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Four solutions for $2m-$Laplacian jumping problem crossing two eigenvalues

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Abstract This paper is dealt with $2m-$Laplacian jumping problem with nonlinearities crossing eigenvalues by using geometric mapping on the finite dimensional reduced subspace. We get one theorem which shows at least four solutions for $2m-$Laplacian jumping problem with nonlinearities crossing two eigenvalues. We obtain this result by finite dimensional reduction method and geometric mapping on the finite reduced subspace.

Key Words and Phrases: $2m$-Laplacian boundary value problem; $2m-$Laplacian eigenvalue problem; jumping nonlinearity; finite dimensional reduction method; geometric mapping on the finite reduced subspace.

AMS 2000 Mathematics Subject Classifications: 35A01, 35A16, 35J30, 35J40, 35J60
[2]

1. Introduction

In this paper we consider multiplicity of solutions for the following $2m$-Laplacian problem with Dirichlet boundary condition and jumping nonlinearities;

$$-\text{div}(|\nabla u|^{2m-2}\nabla u) = b|u|^{2m-2}u^{+} - a|u|^{2m-2}u^{-} + s\phi_1^{2m-1} \qquad \text{in} \;\; \Omega, \tag{1.1}$$

$$u = 0 \qquad \text{on } \partial\Omega,$$

where Ω is a bounded domain in R^n, $n \geq 2$, with smooth boundary $\partial\Omega$,$s \in R$, $m \in N$, $m < \infty$, $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$.

$p-$Laplacian boundary value problems with $p-$growth conditions arise in applications of nonlinear elasticity theory, electro rheological fluids, non-Newtonian fluid theory in a porous medium (cf. [5], [11]. Our problems are characterized as a jumping problem.

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Jumping problem was first suggested in the suspension bridge equation as a model of the nonlinear oscillations in differential equation

$$u_{tt} + K_1 u_{xxxx} + K_2 u^+ = W(x) + \epsilon f(x,t), \tag{1.2}$$

$$u(0,t) = u(L,t) = 0, \qquad u_{xx}(0,t) = u_{xx}(L,t) = 0.$$

This equation represents a bending beam supported by cables under a load f. The constant b represents the restoring force if the cables stretch. The nonlinearity u^+ models the fact that cables resist expansion but do not resist compression. Choi and Jung (cf. [1], [3], [4]) and McKenna and Walter (cf.[10]) investigate existence and multiplicity of solutions for the single nonlinear suspension bridge equation with Dirichlet boundary condition. In [2], the authors investigate the multiplicity of solutions of a semilinear equation

$$Au + bu^+ - au^- = f(x) \qquad \text{in} \ \ \Omega,$$

$$u = 0 \qquad \text{on} \ \ \Omega,$$

where Ω is a bounded domain in R^n, $n \geq 1$, with smooth boundary $\partial\Omega$ and A is a a second order linear partial differential operator when the forcing term is a multiple $s\phi_1$, $s \in R$, of the positive eigenfunction and the nonlinearity crosses eigenvalues.

We know that the eigenvalue problem

$$-\Delta u = \lambda u \qquad \text{in} \ \ \Omega,$$

$$u = 0 \qquad \text{on} \ \ \partial\Omega$$

has infinitely many positive eigenvalues λ_j, $j = 1, 2, \cdots$, $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ and the corresponding normalized eigenfunctions ϕ_j, $j = 1, 2, \cdots$, where the first eigenfunction ϕ_1 is positive. We note that the $2m-$Laplacian eigenvalue problem

$$-\text{div}(|\nabla u|^{2m-2}\nabla u) = \Lambda |u|^{2m-2} u \qquad \text{in} \ \ \Omega,$$

$$u = 0 \qquad \text{on} \ \ \Omega$$

has infinitely many eigenvalues $\Lambda_j = \lambda_j^m$, $0 < \Lambda_1 = \lambda_1^m \leq \Lambda_2 = \lambda_2^m \leq \cdots \leq \Lambda_k = \lambda_k^m \leq \cdots$ and the corresponding normalized eigenfunctions ϕ_j, $j = 1, 2, \cdots$, where the first eigenfunction $\phi_1 > 0$..

