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GROUP PSEUDOREPRESENTATIONS THAT ARE TRIVIAL ON A NORMAL SUBGROUP

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Abstract. Continuing the study of general group pseudorepresentations, we prove that, if π is a pseudorepresentation of a group G in a Banach space E with sufficiently small defect and if N is a normal subgroup of G for which $\pi(n) = 1|_E$ for all $n \in N$, then there is a pseudorepresentation ρ of the quotient group G/N such that the pseudorepresentation π is locally equivalent to the pseudorepresentation of G defined by the rule $g \mapsto \rho(gN)$, $g \in G$.

§ 1. Introduction

For the definitions, notation, and generalities concerning pseudorepresentations, see [1–4]. Recall that a mapping π of a given group G into the family of invertible operators in the algebra $\mathcal{L}(E)$ of bounded linear operators on a Banach space E is said to be a *quasirepresentation* (an ε-quasirepresentation) of G on E if $\pi(e_G) = 1_E$, where e_G stands for the identity element of G and 1_E for the identity operator on E, and if

$$\|\pi(g_1g_2) - \pi(g_1)\pi(g_2)\|_{\mathcal{L}(E)} \le \varepsilon, \qquad g_1, g_2 \in G,$$

for some ε, which is usually assumed to be sufficiently small and its greatest lower bound for π is referred to as the *defect* of π; an ε-quasirepresentation π of G on a Banach space E is said to be a *pseudorepresentation* of G (to be

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more precise, an ε-θ-pseudorepresentation of G on E, where $\theta < 1$) if there is a continuous linear operator $A(g,n)$ on E such that

(1)
$$A(g,n)\pi(g^n) = \pi(g)^n A(g,n), \quad g \in G,\ n \in \mathbb{Z}, \text{ where } \|A(g,n) - 1_E\|_{\mathcal{L}(E)} \le \theta.$$

Below we omit the subscript $\mathcal{L}(E)$ for the operator norms as a rule.

If a pseudorepresentation π of G is one-dimensional and has a sufficiently small defect (less than 0.24) and if π is trivial on a normal subgroup N of G (i.e., $\pi(n) = 1 \in \mathbb{C}$ for every $n \in N$, where $\mathbb{C}$ stands for the field of complex numbers), then, as was shown in [6], the one-dimensional pseudorepresentation π is completely determined by some one-dimensional pseudorepresentation of the quotient group G/N. Here we obtain a weaker result for general pseudorepresentations. For conditions ensuring that π is trivial on a normal subgroup N of G, see [7].

§ 2. Preliminaries

The following result can be proved by repeating the proof of the similar result in [8] almost verbatim, omitting references to topology and continuity and using an invariant mean instead of integration.

Theorem 1. *Let $\varepsilon < 1$. Let G be an amenable group. Let π be a bounded δ-quasirepresentation of G on a reflexive Banach space E with defect δ, let $\|\pi(g)\| \le C$ and $\|\pi(g)^{-1}\| \le C$ for any $g \in G$, and let $\delta \le ((6 + 4C + 2C^2)^{-1}\varepsilon$, where $\varepsilon < 1$. Assume that $\pi(e) = 1_E$. Then there is an ordinary representation ρ of G on E such that $\|\rho(g) - \pi(g)\| \le \varepsilon$ for all $g \in G$.*

Lemma 1. *Let U and V be invertible operators on a dual Banach space E such that U is dual and $\|V^n\| \le C$ for all $n \in \mathbb{Z}$ and $\|U^n - V^n\| < \varepsilon/C$ for all $n \in \mathbb{Z}$, where $\varepsilon < 1$. Then there is an operator A on E such that $\|A - 1_E\| \le \varepsilon$ and $UA = AV$, i.e., $U = AVA^{-1}$. In particular, $\|U - V\| \le 2\varepsilon C/(1-\varepsilon)$.*

Proof. It follows from the condition immediately that $\|U^nV^{-n} - 1_E\| < \varepsilon$. Applying any invariant mean μ on $\mathbb{Z}$ to $f(U^nV^{-n}x)$ for every $x \in E$ and $f \in E_*$, where E_* stands for a predual space of E, we obtain a bilinear form on $E \times E_*$, which defines a continuous linear operator $A \in \mathcal{L}(E)$ such that $\|A - 1_E\| \le \varepsilon$ since μ is a mean. Since $\mu(f(U^{n+1}V^{-n-1}x)) = \mu(f(U^nV^{-n}x))$ by the invariance of μ on $\mathbb{Z}$, it follows that $UAV^{-1} = A$, which implies that $U = AVA^{-1}$. Finally, $\|U - V\| \le \|AVA^{-1} - AV\| + \|AV - V\| \le \|AV\|(\varepsilon/(1-\varepsilon)) + \varepsilon C \le (1+\varepsilon)C(\varepsilon/(1-\varepsilon)) + \varepsilon C = 2\varepsilon C/(1-\varepsilon)$, as was to be proved.

