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GROUPS OF ONE-DIMENSIONAL PURE PSEUDOREPRESENTATIONS OF GROUPS

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Abstract. The group of bounded one-dimensional pure pseudorepresentations of a group is introduced together with its subgroup generated by bounded one-dimensional pure pseudorepresentations with sufficiently small defects. This subgroup of "good" one-dimensional pseudorepresentations is described for connected Lie groups.

§ 1. Introduction

Let G be a group and let π be a one-dimensional pseudorepresentation of G, i.e., $\pi\colon G \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $\pi(e) = 1$, where π is the identity element of G, and

$$(1)\quad |\pi(gh)-\pi(g)\pi(h)| \le \varepsilon, \qquad g, h \in G, \quad \text{and} \quad \pi(g^k) = \pi(g)^k, \qquad k \in \mathbb{Z}.$$

The minimum number ε satisfying (1) is called the *defect* of the pseudorepresentation π. A pseudorepresentation is said to be *pure* if its restriction to every amenable subgroup of G is an ordinary complex character of the subgroup. For the generalities concerning pseudorepresentations, see [1–5]; for the specific features concerning one-dimensional pseudorepresentations, see [6].

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§ 2. Preliminaries

Lemma 1. *Let G be a group, and let π and ρ be bounded one-dimensional pure pseudorepresentations of G with the defects ε_π and ε_ρ, respectively. Then the mapping $\pi\rho\colon G \to \mathbb{T}$, where $\mathbb{T} = \{z : z \in \mathbb{C}, |z| = 1\}$, defined by the rule*

$$\pi\rho(g) = \pi(g)\rho(g), \qquad g \in G,$$

is a bounded one-dimensional pure pseudorepresentation of G whose defect $\varepsilon_{\pi\rho}$ does not exceed $\varepsilon_\pi + \varepsilon_\rho$. In particular, the family of bounded one-dimensional pure pseudorepresentations of G is a group with respect to the ordinary pointwise multiplication of mappings.

Proof. Since $\pi(e) = \rho(e) = 1$, it follows that $\pi\rho(e) = 1$. Since π and ρ are bounded, it follows that $\pi\rho$ is also bounded. Since $|\pi(g)| = |\rho(g)| = 1$ for every $g \in G$, it follows that, for every $g_1, g_2 \in G$, we have

$$\begin{aligned}
|\pi\rho(g_1g_2) - \pi\rho(g_1)\pi\rho(g_2)| &= |\pi(g_1g_2)\rho(g_1g_2) - \pi(g_1)\pi(g_2)\rho(g_1)\rho(g_2)| \\
&\le |\pi(g_1g_2)\rho(g_1g_2) - \pi(g_1)\pi(g_2)\rho(g_1g_2)| \\
&\quad + |\pi(g_1)\pi(g_2)\rho(g_1g_2) - \pi(g_1)\pi(g_2)\rho(g_1)\rho(g_2)| \\
&= |\pi(g_1g_2) - \pi(g_1)\pi(g_2)| + |\rho(g_1g_2) - \rho(g_1)\rho(g_2)| \le \varepsilon_\pi + \varepsilon_\rho.
\end{aligned}$$

which proves that $\pi\rho$ is a one-dimensional pseudorepresentation and $\varepsilon_{\pi\rho} \le \varepsilon_\pi + \varepsilon_\rho$. Since the restrictions of π and ρ to any amenable subgroup H of G are ordinary unitary characters of H, it follows that the restriction of $\pi\rho$ to H is a product of two unitary characters of H, and hence a unitary character of‘H. Therefore, $\pi\rho$ is a bounded pure pseudorepresentation of G whose defect satisfies the inequality $\varepsilon_{\pi\rho} \le \varepsilon_\pi + \varepsilon_\rho$.

Definition 1. Denote the group of bounded one-dimensional pure pseudorepresentations of a group G by $\mathrm{BODPP}(G)$.