In general, It was proved in [7] that when $1 < p < \infty$, the eigenvalue problem

$$-\text{div}(|\nabla u|)^{p-2}\nabla u) = \lambda |u|^{p-2} u \qquad \text{in} \ \ \Omega,$$

$$u = 0 \qquad \text{on} \ \ \partial\Omega$$

has a nondecreasing sequence of nonnegative eigenvalues ν_j obtained by the Ljusternik-Schnirelman principle tending to ∞ as $j \to \infty$, where the first eigenvalue ν_1 is simple and only eigenfunctions associated with ν_1 do not change sign, the set of eigenvalues is closed, the first eigenvalue ν_1 is isolated.

Let $L^p(\Omega, R)$ be the p-Lebesgue space defined by

$$L^p(\Omega, R) = \{u| \ u : \Omega \to R \text{ is measurable, } \int_c^d |u|^p dx < \infty\}$$

and $W^{1,p}(\Omega, R)$ be the $p-$Lebesgue Sobolev space defined by

$$W^{1,p}(\Omega, R) = \{u \in L^p(\Omega, R)| \ \nabla u(x) \in L^p(\Omega, R)\}.$$

We introduce norm on $L^p(\Omega, R)$ and $W^{1,p}(\Omega, R)$ respectively, by

$$\|u\|_{L^p(\Omega)} = \inf\{\lambda > 0| \ \int_c^d |\frac{u(x)}{\lambda}|^p \le 1\},$$

$$\|u\|_{W^{1,p}(\Omega,R)} = [\int_c^d |\nabla u(x)|^p dx]^{\frac{1}{p}}.$$

By [6], when $1 < p < \infty$, the embedding

$$W^{1,p}(\Omega, R) \hookrightarrow L^p(\Omega, R)$$

is continuous and compact and for every $u \in C_0^\infty(\Omega, R)$, we have

$$\|u\|_{L^p(\bar{\Omega},R)} \le C\|u\|_{W^{1,p}(\bar{\Omega},R)}$$

for a positive constant C independent of u. Thus we have that the solutions of the problem

$$-\text{div}(|\nabla u|)^{p-2}\nabla u) = f(x, u) \qquad \text{in} \ \ L^p(\Omega),$$

$$u = 0 \qquad \partial\Omega$$

belong to $W^{1,p}(\Omega)$.

Let us set the operator $-\Delta_{2m}$ by

$$-\Delta_{2m}u = -\text{div}(|\nabla u|^{2m-2}\nabla u).$$

Then (1.1) is equivalent to the equation

$$u = (-\Delta_{2m})^{-1}(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- + s\phi_1^{2m-1}).$$

Our main theorem is as follows:

Theorem 1.1. *Let $m \in N$, $m < \infty$, $a < b$, $-\infty < a < \lambda_1^m$, $\lambda_2^m < b < \lambda_3^m$ and $s < 0$. Then (1.1) has at least four solutions.*

For the proof of Theorem 1.1 we use the finite dimensional reduction method to reduce the problem from an infinite dimensional one on $L^{2m}(\Omega)$ to a finite dimensional one, and geometric mapping on the finite reduced subspace. The outline of the proof of Theorem 1.1 is as follows: In Section 2, we introduce some preliminaries. In Section 3, we prove

Theorem 1.1 by using finite dimensional reduction method and geometric mapping on the finite reduced subspace.

2. Finite dimensional reduction and mapping on finite dimensional subspace

We assume that $m \in N$, $m < \infty$, $a < b$, $-\infty < a < \lambda_1^m$, $\lambda_2^m < b < \lambda_3^m$. Under these assumptions, we are concerned with multiplicity of solutions of $2m-$Laplacian Dirichlet boundary value problem

$$-\text{div}(|\nabla u|)^{2m-2}\nabla u) = b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- + f(x) \qquad \text{in} \;\; \Omega, \tag{2.1}$$

$$u = 0 \qquad\qquad \text{on} \;\; \partial\Omega,$$

where we suppose that $f(x) = s\phi_1^{2m-1}$, $s \in R$. To study equation (2.1), we shall reduce an infinite dimensional problem on $L^{2m}(\Omega)$ to a finite dimensional one.