The following remark is a version of Theorem 5.4 of [1].

Remark 1. Let $\varepsilon < 1$. Let G be a group. Let π be a bounded δ-quasirepresentation of G in a reflexive Banach space E with defect δ, let $\|\pi(g)\| \le C$ and $\|\pi(g)^{-1}\| \le C$ for any $g \in G$, and let $\delta \le ((12+8C+4C^2)^{-1}\varepsilon$, where $C\varepsilon < 1$. Assume that $\pi(e) = 1_E$. Then there is a pseudorepresentation ρ of G on E such that $\|\rho(g) - \pi(g)\| \le \varepsilon/2$ for all $g \in G$ and, for all $g \in G$ and $n \in \mathbb{Z}$, there is an $A(g,n) \in \mathcal{L}(E)$ for which $A(g,n)\rho(g^n) = \rho(g)^n A(g,n)$ and $\|A(g,n) - 1_E\| \le C\varepsilon$ for all $g \in G$ and $n \in \mathbb{N}$. In particular, $\|\rho(g^n) - \rho(g)^n\| \le 2\varepsilon C^2/(1 - C\varepsilon)$ for all $g \in G$ and $n \in \mathbb{N}$.

Proof. Let π be a δ-quasirepresentation of G on E by invertible operators such that $\|\pi(g)\| \le C$ and $\|\pi(g)^{-1}\| \le C$ for all $g \in G$. Choose an element $g \in G$. Let G_g be the subgroup of G generated by g; then it is Abelian, and hence amenable; let μ be an invariant mean on G_g. Applying Theorem 1, we see that there is a representation $\rho_1 = \rho(g, G_g, \mu)$ of the subgroup G_g on E such that $\|\rho_1(g) - \pi(g)\| \le \varepsilon/2$ for all $g \in G$. Define the operator $\rho(g)$ for every $g \in G$ by the rule $\rho(g) = \rho_1(g)$. Let $k \in \mathbb{Z}$. For g^k, the corresponding representation is $\rho_k = \rho(g^k, G_{g^k}, \mu)$ (of the subgroup G_{g^k}), and thus $\rho(g^k) = \rho_k(g^k)$. Since the restriction of the representation ρ to G_{g^k} and the representation ρ_k define representations of a cyclic group G_{g^k} and $\|\rho(g^{kn}) - \pi(g^{kn})\| \le \varepsilon/2$ for all $k, n \in \mathbb{Z}$ and $\|\rho_k(g^{kn}) - \pi(g^{kn})\| \le \varepsilon/2$ for all $k, n \in \mathbb{Z}$, it follows that $\|\rho_k(g^{kn}) - \rho(g^{kn})\| \le \varepsilon$ for all $k, n \in \mathbb{Z}$, and thus the restrictions of the representations ρ_k and ρ to G_k satisfy the conditions of Lemma 1. Therefore, there is an operator $A(g,n) \in \mathcal{L}(E)$ for which $A(g,n)\rho(g^n) = \rho(g)^n A(g,n)$ and $\|A(g,n) - 1_E\| \le C\varepsilon$, as was to be proved. The inequality for $\|\rho(g^n) - \rho(g)^n\|$ follows from Lemma 1.

Thus, under the assumptions of Remark 1, there is an ε-θ-pseudorepresentation of G approximating the given δ-quasirepresentation with the accuracy $\varepsilon/2$, where $\delta \le ((12+8C+4C^2)^{-1}\varepsilon$ and $\theta = C\varepsilon$.

§ 3. Main result

We need the following definition.

Definition 1. Let G be a group and let π and ρ be pseudorepresentations of G in Banach spaces E_π and E_ρ, respectively. The pseudorepresentations π and ρ are said to be *locally equivalent* if there is a family of bounded linear operators $\{A(g), g \in G\}$ with bounded inverse such that $A(g)\colon E_\pi \to E_\rho$ and $A(g)\pi(g) = \rho(g)A(g)$ for all $g \in G$.