Remark 1. Obviously, an arbitrary mapping $f\colon G \to \mathbb{T}$ satisfies the condition

$$|f(gh) - f(g)f(h)| \le 2, \qquad g, h \in G,$$

and thus, if the defect of a pure pseudorepresentation f is not less than 2, then the only meaningful condition on f is the condition of purity claiming that the restriction of f to every amenable subgroup H of G is an ordinary unitary character of H.

Remark 2. Let G be an amenable group and let f be a one-dimensional bounded ε-quasirepresentation of G satisfying the conditions $f(e) = 1$ and $\varepsilon < 1/5$. Then there is an ordinary unitary character ψ of G such that $|f(g) - \psi(g)| < 1/2$ for all $g \in G$, and the character ψ is uniquely defined by these conditions.

Proof. According to Lemma 3.1 of [6], if G is an amenable group and f is a one-dimensional bounded ε-quasirepresentation of G satisfying the conditions $f(e) = 1$ and $\varepsilon < 1/3$, then there is an ordinary unitary character ψ of G for which

$$|f(g) - \psi(g)| < \varepsilon/(1 - 3\varepsilon) \quad \text{for any} \quad g \in G.$$

If $\varepsilon < 1/5$, then

$$|f(g) - \psi(g)| \leq q < 1/2$$

for some q and every $g \in G$, and therefore, if there is another unitary character χ of G such that

$$|f(g) - \chi(g)| \leq q < 1/2$$

for any $g \in G$, then

$$|\psi(g) - \chi(g)| \leq 2q < 1$$

for any $g \in G$; hence $\psi = \chi$, and thus the character ψ is defined uniquely.

Remark 3. Recall that, by Corollary 3.2 of [6], if G is a group and f is a one-dimensional bounded pseudorepresentation of G with defect $\varepsilon < 0.24$, then the restriction of f to every amenable subgroup of G is a homomorphism of this subgroup into $\mathbb{T}$, and thus f is a pure pseudorepresentation.

Definition 2. Let GBODP(G) (the first "G" in this notation stays for "good") be the family of bounded one-dimensional pseudorepresentations of a group G whose defect is less than 0.24 and which are thus pure pseudorepresentations by Remark 3. Let BODP(G) be the subgroup of BODPP(G) generated by GBODP(G).

§ 3. Main theorem

For the terminology used in the statement of the following theorem, see [4].

Theorem 1. *Let G be a connected Lie group, let R be the radical of G, let S be a Levi subgroup of G, and let* $\mathrm{BODP}(G)$ *be the group introduced in Definition 2. If S has no Hermitian symmetric subgroups, then* $\mathrm{BODP}(G)$ *coincides with the group of characters of G defined by the central characters of R (i.e., the characters of R invariant with respect to the inner automorphisms of G). If S has Hermitian symmetric subgroups, then the group* $\mathrm{BODP}(G)$ *is generated by the family of one-dimensional pure pseudorepresentations F whose defect is less than* 0.24 *and whose restriction to R is some central character χ of R and the restriction to S is a function of the form*

$$f\colon s \mapsto e^{i\Phi(s)}, \qquad s \in S,$$

where Φ stands for a Guichardet–Wigner pseudocharacter on G and f agrees with χ on $S \cap R$, by the rule

$$F(sr) = f(s)\chi(r), \qquad s \in S, \quad r \in R. \tag{2}$$

Proof. The assertion follows immediately from the definition of the group $\mathrm{BODP}(G)$ and from Lemma 3.3.10 and Theorem 3.3.17 of [4]. Formula (2) follows from the fact that the mapping $n \mapsto F((sr)^n)$, $n \in \mathbb{Z}$, where $s \in S$ and $r \in R$, is a homomorphism:

$$F((sr)^n) = F(s^n(s^{-(n-1)}rs^{n-1}s^{-(n-2)}rs^{n-2}\cdots s^{-1}rsr)) = f(s^n)\chi(r^n) = F(sr)^n,$$

and from Remark 2.

§ 4. Concluding remarks

Using the results of [5], we can extend the result of Theorem 1 to connected locally compact groups.

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