Let V be the two dimensional subspace of $L^{2m}(\Omega)$ spanned by ϕ_1 and ϕ_2 and W be the orthogonal complement of V in $L^{2m}(\Omega)$. Let P be an orthogonal projection from $L^{2m}(\Omega)$ onto V. Then every element $u \in L^{2m}(\Omega)$ is expressed by

$$u = v + z,$$

where $v = Pu$, $z = (I - P)u$. Hence equation (2.1) is equivalent to a pair of equations

$$(I-P)\big(-\text{div}(|\nabla(v+z)|^{2m-2}\nabla(v+z))\big) = (I-P)\big(b|v+z|^{2m-2}(v+z)^+ - a|v+z|^{2m-2}(v+z)^-\big), \tag{2.2}$$

$$P\big(-\text{div}(|\nabla(v+z)|^{2m-2}\nabla(v+z))\big) = P\big(b|v+z|^{2m-2}(v+z)^+ - a|v+z|^{2m-2}(v+z)^- + s\phi_1^{2m-1}\big). \tag{2.3}$$

We can consider (2.2) and (2.3) as a system of two equations in two unknowns v, z.

Lemma 2.1. *Let Let $m \in N$, $m < \infty$, $a < b$, $-\infty < a < \lambda_1^m$, $\lambda_2^m < b < \lambda_3^m$. For fixed $v \in V$, (2.2) has a unique solution $z = \theta(v)$. Furthermore, $\theta(v)$ is Lipschitz continuous (with respect to L^{2m} norm) in terms of v.*

Proof. We suppose that for fixed $v \in V$, (2.2) has two solutions z_1, z_2. Then we have

$$\begin{aligned}
&(I-P)[\big(-\text{div}(|\nabla(v+z_1)|^{2m-2}\nabla(v+z_1))\big) - \big(-\text{div}(|\nabla(v+z_2)|^{2m-2}\nabla(v+z_2))\big)] \\
&= (I-P)[\big(b|v+z_1|^{2m-2}(v+z_1)^+ - a|v+z_1|^{2m-2}(v+z_1)^-\big) \\
&\quad -\big(b|v+z_2|^{2m-2}(v+z_2)^+ - a|v+z_2|^{2m-2}(v+z_2)^-\big)].
\end{aligned} \tag{2.4}$$

Taking the inner product of (2.4) with $z_1 - z_2$, we have

$$
\begin{aligned}
&\big\langle (I-P)[\big(-\operatorname{div}(|\nabla(v+z_1)|^{2m-2}\nabla(v+z_1))\big) \\
&\qquad -\big(-\operatorname{div}(|\nabla(v+z_2)|^{2m-2}\nabla(v+z_2))\big)], z_1 - z_2\big\rangle \\
&= \big\langle (I-P)[\big(b|v+z_1|^{2m-2}(v+z_1)^+ - a|v+z_1|^{2m-2}(v+z_1)^-\big) \\
&\qquad -\big(b|v+z_2|^{2m-2}(v+z_2)^+ - a|v+z_2|^{2m-2}(v+z_2)^-\big)], z_1 - z_2\big\rangle. \qquad (2.5)
\end{aligned}
$$

The left hand side of (2.5) is equal to

$$
\begin{aligned}
&\big\langle (I-P)[\big(-\operatorname{div}(|\nabla(v+z_1)|^{2m-2}\nabla(v+z_1))\big) \\
&\qquad\qquad -\big(-\operatorname{div}(|\nabla(v+z_2)|^{2m-2}\nabla(v+z_2))\big)], z_1 - z_2\big\rangle \\
&= (2m-1)\int_\Omega [(I-P)[\big((|\nabla(v+z_2+\theta(z_1-z_2))|^{2m-2} \\
&\qquad\qquad \nabla(v+z_2+\theta(z_1-z_2))(\nabla(z_1-z_2))^2\big)]dx \\
&\geq (2m-1)\lambda_3^m \int_\Omega [(I-P)\big(|(v+z_2)+\theta(z_1-z_2)|^{2m-2}(z_1-z_2)^2\big)]dx. \qquad (2.6)
\end{aligned}
$$

by mean value theorem. On the other hand, the right hand side of (2.5) is equal to