Obviously, if π and ρ are finite-dimensional and locally equivalent, then their characters coincide [8].

Theorem. *Let G be a group, let N be a normal subgroup of G, let π be a δ-θ-pseudorepresentation of G on a reflexive Banach space E such that $\|\pi(g)\| \le C$ and $\|\pi(g)^{-1}\| \le C$ for all $g \in G$, and let $\pi(n) = 1_E$ for every $n \in N$. Let $\kappa\colon G \to G/N$ be the canonical epimorphism. If δ is sufficiently small ($C(\varepsilon/2+\delta) < 1$, where $2\delta \le (12+8C+4C^2)^{-1}\varepsilon$ and $C\varepsilon < 1$), then there is a pseudorepresentation ρ of G/N on E such that the pseudorepresentations π and $\rho \circ \kappa$ are locally equivalent.*

Proof. Let $s\colon G/N \to G$ be a section of the canonical epimorphism κ, i.e., $\kappa(s(gN)) = gN$ for every $g \in G$. Introduce a mapping $\sigma\colon G/N \to \mathcal{L}(E)$ by the rule $\sigma(gN) = \pi(s(gN))$ for every $g \in G$; it is clear that

$$\begin{aligned}|\pi(s(g_1N))\pi(s(g_2N)) - \pi(s(g_1g_2N)| \le |\pi(s(g_1N))\pi(s(g_2N))\\ - \pi(s(g_1N)s(g_2N))| + |\pi(s(g_1N)s(g_2N)) - \pi(s(g_1g_2N))|,\end{aligned}$$

where $s(g_1g_2N) = s(g_1N)s(g_2N)n(g_1,g_2)$ for some $n = n(g_1,g_2) \in N$, and therefore

$$\begin{aligned}&|\pi(s(g_1N))\pi(s(g_2N)) - \pi(s(g_1g_2N)|\\ &\quad \le \delta + ||\pi(s(g_1N)s(g_2N)) - \pi(s(g_1N)s(g_2N)n)| \le 2\delta,\end{aligned}$$

which means that $|\sigma(g_1N)\sigma(g_2N) - \sigma(g_1g_2N)| \le 2\delta$ for all $g_1, g_2 \in G$, and therefore σ is a 2δ-quasirepresentation of G/N on E. Let ρ be a pseudorepresentation of G/N on E corresponding to the quasirepresentation σ and some invariant mean on $\mathbb{Z}$. Then, according to Remark 1, ρ is an ε-θ-pseudorepresentation of G approximating the 2δ-quasirepresentation σ with the accuracy $\varepsilon/2$, where $2\delta \le ((12+8C+4C^2)^{-1}\varepsilon$ and $\theta = C\varepsilon$. In particular, $|\rho(gN) - \sigma(gN)| \le \varepsilon/2$ for all $g \in G$. In turn, $|\sigma(gN) - \pi(g)| = |\pi(s(g)) - \pi(g)| = |\pi(gn(g)) - \pi(g)| \le \delta$, and thus, for every $g \in G$, we have $|\rho(\kappa(g)) - \pi(g)| \le |\rho(gN) - \sigma(gN)| + |\sigma(gN) - \pi(g)| \le \varepsilon/2 + \delta < 1/C$. By Lemma 1, this means that, for every $g \in G$, the cyclic representations $n \mapsto \rho(\kappa(g^n))$ and $n \mapsto \pi(g^n)$, $n \in \mathbb{N}$, are similar; more precisely, there is a bounded linear operator $A(g) \in \mathcal{L}(E)$ with bounded inverse such that $A(g)\pi(g) = \rho(\kappa(g))A(g)$ for every $g \in G$ and $\|A(g) - 1_E\| \le C\varepsilon$. This completes the proof.

§ 4. Comments

It seems that the assertion of the theorem, claiming only the local equivalence of the pseudorepresentations in question, cannot be strengthened to similarity.

Question. Let the conditions of the theorem hold. What conditions ensure the existence of a pseudorepresentation ρ of G/N for which $\rho \circ \kappa = \pi$?

As is known, for one-dimensional pseudorepresentations, the answer is very simple: no additional conditions are needed for this coincidence [6].

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