS}}(KG_{p,k}) = \frac{1}{2}\Bigg[2W(KG_{p,k})\left(\binom{p}{k} - \binom{p-k}{k} - 1\right) + 2(W(KG_{p,k}))^2$$
$$- W(KG_{p,k}) - \binom{p-k}{k}\left(\frac{2(W(KG_{p,k}))^2}{\binom{p}{k}}\right)\Bigg].$$

A nanostructure is called achiral polyhex nanotorus $T[p,q]$ of perimeter p and length q. In fact $T[p,q]$ is regular of degree 3 and has pq vertices, $\frac{3pq}{2}$ edges. Inspired by the work on Operation of nanostructures via SDD, ABC_4 and GA_5 indices [10], *SK* indices, forgotten topological indices and hyper Zagreb index of Q operator of carbon nanocone [11] and some computational aspects of carbon nanocone using $Q(G)$ operator, hexagonal network and probabilistic neural network [12], we get the following result.

Lemma 3.10 ([3, 21]). *The achiral polyhex nanotorus $T = T[p,q]$ is vertex transitive such that for an arbitrary vertex $u \in V(T)$, we have*

$$\sigma_T(u) = \begin{cases} \frac{q}{12}(6p^2 + q^2 - 4), & q < p\,, \\ \frac{p}{2}(p^2 + 3q^2 + 3pq - 4), & q \geq p\,. \end{cases}$$

The following is a direct consequence of Lemmas 3.2 and 3.10.

Theorem 3.11. *Let $T = T[p,q]$ be a achiral polyhex nanotorus. Then $VLS(T)$ is equal to*

$$\begin{cases} \frac{pq}{192}[(6p^2 + q^2 - 4)(q^4 - 4q^2 + 24q + 6p^2q^2)], & q < p\,, \\ \frac{pq}{2}[(p^2 + 3q^2 + 3pq - 4)(p^4 - 4p^2 + 24p + 3p^2q^2 + 3p^3q)], & q \geq p\,. \end{cases}$$

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