$$
\begin{aligned}
&\big\langle (I-P)[\big(b|v+z_1|^{2m-2}(v+z_1)^+ - a|v+z_1|^{2m-2}(v+z_1)^-\big) \\
&\qquad -\big(b|v+z_2|^{2m-2}(v+z_2)^+ - a|v+z_2|^{2m-2}(v+z_2)^-\big)], z_1 - z_2\big\rangle \\
&\leq (2m-1)b\int_\Omega [[I-P]|v+z_2+\theta(z_1-z_2)|^{2m-2}(z_1-z_2)^2]dx \qquad (2.7)
\end{aligned}
$$

for $0 < \theta < 1$. On the other hand, by (2.6) and (2.7), we have

$$
\begin{aligned}
&(2m-1)\lambda_3^m \int_\Omega [(I-P)\big(|(v+z_2)+\theta(z_1-z_2)|^{2m-2}(z_1-z_2)^2\big)]dx \\
\leq\;& (2m-1)b\int_\Omega [[I-P]|v+z_2+\theta(z_1-z_2)|^{2m-2}(z_1-z_2)^2]dx,
\end{aligned}
$$

which is a contradiction because $b < \lambda_3^m$. Thus $z_1 = z_2$. Thus for fixed $v \in V$, every solution of (2.2) is a unique solution $z = \theta(v) \in W$ which satisfies (2.2). It follows that, by the standard argument principle, that $\theta(v)$ is Lipschitz continuous in v. ∎

By Lemma 2.1, the study of multiplicity of solutions of (2.1) is reduced to the study of multiplicity of solutions of an equivalent problem

$$
\begin{aligned}
&P\big(-\operatorname{div}(|\nabla(v+\theta(v))|^{2m-2}\nabla(v+\theta(v)))\big) \\
&= P\big(b|v+\theta(v)|^{2m-2}(v+\theta(v))^+ - a|v+\theta(v)|^{2m-2}(v+\theta(v))^- + s\phi_1^{2m-1}\big) \qquad (2.8)
\end{aligned}
$$

defined on the two-dimensional subspace V spanned by $\{\phi_1,\ \phi_2\}$. For some special case $u's$, we know $\theta(v)$ as follows: If $v \geq 0$ or $v \leq 0$, then $\theta(v) = 0$. In fact, for example, take $v \geq 0$ and $\theta(v) = 0$. Then (2.2) is reduced to

$$
(I-P)\big(-\operatorname{div}(|\nabla(v)|^{2m-2}\nabla v)\big) = (I-P)\big(b|v|^{2m-2}v^+ - a|v|^{2m-2}v^-\big) = 0
$$

because $v^+ = v$, $v^- = 0$, $(I-P)\big(-\operatorname{div}(|\nabla(v)|^{2m-2}\nabla v)\big) = 0$ and $(I-P)b|v|^{2m-2}v = 0$. If $v < 0$ and $\theta(v) = 0$, then

$$\begin{aligned}(I-P)\big(-\operatorname{div}(|\nabla(v)|^{2m-2}\nabla v)\big) &= (I-P)\big(b|v|^{2m-2}v^+ - a|v|^{2m-2}v^- + s\phi_1^{2m-1}\big)\\ &= (I-P)(a|v|^{2m-2}v + s\phi_1^{2m-1}) = 0\end{aligned}$$

because $v^+ = 0$, $v^- = -v$, $(I-P)\big(-\operatorname{div}(|\nabla(v)|^{2m-2}\nabla v)\big) = 0$ and $(I-P)a|v|^{2m-2}v = 0$. Thus (2.1) is reduced to

$$P\big(-\operatorname{div}(|\nabla v|^{2m-2}\nabla v\big) = P\big(b|v|^{2m-2}v^+ - a|v|^{2m-2}v^- + s\phi_1^{2m-1}\big),$$

where $v = c\phi_1$, $c \in R$.

We define a map $h : V \to V$ given by

$$h(v) = P(-\Delta_{2m}(v+\theta(v)) - P\big(b|v+\theta(v)|^{2m-2}(v+\theta(v))^+ - a|v+\theta(v)|^{2m-2}(v+\theta(v))^-\big) \quad (2.9)$$

for $v \in V$. Then h is continuous on V, since θ is continuous on V.

LEMMA 2.2. *$h(dv) = d^{2m-1}h(v)$ for $d \geq 0$ and $v \in V$.*

Proof. We can easily check that $\theta(dv) = d\theta(v)$. It follows the lemma. ∎

LEMMA 2.3. *Let $m \in N$ and $m < \infty$. Then there exists $\tau > 0$ such that*

$$\big\langle h(d_1\phi_1 + d_2\phi_2), \phi_1^{2m-1}\big\rangle \leq -\tau|d_2|^{2m-1}.$$

Proof. Let $u = d_1\phi_1 + d_2\phi_2 + \theta(d_1, d_2)$. Then

$$\begin{aligned}&\big\langle h(d_1\phi_1 + d_2\phi_2), \phi_1^{2m-1}\big\rangle\\ &= \big\langle P(-\Delta_{2m}(d_1\phi_1 + d_2\phi_2 + \theta(d_1,d_2)) - \lambda_1^m(d_1\phi_1 + d_2\phi_2 + \theta(d_1,d_2))^{2m-1}), \phi_1^{2m-1}\big\rangle\\ &\quad - \big\langle (P\big(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- + \lambda_1^m u^{2m-1}\big), \phi_1^{2m-1}\big\rangle.\end{aligned}$$

The first part of the right hand side is equal to 0 because $(P(-\Delta_{2m}(d_1\phi_1 + d_2\phi_2 + \theta(d_1,d_2)) - \lambda_1^m(d_1\phi_1 + d_2\phi_2 + \theta(d_1,d_2))^{2m-1})\phi_1^{2m-1} = 0$. Since

$$P\big(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- + \lambda_1^m u^{2m-1}\big) \geq \min\{b - \lambda_1^m, \lambda_1^m - a\}|u|^{2m-1},$$

we have

$$\big\langle h(d_1\phi_1 + d_2\phi_2), \phi_i^{2m-1}\big\rangle \leq -\min\{b - \lambda_1^m, \lambda_1^m - a\}\int |u|^{2m-1}\phi_1^{2m-1}.$$

Since $\min\{b - \lambda_1^m, \lambda_1^m - a\} > 0$, there exists a constant $\tau > 0$ such that

$$\min\{b - \lambda_1^m, \lambda_1^m - a\}\phi_1^{2m-1} \geq \tau|\phi_2|^{2m-1}$$

for some $\tau > 0$. It follows that

$$\big\langle h(d_1\phi_1 + d_2\phi_2), \phi_1^{2m-1}\big\rangle \leq -\tau\int |u|^{2m-1}|\phi_2^{2m-1}| \leq -\tau|\int (u\phi_2)^{2m-1}| = -\tau|(u,\phi_2)|^{2m-1}.$$

3. Proof of Theorem 1.1

By Lemma 2.2, h maps a cone with vertex 0 onto a cone with vertex 0.

Let us split V into four regions as follows: Since the subspace V is spanned by $\{\phi_1, \phi_2\}$ and $\phi_1(x) > 0$ in Ω, there exists a cone D_1 defined by

$$D_1 = \{v = d_1\phi_1 + d_2\phi_2 : d_1 \geq 0,\ |d_2| \leq \epsilon_0 d_1\}$$

for some small number $\epsilon_0 > 0$ so that $v \geq 0$ for all $v \in D_1$ and a cone D_3 defined by

$$D_3 = \{v = d_1\phi_1 + d_2\phi_2 : d_1 \leq 0,\ |d_2| \leq \epsilon_0 |d_1|\}$$

so that $v \leq 0$ for all $v \in D_3$. Thus by the above statement, $\theta(v) = 0$ for $v \in D_1 \cup D_3$. Let us set

$$D_2 = \{v = d_1\phi_1 + d_2\phi_2 : d_2 > 0,\ \epsilon_0|d_1| \leq d_2\}$$

and

$$D_4 = \{v = d_1\phi_1 + d_2\phi_2 : d_2 < 0,\ \epsilon_0|d_1| \leq |d_2|\}.$$

Then the union of four cones D_i $(1 \leq i \leq 4)$ is the space V. Now we investigate the images of the cones D_1 and D_3 under h. First we consider the image of the cone D_1. If $v = d_1\phi_1 + \epsilon_0\phi_2 \geq 0$, then $v > 0$ and $\theta(v) = 0$. It follows that $(v + \theta(v))^+ = v$ and $(v + \theta(v))^- = 0$. Thus we have

$$\begin{aligned} h(v) &= P(-\Delta_{2m}(v + \theta(v)) - P\big(b|v + \theta(v)|^{2m-2}(v + \theta(v))^+ - a|v + \theta(v)|^{2m-2}(v + \theta(v))^-\big) \\ &= \lambda_1^m d_1^{2m-1}\phi_1^{2m-1} + \lambda_2^m d_2^{2m-1}\phi_2^{2m-1} - b(d_1^{2m-1}\phi_1^{2m-1} + d_2^{2m-1}\phi_2^{2m-1}) \\ &= (\lambda_1^m - b)d_1^{2m-1}\phi_1^{2m-1} + (\lambda_2^m - b)d_2^{2m-1}\phi_2^{2m-1}. \end{aligned}$$

Thus the images of the rays $d_1\phi_1 \pm \epsilon_0 d_1\phi_2 (d_1 \geq 0)$ can be explicitly calculated and they are

$$d_1^{2m-1}(\lambda_1^m - b)\phi_1^{2m-1} \pm \epsilon_0^{2m-1} d_1^{2m-1}(\lambda_2^m - b)\phi_2^{2m-1} \qquad (d_1 \geq 0).$$

Therefore h maps D_1 onto the cone

$$E_1 = \left\{ e_1\phi_1^{2m-1} + e_2\phi_2^{2m-1} : e_1 \leq 0, |e_2| \leq \epsilon_0^{2m-1}\left(\frac{\lambda_2^m - b}{\lambda_1^m - b}\right) e_1 \right\}.$$

The cone E_1 is in the left half-plane of V and the restriction $h|_{D_1} : D_1 \to E_1$ is bijective.

Next We determine the image of the cone D_3. If $v = -d_1\phi_1 + d_2\phi_2 \leq 0$, we have

$$\begin{aligned} h(v) &= P(-\Delta_{2m}(v + \theta(v)) - P\big(b|v + \theta(v)|^{2m-2}(v + \theta(v))^+ - a|v + \theta(v)|^{2m-2}(v + \theta(v))^-\big) \\ &= \lambda_1^m(-d_1^{2m-1})\phi_1^{2m-1} + \lambda_2^m d_2^{2m-1}\phi_2^{2m-1} - a(-d_1^{2m-1})\phi_1^{2m-1} + d_2^{2m-1}\phi_2^{2m-1}) \\ &= (\lambda_1^m - a)(-d_1^{2m-1})\phi_1^{2m-1} + (\lambda_2^m - a)d_2^{2m-1}\phi_2^{2m-1}. \end{aligned}$$

Thus the images of the rays $-d_1\phi_1 \pm \epsilon_0 d_1\phi_2$. $(d_1 \geq 0)$ can be explicitly calculated and they are

$$-d_1^{2m-1}(\lambda_1^m - a)\phi_1^{2m-1} \pm \epsilon_0^{2m-1} d_1^{2m-1}(\lambda_2^{2m-1} - a)\phi_2^{2m-1} \qquad (d_1 \geq 0).$$

Therefore h maps D_3 onto the cone

$$E_3 = \left\{ e_1\phi_1^{2m-1} + e_2\phi_2^{2m-1} : e_1 \leq 0, |e_2| \leq \epsilon_0^{2m-1} |\frac{\lambda_2^m - a}{\lambda_1^m - a}| \, |e_1| \right\}.$$

The cone E_3 is in the left half-plane of V and the restriction $h|_{D_3} : D_3 \to E_3$ is bijective. We note that $E_1 \subset E_3$ since $a < \lambda_1^m < \lambda_2^m < b < \lambda_3^m$.

Thus $h(v) = s\phi^{2m-1}$, $s < 0$, has one solution in each of the cones D_1, D_3, namely

$$(\frac{s}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1 > 0 \qquad -(\frac{s}{a - \lambda_1^m})^{\frac{1}{2m-1}}\phi_1 < 0.$$

Now we investigate the images of the cone D_2 and D_4 under the map h. Let us consider the image under h of the line L in D_2: $L : v = d_1\phi_1 + d_2\phi_2 \in D_2$ with $d_2 \geq \epsilon_0|d_1|$, $d_2 = k$ for some $k > 0$.

By Lemma 2.3, we have

$$\left\langle h(v), \phi_1^{2m-1} \right\rangle \leq -\tau|d_2|^{2m-1}.$$

Therefore the image of $h(L)$ of $L : d_2 = k$, $d_1 \leq \frac{1}{\epsilon_0}k$ must lie to the left of the line $e_1 = -\tau k^{2m-1}$. Thus we have shown that if $u = d_1\phi_1 + k\phi_2 + \theta(d_1, k)$, $k > 0$, $|d_1| \leq \frac{k}{\epsilon_0}$, then u satisfies, for some d_1,

$$-\text{div}(|\nabla u|)^{2m-2}\nabla u) - b|u|^{2m-2}u^+ + a|u|^{2m-2}u^- = s\phi_1^{2m-1}$$

for some $s < -\tau k^{2m-1}$ and k is positive.

Similarly we can get one solution of (1.1) in the region D_4 as follows: Let us consider the image under h of the line $\bar{L}$ in D_4:$\bar{L} : v = d_1\phi_1 + d_2\phi_2 \in D_4$ with $|d_2| \geq \epsilon_0|d_1|$, $d_2 = -k$ for some $k > 0$.

By Lemma 2.3, we also have

$$\left\langle h(v), \phi_1^{2m-1} \right\rangle \leq -\tau|d_2|^{2m-1} = -\tau k^{2m-1}.$$

Therefore the image of $h(\bar{L})$ of $\bar{L} : d_2 = -k$, $|d_1| \leq \frac{1}{\epsilon_0}| - k|$ must lie to the left of the line $e_1 = -\tau| - k|^{2m-1}$. Thus we have shown that if $\bar{u} = d_1\phi_1 - k\phi_2 + \theta(d_1, -k)$, $k > 0$, $|d_1| \leq \frac{|-k|}{\epsilon_0}$, then $\bar{u}$ satisfies, for some d_1,

$$-\text{div}(|\nabla u|)^{2m-2}\nabla u) - b|u|^{2m-2}u^+ + a|u|^{2m-2}u^- = s\phi_1^{2m-1}$$

for some $s < -\tau| - k|^{2m-1}$ and $-k$ is negative. Thus for some $s < -\tau| \pm k|^{2m-1}$, $k > 0$, one solution $(\frac{s}{b-\lambda_1^m})^{\frac{1}{2m-1}}\phi_1^{2m-1}$ is in D_1, another solution $-(\frac{s}{\lambda_1^m - a})^{\frac{1}{2m-1}}\phi_1^{2m-1}$ is in D_3, the third one is in D_2 and the fourth one is in D_4. Thus we prove that (1.1) has at least four solutions, one in each of the four cones, which D_1 and D_3 divide the , ϕ_1, ϕ_2 plane into. Thus we prove Theorem 1.1. ▪

Declarations

List of abbreviations

Not applicable

Availability of data and materials

Not applicable

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Authors's contributions

Tacksun Jung introduced the main ideas of multiplicity study for this problem. Q-Heung Choi participate in applying the method for solving this problem and drafted the manuscript. All authors contributed equally to read and approved the final manuscript.

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GROUPS OF ONE-DIMENSIONAL PURE PSEUDOREPRESENTATIONS OF GROUPS

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Abstract. The group of bounded one-dimensional pure pseudorepresentations of a group is introduced together with its subgroup generated by bounded one-dimensional pure pseudorepresentations with sufficiently small defects. This subgroup of "good" one-dimensional pseudorepresentations is described for connected Lie groups.

§ 1. Introduction

Let G be a group and let π be a one-dimensional pseudorepresentation of G, i.e., $\pi\colon G \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $\pi(e) = 1$, where π is the identity element of G, and

$$(1)\quad |\pi(gh)-\pi(g)\pi(h)| \le \varepsilon, \qquad g, h \in G, \quad \text{and} \quad \pi(g^k) = \pi(g)^k, \qquad k \in \mathbb{Z}.$$

The minimum number ε satisfying (1) is called the *defect* of the pseudorepresentation π. A pseudorepresentation is said to be *pure* if its restriction to every amenable subgroup of G is an ordinary complex character of the subgroup. For the generalities concerning pseudorepresentations, see [1–5]; for the specific features concerning one-dimensional pseudorepresentations, see [6].

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